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A FUNCTIONAL EQUATION CHARACTERIZING MONOMIAL FUNCTIONS USED IN PERMANENCE THEORY FOR ECOLOGICAL DIFFERENTIAL EQUATIONS

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Abstract. It is well known that monomial average Liapunov functions of the form $R(x_1, x_2, \ldots, x_n) = r_0 \prod_{i=1}^{n} x_i^{r_i}$ ($r_i > 0,\ i = 0, 1, 2, \ldots, n$) play an eminent role in the permanence theory of ecological (or Kolmogorov) differential equations. A functional equation characterizing the above class of functions is presented.

1. Introduction. We consider an autonomous differential equation of the Kolmogorov type

$$(1) \qquad \dot{x}_i = x_i f_i(x) \quad (i = 1, 2, \ldots, n)\ , \quad x = (x_1, x_2, \ldots, x_n) \in X$$

where X is the probability simplex $\{x \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1\}$ and $f\colon X \to \mathbb{R}^n$ is a C^ℓ function satisfying $\sum_i x_i f_i(x) = 0$ for each $x \in X$ ($\ell \geq 1$ is an integer). The solving dynamical system is denoted by $\Phi : \mathbb{R} \times X \to X$. It is immediate that ∂X, the boundary of X, is Φ–invariant. System (1) is called *permanent* (or uniformly persistent) [**4**] if ∂X is a repellor. The standard biological interpretation is that x_i represents the relative frequency of the i–th species ($i = 1, 2, \ldots, n$) in a given ecosystem whereas permanence means the ultimate survival of all the species involved.

Besides stability theory, criteria ensuring permanence involve various branches of mathematics (index theory [**11**], [**10**], ergodic theory [**9**], [**8**], Morse decompositions [**3**], linear programming [**2**]) and have been intensively studied

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in recent years (for more results, see the references in [**4**], [**5**]). Monomial functions of the form

(2) $$R(x) = r_0 \prod_{i=1}^{n} x_i^{r_i} \ ; \quad r_i \text{ is a positive constant } (i = 0, 1, 2, \ldots, n) \ , \quad x \in X$$

have played a distinguished role in this development.

The aim of this paper is to characterize functions of the form (2) by a functional equation. When doing this, we follow a tradition which goes back to Cauchy, who characterized linear functions of the form $L : \mathbb{R} \to \mathbb{R}, x \to \text{const} \cdot x$ as continuous solutions of the functional equation $L(x + a) = L(x) + L(a)$, $x, a \in \mathbb{R}$. As for the second example, we refer to the important contribution of functional equations theory in characterizing (multivariate) normal distributions. These and many more examples can be found in the monographs by Kuczma [**7**] and by Aczél and Dhombres [**1**].

Ecological differential systems of the form $\dot{x}_i = x_i f_i(x)$ $(i = 1, 2, \ldots, n)$ can be defined on $\mathbb{R}^n_+$ as well. The permanence theory for ecological equations on X is more or less parallel to the permanence theory for *dissipative* ecological equations on $\mathbb{R}^n_+$: the non-compactness of the phase space can be counterbalanced by a compactness condition on the flow. On the other hand, from the viewpoint of functional equations, the lack of compactness of $\mathbb{R}^n_+$ is irrelevant. Moreover, when solving functional equations on $\mathbb{R}^n_+$, the lack of constraints like $\sum_i x_i = 1$ or $\sum_i x_i f(x) = 0$ is truly advantageous and makes various simplifications possible.

Throughout this paper, we shall be working within the framework of the probability simplex X. The case of $\mathbb{R}^n_+$ will be settled by Remark 2.

2. Average Liapunov functions and a functional equation. Recall that a continuous mapping $P : \mathbb{R}^n_+ \to \mathbb{R}$ is a *good average Liapunov function for* (1) (or, equivalently, *for the induced continuous–time dynamical system* Φ) [**2**] (a version of earlier concepts in [**4**]) if

(a) $P(x) = 0$ for all $x \in \partial\mathbb{R}^n_+$, $P(x) > 0$ for all $x \in \text{int}(\mathbb{R}^n_+)$;

(b) P is differentiable on $\text{int}(\mathbb{R}^n_+)$ and $p_i(x) := \frac{x_i}{P(x)} \frac{\partial P}{\partial x_i}$ can be extended to a continuous function on X for every i; and

(c) For every $x \in \partial X$ there is a positive constant T_x with the property that $\int_0^{T_x} \sum_i p_i(\Phi(t, x)) f_i(\Phi(t, x)) \, dt > 0$.

