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DIFFERENTIABLE SEMIFLOWS FOR DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS

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Careful modelling of systems governed by delayed feedback often leads to delay differential equations where the delay is not constant but depends on the state of the system and its history. In typical, not too complicated cases one arrives at equations of the form

$$\dot{x}(t) = g(x(t - r(x_t))), \tag{1}$$

with a map $g : \mathcal{O} \to \mathbb{R}^n$, $\mathcal{O} \subset \mathbb{R}^n$ open, and with a delay functional r which is defined on some set of functions $\phi : [-h, 0] \to \mathbb{R}^n$ and has values in $[0, h]$, for some $h > 0$. The function x_t in eq. (1) is defined by

$$x_t(s) = x(t + s) \text{ for } -h \le s \le 0$$

as usual. In more general cases, the right hand side of the differential equation contains more arguments. It also happens that the delay is given only implicitly by an equation which involves the history x_t of the state. Differential equations with state-dependent delay share the property that the results on the uniqueness and dependence on initial data from the theory of retarded functional differential equations (RFDEs) on the state space $C = C([-h, 0], \mathbb{R}^n)$, with

$$\|\phi\| = \max_{-h \le t \le 0} |\phi(t)|,$$

are not applicable. For data in C, the initial value problem (IVP) for eq. (1) given by

$$x_0 = \phi$$

is not well-posed. Also, the linearization at a stationary solution seemed impossible [**2**]. In the present paper we describe results which overcome these difficulties, explain an estimate from the proof of the main theorem, and give an example which is based on elementary physics and satisfies the hypotheses we need. A particular feature is that we find a semiflow with optimal smoothness properties not on an open subset of a Banach space, but on an infinite-dimensional submanifold which is defined by the differential equation.

Let us first see why for the IVP associated with eq. (1) the results from, e.g., [**3, 5**] on the uniqueness and dependence on initial data for RFDEs

$$\dot{x}(t) = f(x_t) \tag{2}$$

with a functional $f : \mathcal{U} \to \mathbb{R}^n$, $\mathcal{U} \subset C$, fail to apply. If the delay functional is a map

$$r : \mathcal{U} \to [0, h]$$

and if g is defined on $\mathbb{R}^n$ (for simplicity) then eq. (1) has the form (2) for

$$f = g \circ ev \circ (id \times (-r)),$$

where

$$ev : C \times [-h, 0] \to \mathbb{R}^n$$

is the evaluation map given by

$$ev(\phi, s) = \phi(s).$$

The problem is now that except for the cases which are not of interest here (e.g., r constant) f does not satisfy the hypotheses required for the associated IVP to be well-posed. In general, f is not even locally Lipschitz continuous, no matter how smooth g and r are. A 'reason' for this may be seen in the fact that the middle composite ev is not smooth: Lipschitz continuity of ev would imply Lipschitz continuity of elements $\phi \in C$. Differentiability would imply that

$$D_2 ev(\phi, s)1 = \dot{\phi}$$

exists.

If C is replaced with the smaller Banach space $C^1 = C^1([-h, 0], \mathbb{R}^n)$ of continuously differentiable functions $\phi : [-h, 0] \to \mathbb{R}^n$, with the norm given by

$$\|\phi\|_1 = \|\phi\| + \|\dot{\phi}\|,$$

then the smoothness problem disappears, since the restricted evaluation map

$$Ev : C^1 \times [-h, 0] \to \mathbb{R}^n$$

is continuously differentiable, with

$$D_1 Ev(\phi, s)\chi = Ev(\chi, s) \quad \text{and} \quad D_2 Ev(\phi, s)1 = \dot{\phi}(s).$$

So, for $g : \mathbb{R}^n \to \mathbb{R}^n$ and $r : U \to [0, h]$, $U \subset C^1$ open, both continuously differentiable, the resulting functional

$$f = g \circ Ev \circ (id \times (-r))$$

is continuously differentiable from U to $\mathbb{R}^n$.

Let us now abandon the special case of eq. (1) and consider eq. (2), with $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, continuously differentiable.

A look at the associated IVP

$$\dot{x}(t) = f(x_t), x_0 = \phi$$

reveals that also this problem is not well-posed for arbitrary data in the open subset $U \subset C^1$: A solution $x : [-h, t_e) \to \mathbb{R}^n$, $0 < t_e \leq \infty$, would have continuously differentiable segments x_t, $0 \leq t < t_e$. Hence the solution itself would be continuously differentiable, and the curve $[0, t_e) \ni t \mapsto x_t \in C^1$ would be continuous. Continuity at $t = 0$ yields

$$\dot{\phi}(0) = \dot{x}(0) = f(x_0) = f(\phi),$$

an equation which is in general not satisfied on the entire set U. In any case, we are led to consider the closed subset

$$X = X_f = \{\phi \in U : \dot{\phi}(0) = f(\phi)\}$$

of $U \subset C^1$.

Notice that X is a nonlinear version of the positively invariant domain

$$\{\phi \in C^1 : \dot{\phi}(0) = L\phi\}$$

of the generator G of the semigroup defined by the linear autonomous RFDE

$$\dot{y}(t) = Ly_t$$

on the larger space C, for $L : C \to \mathbb{R}^n$ linear continuous.

For a class of differential equations with state-dependent delay, Louihi, Hbid, and Arino [**10**] identified the set X as the domain of the generator of a nonlinear semigroup in a state space different from C^1. They mention without proof that X is a Lipschitz manifold. In [**9**] a complete metric space analogous to X serves as a state space for neutral functional differential equations.

Notice also that in the case of a locally Lipschitz continuous map $f_* : U_* \to \mathbb{R}^n$, $U_* \subset C$ open, all solutions $x : [-h, b) \to \mathbb{R}^n$, $h < b$, of the RFDE

$$\dot{x}(t) = f_*(x_t)$$

satisfy

$$x_t \in X_{f_*} = \{\phi \in U_* \cap C^1 : \dot{\phi}(0) = f_*(\phi)\} \text{ for } h \leq t < b.$$

Thus the set X_{f_*} absorbs all flowlines on intervals $[-h, b)$ which are long enough. In particular, X_{f_*} contains all segments of solutions on intervals $(-\infty, b)$, $b \leq \infty$ – equilibria, periodic orbits, local unstable manifolds, and the global attractor if the latter is present.

In order to have a semiflow on X with differentiability properties, we need X to be smooth. This requires an additional condition on f. A suitable condition, which is satisfied in a variety of examples coming from differential equations with state-dependent delay, is that

(P1) every derivative $Df(\phi)$, $\phi \in U$, has a linear extension

$$D_e f(\phi) : C \to \mathbb{R}^n$$

which is continuous with respect to the norm on C.

Property (P1) is a special case of the condition used in Krisztin's recent work on local unstable manifolds [**7**]. Earlier, it appeared under the name of *almost Frechet differentiability* in Mallet-Paret, Nussbaum, and Paraskevopoulos's work [**12**] on the existence of periodic solutions.

It is then easy to prove that in case (P1) holds and $X \neq \emptyset$ the set X is a continuously differentiable submanifold of C^1 with codimension n.

The argument is the following. We have

$$X = (p - f)^{-1}(0),$$

with the continuous linear map

$$p : C^1 \ni \phi \mapsto \dot{\phi}(0) \in \mathbb{R}^n.$$

The Implicit Function Theorem yields local graph representations of X, provided all derivatives $D(p - f)(\phi)$ are surjective. Proof of this, for $n = 1$: Let $\phi \in X$ be given. Due to (P1) there exists $\delta > 0$ such that

$$|D_e f(\phi)\psi| < 1 \text{ for all } \psi \in C \text{ with } \ |\psi\| < \delta.$$

There exists $\psi \in C^1$ with $\|\psi\| < \delta$ and $\dot{\psi}(0) = 1$. Hence

$$0 < \dot{\psi}(0) - Df(\phi)\psi = D(p - f)(\phi)\psi,$$

$$\mathbb{R} \subset D(p - f)(\phi)C^1.$$

For $t_0 < t_e \leq \infty$ we define a solution of eq. (2) on $[t_0 - h, t_e)$ to be a continuously differentiable map $x : [t_0 - h, t_e) \to \mathbb{R}^n$ such that $x_t \in U$ for $t_0 \leq t < t_e$ and eq. (2) is satisfied for $0 < t < t_e$. We also consider solutions on unbounded intervals $(-\infty, t_e)$. Maximal solutions of IVPs are defined as usual.