The existence of a good average Liapunov function for (1) implies a particularly strong form of permanence: it implies that ∂X is robustly and exponentially repulsive [**2**]. The standard candidate for a good average Liapunov function belongs to the function class (2). In this case $p_i(x) = r_i$ $(i = 1, 2, \ldots, n)$, assumptions (a) and (b) are automatically satisfied, and the inequality in assumption (c) simplifies to $\int_0^{T_x}\sum_i r_i f_i(\Phi(t,x))\,dt > 0$.

The functional equation characterizing function class (2) will be derived by a heuristic application of Euler's discretization method to the differential identity

$$\frac{d}{dt}\log(P(\Phi(t,x))) = \sum_{i=1}^{n} p_i(\Phi(t,x))f_i(\Phi(t,x)) \ , \quad (t,x) \in \mathbb{R} \times (X \setminus \partial X).$$

By the definition of the time derivative at $t_0 = 0$, it follows that

$$\text{(3)} \qquad \log\left(\frac{P(\Phi_1(t,x),\Phi_2(t,x),\ldots,\Phi_n(t,x))}{P(x_1,x_2,\ldots,x_n)}\right) = t\sum_{i=1}^{n} p_i(x)f_i(x) + o(t)$$

whenever $(t,x) \in \mathbb{R} \times (X \setminus \partial X)$. On the other hand, system (1) can be reformulated on $X \setminus \partial X$ as

$$\frac{d}{dt}\log(\Phi_i(t,x)) = f_i(\Phi(t,x)) \quad (i = 1,2,\ldots,n) \ , \quad (t,x) \in \mathbb{R} \times (X \setminus \partial X).$$

It is not hard to show that $\Phi_i(t,x) = x_iQ_i(t,x)$ where $Q = (Q_1,Q_2,\ldots,Q_n) : \mathbb{R} \times X \to \mathbb{R}_+ \setminus \{0\}$ is a C^ℓ function. (The existence and continuity of the highest-order derivative is a consequence of the C^ℓ parametrized version of the Picard–Lindelöf theorem. Indeed, with $x \in X$ as a parameter, let $z(\cdot;x)$ denote the solution of the initial value problem

$$\dot{z}_i = z_if_i(x_1z_1,x_2z_2,\ldots,x_nz_n) \text{ and } z_i(0) = 1, \quad i = 1,2,\ldots,n.$$

Since $(x_1z_1(\cdot;x),x_2z_2(\cdot;x),\ldots,x_nz_n(\cdot;x))$ is a solution to (1), we have by uniqueness that $z(t;x) = Q(t,x)$ for all $t \in \mathbb{R}$ and $x \in X$.) Thus $\sum_i x_iQ_i(t,x) = 1$ for each $(t,x) \in \mathbb{R} \times X$ by invariance and

$$\text{(4)} \quad \log(Q_i(t,x)) = t\,f_i(x) + o(t) \quad (i = 1,2,\ldots,n) \ , \quad (t,x) \in \mathbb{R} \times (X \setminus \partial X).$$

Omitting the $o(t)$ terms in (3)–(4), we conclude that

$$\log\left(\frac{P(x_1Q_1(1,x),x_2Q_2(1,x),\ldots,x_nQ_n(1,x))}{P(x_1,x_2,\ldots,x_n)}\right) \approx \sum_{i=1}^{n} p_i(x)\cdot\log(Q_i(1,x)).$$

Thus we feel motivated enough to investigate the class of continuous functions $R : X \to \mathbb{R}$ with the following properties:

(A) $R(x) = 0$ for all $x \in \partial X$, $R(x) > 0$ for all $x \in X \setminus \partial X$; and

(B) There exist continuous functions $r_i : X \to \mathbb{R}$ $(i = 1,2,\ldots,n)$ such that

$$\log\left(\frac{R(x_1F_1,x_2F_2,\ldots,x_nF_n)}{R(x_1,x_2,\ldots,x_n)}\right) = \sum_{i=1}^{n} r_i(x_1,x_2,\ldots,x_n)\cdot\log(F_i)$$

whenever $(x_1,x_2,\ldots,x_n),(x_1F_1,x_2F_2,\ldots,x_nF_n) \in X \setminus \partial X$.