In order to obtain maximal solutions for data in X and a nice semiflow on X we need a local Lipschitz condition on f, namely that for every $\phi \in U$ there exist a neigbourhood N of ϕ in C^1 and $L \geq 0$ with

$$\text{(P2) } |f(\psi) - f(\overline{\psi})| \leq L\|\psi - \overline{\psi}\| \text{ for all } \psi, \overline{\psi} \text{ in } N.$$

Notice that this Lipschitz estimate involves the smaller norm

$$\| \cdot \| \leq \| \cdot \|_1$$

on the larger space $C \supset C^1$; it is not a consequence of continuous differentiability of f. Property (P2) is closely related to the idea of being *locally almost Lipschitzian* from [**12**]. In [**13**] it is used in a proof that certain differential equations with state-dependent delay generate stable periodic motion.

Theorem 1. [**14**] *Suppose $X \neq \emptyset$, and (P1) and (P2) hold. Then the maximal solutions $x^{\phi} : [-h, t_e(\phi)) \to \mathbb{R}^n$ of eq. (2) which start at points $x_0^{\phi} = \phi \in X$ define a continuous semiflow*

$$F : \Omega \to X$$

by

$$F(t, \phi) = x_t^{\phi}, 0 \leq t < t_e(\phi).$$

All solution maps

$$F_t : \{\phi \in X : 0 \leq t < t_e(\phi)\} \ni \phi \mapsto F(t, \phi) \in X,\ t \geq 0,$$

on nonempty domains are continuously differentiable. For all $\phi \in X$, $t \in [0, t_e(\phi))$, and $\chi \in T_\phi X$,

$$DF_t(\phi)\chi = v_t^{\phi,\chi}$$

with a continuously differentiable solution $v^{\phi,\chi} : [-h, t_e(\phi)) \to \mathbb{R}^n$ of the IVP

$$\dot{v}(t) = Df(F_t(\phi))v_t,\ v_0 = \chi.$$

A condition which implies both (P1) and (P2) and which can be verified for a large variety of differential equations with state-dependent delay is that (P) each map $Df(\phi), \phi \in U$, has a linear extension $D_e f(\phi) : C \to \mathbb{R}^n$, and the map

$$U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

Let us mention that the simpler and stronger condition of continuity of the map $U \ni \phi \mapsto D_e f(\phi) \in L_c(C, \mathbb{R}^n)$ is typically violated by differential equations with state-dependent delay.

For a proof that (P) implies (P1) and (P2), see [**15**]. The main result of [**15**] says that under hypothesis (P) the restriction of the semiflow to the open subset $\{(t, \phi) \in \Omega : h < t\}$ of $(0, \infty) \times X$ is continuously differentiable, with

$$D_1F(t, \phi)1 = \dot{x}_t^{\phi} \in C^1.$$

A comparison with results for RFDEs shows that a better smoothness can not be expected.

Comments on Theorem 1. (1) The first remark concerns linearization at a stationary point $\phi_0 \in X$. We know from Theorem 1 that time-t maps can be differentiated and that their derivatives are given by a variational equation on the tangent bundle TX. When Theorem 1 was not available, authors studying solutions close to equilibria used an auxiliary linear RFDE on the space C instead of the variational equation on TX. See, e.g., Cooke and Huang's work [**2**] on the principle of linearized stability, [**12**], and Krishnan's [**6**] and Krisztin's [**7**] works on local unstable manifolds. The method to obtain this auxiliary equation is heuristic: In equations like (1), where the delay appears explicitly, one can freeze the delay at the equilibrium and linearize the resulting RFDE with constant delay. The question arises how the auxiliary equation is related to the variational equation from Theorem 1. A look at the relevant examples shows that the auxiliary equation on the space C coincides with the equation

$$\dot{v}(t) = D_e f(\phi_0) v_t \tag{3}$$

in our framework. In other words, the true linearization is given by the restriction of the auxiliary equation to the tangent bundle of X.

It is also true that $T_{\phi_0}X$ coincides with the domain of the generator G of the semigroup $(T_t)_{t\geq 0}$ on C defined by the solutions of eq. (3), and

$$DF_t(\phi_0)\chi = T(t)\chi \quad \text{on} \quad T_{\phi_0}X.$$

(2) Theorem 1 yields continuously differentiable local unstable, center, and stable manifolds

$$W_u,\ W_c,\ W_s$$

of the solution maps F_t at fixed points; in particular, at stationary points ϕ_0 of the semiflow. In the last case the tangent spaces of the local invariant manifolds at ϕ_0 are the unstable, center, and restricted stable spaces

$$C^u, C^c, \quad \text{and} \quad C^s \cap T_{\phi_0}X$$

of the generator G, respectively. At a stationary point ϕ_0, the local unstable and stable manifolds W_u, W_s of the time–t maps coincide with local unstable and stable manifolds of the semiflow F. For center manifolds the analogue of the previous statement is in general false [**8**], and continuously differentiable local center manifolds for the semiflow from Theorem 1 have not yet been obtained.

The approach to local invariant manifolds via Theorem 1 obviously avoids any additional spectral hypothesis, while the results on unstable manifolds in [**6, 7**] require that the auxiliary linear RFDE be hyperbolic. Hyperbolicity is also necessary for Arino and Sanchez's recent result [**1**], which captures a part of the saddle point behaviour of solutions close to an equilibrium, for certain differential equations with state-dependent delays.

Let us turn to the proof of Theorem 1. An essential part is solving the equation

$$x(t) = \phi(0) + \int_0^t f(x_s)ds,\ 0 \leq t \leq T,$$

$$x_0 = \phi \in X,$$

by a continuously differentiable map

$$x : [-h, T] \to R^n,$$

with $\phi \in X$ given; x should also be continuously differentiable with respect to ϕ. In order to achieve this, we first rewrite the fixed point equation so that the dependence of the integral on ϕ becomes explicit. For $\phi \in C^1$, let $\hat{\phi}$ denote the continuously differentiable extension to $[-h, T]$ given by

$$\hat{\phi}(t) = \phi(0) + \dot{\phi}(0)t \text{ on } [0, T].$$

Set $u = x - \hat{\phi}$. Then u belongs to the Banach space C^1_{0T} of continuously differentiable maps $y : [-h, T] \to \mathbb{R}^n$ with

$$y(t) = 0 \text{ on } [-h, 0];$$

the norm on C^1_{0T} is given by

$$\|y\|_{C^1_{0T}} = \max_{-h \le t \le T} |y(t)| + \max_{-h \le t \le T} |\dot{y}(t)|.$$

u and $\hat{\phi}$ satisfy

$$u(t) + \hat{\phi}(t) = \phi(0) + \int_0^t f(u_s + \hat{\phi}_s) ds$$

and

$$\hat{\phi}(t) = \phi(0) + \dot{\phi}(0)t = \phi(0) + t\, f(\phi)$$

(since $\phi \in X$)

$$= \phi(0) + \int_0^t f(\phi) ds.$$

For $u \in C^1_{0T}$ this yields the fixed point equation

$$u(t) = \int_0^t (f(u_s + \hat{\phi}_s) - f(\phi)) ds, \; 0 \le t \le T,$$

with a parameter $\phi \in X$.