Theorem. *Let $R : X \to \mathbb{R}$ be a continuous mapping and assume that conditions (A) and (B) are satisfied. Then there are positive constants $\{r_i\}_{i=0}^n$ such that $r_i(x) = r_i$ for $i = 1, 2, \ldots, n$ and $R(x) = r_0 \prod_{i=1}^n x_i^{r_i}$, $x \in X$. In other words, R belongs to the function class defined by (2).*

The proof of the Theorem is postponed to Section 3. The result itself has already been announced in [**2**].

Remark 1. Discrete–time dynamical systems of the Kolmogorov type on X are iterates of self–homeomorphisms $\mathcal{F}$ of X of the form

$$\mathcal{F}(x) = (x_1 F_1(x), x_2 F_2(x), \ldots, x_n F_2(x))$$

where $F_i : X \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ is a continuous function with $F_i(x) = F_i(x_1, x_2, \ldots, x_n) > 0$ whenever $x_i > 0$. A continuous mapping $R : X \to \mathbb{R}$ is an *average Liapunov function for the discrete–time dynamical system* $(\mathcal{F})$ [**6**] if

(A)-(d) $R(x) = 0$ for all $x \in \partial X$, $R(x) > 0$ for all $x \in X \setminus \partial X$;

(B)-(d) There exists a continuous function $r : X \to \mathbb{R}$ such that

$$r(x) = \log\left(\frac{R(x_1 F_1(x), x_2 F_2(x), \ldots, x_n F_n(x))}{R(x_1, x_2, \ldots, x_n)}\right)$$

whenever $x \in X \setminus \partial X$; and

(C)-(d) For every $x \in \partial X$ there is a positive integer N_x with

$$\sum_{k=1}^{N_x} r(\mathcal{F}^{k-1}(x)) \cdot \log(F_i(\mathcal{F}^{k-1}(x))) > 0.$$

The existence of an average Liapunov function for the discrete–time dynamical system $\mathcal{F}$ implies that ∂X is a repellor [**6**]. Robust and exponential repulsivity is implied in the special case with $R(x) = r_0 \prod_{i=1}^n x_i^{r_i}$, $x \in X$. For this and other permanence results on continuous–time, discrete–time, and discretized systems of the Kolmogorov type, see [**2**]. Nevertheless, we have to admit that we are still unable to define — and this would be a better analogy to the concept of good average Liapunov functions — a larger class of average Liapunov functions implying robust and exponential repulsiveness for ∂X in discrete–time dynamical systems of Kolmogorov type. Our theorem above is a negative result in the search for such a larger class of average Liapunov functions.

The next remark has already been announced at the end of the Introduction.

Remark 2. *The Theorem remains valid if X and ∂X are replaced by $\mathbb{R}_+^n$ and $\partial\mathbb{R}_+^n$, respectively.*

3. The proof of the Theorem. In order to keep the technicalities limited, we restrict ourselves to a proof of the special case $n = 4$ and only indicate that the general case follows from the very same considerations.

With a self–explanatory notation, we pass to the functional equation

$$\text{(5)} \qquad \log\left(\frac{R(xF, yG, zH, 1-xF-yG-zH)}{R(x,y,z,1-x-y-z)}\right) = \rho_1(x,y,z)\cdot\log(F)$$

$$+\rho_2(x,y,z)\cdot\log(G)+\rho_3(x,y,z)\cdot\log(H)+\rho_4(x,y,z)\cdot\log\left(\frac{1-xF-yG-zH}{1-x-y-z}\right).$$

Taking $G = H = 1$ and $F = a$, (5) goes over into the simplified functional equation

$$\text{(6)} \qquad \log\left(\frac{R(xa,y,z,1-xa-y-z)}{R(x,y,z,1-x-y-z)}\right)$$

$$= \rho_1(x,y,z)\cdot\log(a) + \rho_4(x,y,z)\cdot\log\left(\frac{1-xa-y-z}{1-x-y-z}\right).$$