Now let some $\phi_0 \in X$ be given. For $\phi \in X$ close to ϕ_0, $u \in C^1_{0T}$ small, and $0 \le t \le T$ with $T > 0$ small, define

$$A(\phi, u)(t)$$

to be the right hand side of the latest equation. Property (P2) is used to show that the maps $A(\phi, \cdot)$ are contractions with respect to the norm on C^1_{0T}, with a contraction factor independent of ϕ: Let $v = A(\phi, u), \overline{v} = A(\phi, \overline{u})$. For $0 \le t \le T$,

$$|\dot{v}(t) - \dot{\overline{v}}(t)| = |f(u_t + \hat{\phi}_t) - f(\overline{u}_t + \hat{\phi}_t)| \le L \|u_t - \overline{u}_t\|$$

(due to (P2), L may be large)

$$\le L \max_{0 \le s \le T} |u(s) - \overline{u}(s)|.$$

We exploit the fact that the last term does not contain derivatives. For $0 \le s \le T$,

$$|u(s) - \overline{u}(s)| = |0 - 0 + \int_0^s (\dot{u}(r) - \dot{\overline{u}}(r)) dr| \le T \|u - \overline{u}\|_{C^1_{0T}}.$$

Hence

$$|\dot{v}(t) - \dot{\overline{v}}(t)| \le \; L\, T\, \|u - \overline{u}\|_{C^1_{0T}}.$$

Also,

$$|v(t) - \overline{v}(t)| = |\int_0^t (f(u_s + \hat{\phi}_s) - f(\overline{u}_s + \hat{\phi}_s))ds| \leq L\,T \max_{0 \leq s \leq t} \|u_s - \overline{u}_s\|$$
$$\leq L\,T\,\|u - \overline{u}\|_{C^1_{0T}}.$$

(Property (P2) is not necessary here. Alternatively, the local Lipschitz continuity of f with respect to the norm on C^1 can be used to find a suitable upper estimate.)

For $2\,L\,T < 1$, the map $A(\phi, \cdot)$ becomes a contraction.

One finds a closed ball which is mapped into itself by each map $A(\phi, \cdot)$. The formula defining A can be used to show that A is continuously differentiable. It follows that for each ϕ the ball contains a fixed point u_ϕ of $A(\phi, \cdot)$ which is continuously differentiable with respect to ϕ. This completes the first essential step in the proof of Theorem 1.

Example. Consider the motion of an object on a line which attempts to regulate its position by echo. The object emits a signal which is then reflected by an obstacle. The reflected signal is detected and the signal running time is measured. From this, a position is computed; this position is not necessarily the true one. The computed position is followed by an acceleration towards a preferred position, e.g., to an equilibrium position at a certain distance from the obstacle.

Let $c > 0$ denote the speed of the signals, $-w < 0$ the position of the obstacle, and $\mu > 0$ a friction constant. The acceleration is given by a function $a : \mathbb{R} \to \mathbb{R}$; one may think of negative feedback with respect to the position $\xi = 0$ as expressed by the relations

$$a(0) = 0 \text{ and } \xi\, a(\xi) < 0 \text{ for } \xi \neq 0.$$

Let $x(t)$ denote the position of the object at time t, $v(t)$ its velocity, $p(t)$ the computed position, and $s(t)$ the running time of the signal which has been emitted at time $t - s(t)$ and whose reflection is detected at time t. The model equations then are

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = -\mu v(t) + a(p(t))$$
$$p(t) = \frac{c}{2} s(t) - w$$
$$c\, s(t) = x(t - s(t)) + x(t) + 2w.$$

Here the solutions with

$$-w < x(t)$$

are only considered. The justification for the formula defining $p(t)$ is that it yields the true position if

$$x(t) = x(t - s(t)),$$

which holds at least at equilibria.

Let $w_+ > 0$. We restrict our attention further to bounded solutions with

$$-w < x(t) < w_+ \quad \text{and} \quad |\dot{x}(t)| < c.$$

Then necessarily

$$0 < s(t) \leq \frac{2w + 2w_+}{c} = h.$$

The model has not yet the form (2) considered in Theorem 1. In order to reformulate the model, take h as just defined, consider the space

$$C^1 = C^1([-h, 0], \mathbb{R}^2),$$

and the open convex subset

$$U = \{\phi = (\phi_1, \phi_2) \in C^1 : -w < \phi_1(t) < w_+, |\dot{\phi}_1(t)| < c \text{ for } -h \leq t \leq 0\}.$$

Each $\phi \in U$ determines the unique solution $s = \sigma(\phi)$ of

$$s = \frac{1}{c}(\phi_1(-s) + \phi_1(0) + 2w),$$

as the right hand side of this fixed point equation defines a contraction on $[0, h]$. The Implicit Function Theorem shows that the resulting map

$$\sigma : U \to [0, h]$$

is continuously differentiable.

If the response function $a : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. then the map

$$f : U \to \mathbb{R}^2$$

given by

$$f_1(\phi) = \phi_2(0)$$

and

$$f_2(\phi) = -\mu\phi_2(0) + a(\frac{c}{2}\sigma(\phi) - w) = -\mu\phi_2(0) + a\left(\frac{\phi_1(-\sigma(\phi)) + \phi_1(0)}{2}\right)$$

is continuously differentiable, and for bounded solutions as above the model can be rewritten in the form of eq. (2).

In [**14, 15**] it is verified that condition (P) holds. One can easily show that the map $U \ni \phi \mapsto D_e f(\phi) \in L_c(C, \mathbb{R}^n)$ is *not* continuous. In the case $a(0) = 0$, it becomes obvious that the auxiliary linear RFDE at the zero solution is the same as eq. (3) with $\phi_0 = 0$.

Let us mention that the above model is related to the more complicated equations of motion for two charged particles, which were first studied by Driver [**4**]. Recent work [**16**] shows that for suitable parameters our model has a periodic solution whose orbit is exponentially attracting. The analysis makes use of Theorem 1 and of the smoothness result from [**15**].

References

1. Arino O., Sanchez E., *A saddle point theorem for functional state–dependent delay equations,* Preprint, 2002.
2. Cooke K., Huang W., *On the problem of linearization for state-dependent delay differential equations,* Proc. Amer. Math. Soc., **124** (1996), 1417–1426.
3. Diekmann O., van Gils S.A., Verduyn Lunel S.M., Walther H.O., *Delay Equations: Functional-, Complex- and Nonlinear Analysis,* Springer, New York, 1995.
4. Driver R.D., *Existence theory for a delay-differential system,* Contrib. Differential Equations, **1** (1963), 317–336.
5. Hale J.K., Verduyn Lunel S., *Introduction to Functional Differential Equations,* Springer, New York, 1993.
6. Krishnan, H.P., *An analysis of singularly perturbed delay-differential equations and equations with state-dependent delays,* Ph.D. thesis, Brown University, Providence (R.I.), 1998.
7. Krisztin T., *An unstable manifold near a hyperbolic equilibrium for a class of differential equations with state-dependent delay,* Discrete and Continuous Dynamical Systems, **9** (2003), 993–1028.
8. ———, *Invariance and noninvariance of center manifolds of time–t maps with respect to semiflows,* Preprint, 2002.
9. Krisztin T., Wu J., *Monotone semiflows generated by neutral equations with different delays in neutral and retarded parts,* Acta Math. Univ. Comenian., **63** (1994), 207–220.
10. Louihi M., Hbid M.L., Arino O., *Semigroup properties and the Crandall-Liggett approximation for a class of differential equations with state-dependent delays,* J. Differential Equations, **181** (2002), 1–30.
11. Mallet-Paret J., *Morse decompositions for differential delay equations,* J. Differential Equations, **72** (1988), 270–315.
12. Mallet-Paret J., Nussbaum R.D., Paraskevopoulos P., *Periodic solutions for functional differential equations with multiple state-dependent time lags,* Topol. Methods Nonlinear Anal., **3** (1994), 101–162.
13. Walther H.O., *Stable periodic motion of a system with state-dependent delay,* Differential Integral Equations, **15** (2002), 923–944.
14. ———, *The solution manifold and C^1-smoothness of solution operators for differential equations with state dependent delay,* J. Differential Equations, to appear.
15. ———, *Smoothness properties of semiflows for differential equations with state dependent delay,* Proceedings of the International Conference on Differential and Functional Differential Equations, Moscow, 2002, to appear.
16. ———, *Stable periodic motion of a system using echo for position control,* J. Dynamics Differential Equations, to appear.

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