With y and z as parameters (satisfying $y, z > 0$, $y + z < 1$), (6) simplifies further to a functional equation

$$\text{(7)} \qquad \log\frac{r(xa)}{r(x)} = p(x)\cdot\log(a) + q(x)\cdot\log\left(\frac{1-xa-y-z}{1-x-y-z}\right).$$

in two variables. Here of course $0 < x, xa < 1-y-z$, $p(x) = p_{y,z}(x) = \rho_1(x,y,z)$, $q(x) = q_{y,z}(x) = \rho_4(x,y,z)$, $r(x) = r_{y,z}(x) = R(x,y,z,1-x-y-z)$. Note that $p, q, r : [0, 1-y-z] \to \mathbb{R}$ are continuous functions and, in view of assumption (A), $r(0) = r(1-y-z) = 0$ and $r(x) > 0$ for $x \in (0, 1-y-z)$.

Claim 1: We claim that

$$\text{(8)} \qquad r(x) = \kappa x^{\alpha}(1-x-y-z)^{\delta} \quad \text{with some positive constants} \quad \kappa, \alpha, \delta.$$

Indeed, rewrite equation (7) in the form

$$\frac{r(xa)-r(x)}{xa-x} = \frac{r(x)}{x}\cdot\frac{a^{p(x)}\cdot\left(\frac{1-xa-y-z}{1-x-y-z}\right)^{q(x)}-1}{a-1}.$$

By letting $a \to 1$ in the respective difference quotients, the existence of the limit on the right–hand side shows that, at least on the open interval $(0, 1-y-z)$, the function r is differentiable and

$$r'(x) = \frac{r(x)}{x}\cdot\left(p(x) + q(x)\cdot\frac{-x}{1-x-y-z}\right)$$

or, equivalently,

$$\text{(9)} \qquad r(x) = r\left(\frac{1-y-z}{2}\right)\cdot\exp\left(\int_{\frac{1-y-z}{2}}^{x}\left(\frac{p(s)}{s} - \frac{q(s)}{1-s-y-z}\right)ds\right)$$

for $x \in (0, 1-y-z)$. Replacing x by xa in (9), a twofold substitution in (7) yields

$$\int_x^{xa} \left(\frac{p(s)}{s} - \frac{q(s)}{1-s-y-z} \right) ds = p(x) \cdot \log(a) + q(x) \cdot \log \left(\frac{1-xa-y-z}{1-x-y-z} \right)$$

and, a fortiori, via differentiation with respect to a,

$$x \left(\frac{p(xa)}{xa} - \frac{q(xa)}{1-xa-y-z} \right) = \frac{p(x)}{a} + q(x) \cdot \frac{-x}{1-xa-y-z}$$

whenever $x, xa \in (0, 1-y-z)$. By passing to the new pair of variables, x and $c = xa$, we conclude that

$$\frac{p(c)}{c} - \frac{q(c)}{1-c-y-z} = \frac{p(x)}{c} - \frac{q(x)}{1-c-y-z} .$$

By taking $c = c_1$ and $c = c_2$, the last identity goes over into a linear system of equations for $p(x)$ and $q(x)$. It follows immediately via Cramer's rule that functions p and q are constants, say α and δ, respectively. In view of the sign conditions on r, formula (9) simplifies to (8) and this ends the proof of *Claim 1.* Actually, applying Cramer's rule and performing the integration in (9), we conclude that our constants κ, α, δ depend continuously on parameters y, z. The continuity assumption on r_1 and r_4 implies that functions α and δ extend continuously to $\{(y,z) \in \mathbb{R}^2 \mid y \geq 0, z \geq 0, y+z \leq 1\}$, the closure of their previous domain of definition.

Thus we are justified in writing that

$$R(x,y,z,1-x-y-z) = \kappa(y,z) x^{\alpha(y,z)} (1-x-y-z)^{\delta(y,z)} \tag{10}$$

whenever $x > 0$, $y > 0$, $z > 0$, $x+y+z < 1$.

Claim 2: We claim that $\alpha(y,z) = \alpha$, a positive constant. Indeed, assume to the contrary that $\alpha(y_1, z_1) \neq \alpha(y_2, z_2)$ for some $(y_1, z_1), (y_2, z_2) \in \{(y,z) \in \mathbb{R}^2 \mid y \geq 0, z \geq 0, y+z \leq 1\}$. By symmetry and continuity, we see there is no loss of generality in assuming that $z_1 = z_2 = z$, $\alpha(y_1, z) < \alpha(y_2, z)$ and $y_1 > 0$, $y_2 > 0$, $z > 0$, $y_1 + z < 1$, $y_2 + z < 1$. Thus

$$\kappa(y_1, z) x^{\alpha(y_1,z)} (1-x-y_1-z)^{\delta(y_1,z)} = R(x, y_1, z, 1-x-y_1-z),$$
$$\kappa(y_2, z) x^{\alpha(y_2,z)} (1-x-y_2-z)^{\delta(y_2,z)} = R(x, y_2, z, 1-x-y_2-z)$$

whenever $x > 0$, $x + y_1 + z < 1$, $x + y_2 + z < 1$. On the other hand, the derivation of formula (10) shows that

$$R(x, y_1, z, 1-x-y_1-z) = \lambda(x,z) y_1^{\beta(x,z)} (1-x-y_1-z)^{D(x,z)},$$
$$R(x, y_2, z, 1-x-y_2-z) = \lambda(x,z) y_2^{\beta(x,z)} (1-x-y_2-z)^{D(x,z)}$$

where $\lambda, \beta, D : \{(x,z) \in \mathbb{R}^2 \mid x > 0, z > 0, x+z < 1\} \to \mathbb{R}^+ \setminus \{0\}$ are suitably chosen continuous functions. The continuity assumption on r_2 and r_4 implies that functions β and D extend to the closure of their previous domain of definition. In particular, there is

$$\frac{x^{\alpha(y_1,z)}}{x^{\alpha(y_2,z)}} = \frac{\kappa(y_2,z)}{\kappa(y_1,z)} \cdot \frac{y_1^{\beta(x,z)}}{y_2^{\beta(x,z)}} \cdot \frac{(1-x-y_1-z)^{D(x,z)-\delta(y_1,z)}}{(1-x-y_2-z)^{D(x,z)-\delta(y_2,z)}}\,. \tag{11}$$

By letting $x \to 0$, we conclude that the right–hand side of (11) remains bounded but the left–hand side approaches infinity, a contradiction. This ends the proof of *Claim 2* and, by symmetry, shows also that $\rho_i(x,y,z) = \rho_i$ $(i = 1,2,3,4$ with $\rho_1 = \alpha)$, a positive constant.

Thus we are justified in writing that

$$R(x,y,z,1-x-y-z) = \kappa(y,z)x^{\alpha}(1-x-y-z)^{\delta}$$
$$= \lambda(x,z)y^{\beta}(1-x-y-z)^{\delta} = \mu(x,y)z^{\gamma}(1-x-y-z)^{\delta}$$

where $\alpha,\beta,\gamma,\delta$ are positive constants, and functions

$$\kappa : \{(y,z) \in \mathbb{R}^2 \mid y > 0, z > 0, y+z < 1\} \to \mathbb{R}^+ \setminus \{0\},$$
$$\lambda : \{(x,z) \in \mathbb{R}^2 \mid x > 0, z > 0, x+z < 1\} \to \mathbb{R}^+ \setminus \{0\},$$
$$\mu : \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0, x+y < 1\} \to \mathbb{R}^+ \setminus \{0\}$$

are continuous. Thus $\kappa(y,z)x^{\alpha} = \lambda(x,z)y^{\beta} = \mu(x,y)z^{\gamma}$ whenever $x > 0$, $y > 0$, $z > 0$, $x+y+z < 1$. Since $\kappa(y,z)/y^{\beta} = \lambda(x,z)/x^{\alpha}$ and $\lambda(x,z)/z^{\gamma} = \mu(x,y)/y^{\beta}$, there exist continuous functions $\sigma,\tau : (0,1) \to \mathbb{R}^+ \setminus \{0\}$ such that $\lambda(x,z) = x^{\alpha}\sigma(z) = \tau(x)z^{\gamma}$. The desired result $R(x,y,z,1-x-y-z) = \mathrm{const} \cdot x^{\alpha}y^{\beta}z^{\gamma}(1-x-y-z)^{\delta}$ follows immediately.

Mutatis mutandis, the general case $n \geq 2$ is easily settled by the very same considerations.

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