UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLIII
2005

SECOND ORDER CAUCHY PROBLEM WITH A DAMPING OPERATOR

by Teresa Winiarska

Abstract. The purpose of this paper is to present some theorems on existence and uniqueness of solutions for autonomous (with not densely defined operators) and nonatonomous second order Cauchy problem with a damping operator.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space and let $A : X \to X$ be a linear operator. By $\mathcal{D}(A)$, $\varrho(A)$, $R(\lambda, A)$ we will denote the domain, the resolvent set and the resolvent of A, respectively. The graph of A is isomorphic to the space

$$X_1^A := (\mathcal{D}(A), \|\cdot\|_{X_1^A}), \text{ where } \|x\|_{X_1^A} = \|Ax\| + \|x\|$$

which is called the *interpolation space* for A.

For $\lambda \in \varrho(A)$ the space

$$\begin{aligned} X_{-1}^A &:= \text{the completion of the space } (X, \|\cdot\|_{X_{-1}^A}), \text{ where} \\ \|x\|_{X_{-1}^A} &:= \|R(\lambda, A)x\| \end{aligned}$$

is called the *extrapolation space* for A.

Let us recall that

(a) A is closed if and only if X_1^A is a Banach space.
(b) If 0 belongs to the resolvent set $\varrho(A)$ of A then the norms : $\|\cdot\|_{X_1^A}$ and $\mathcal{D}(A) \ni x \mapsto \|Ax\|$ are equivalent.
(c) Since the norms $X \ni x \mapsto \|R(\lambda, A)x\|$ corresponding to $\lambda \in \varrho(A)$ are equivalent, the space X_{-1}^A is independent of λ.

Key words and phrases. Cauchy problem, damping, not densely defined.

Let $(A(t))_{t\in[0,T]}$, $(B(t))_{t\in[0,T]}$ be two families of linear closed operators from $X \to X$. We consider the following abstract semilinear Cauchy problem

$$\text{(1)} \qquad \begin{cases} \dfrac{d^2u}{dt^2} = B(t)\dfrac{du}{dt} + A(t)u + f\left(t, u, \dfrac{du}{dt}\right), & t \in [0,T], \\ u(0) = u_0, \quad \dfrac{du}{dt}(0) = u_1, \quad u_0, u_1 \in X\,, \end{cases}$$

where $f : [0,T] \times X \times X \to X$ is a given function.

Problems of form (1) appear, for example, in studying problems concerning a rod compressed by a time-dependent follower force and made of a Kelvin–Voigt viscoelastic material.

The paper consists of two independent parts. In first part we consider the case of not densely defined operators $A(t) = A$, $B(t) = B$ independent of t. In the second part, the general case is considered.

2. Autonomous Cauchy problem. In this part we consider an autonomous Cauchy problem corresponding to (1), i.e. the following problem

$$\text{(2)} \qquad \begin{cases} \dfrac{d^2u}{dt^2} = B\dfrac{du}{dt} + Au + f\left(t, u, \dfrac{du}{dt}\right), & t \in [0,T], \\ u(0) = u_0, \quad \dfrac{du}{dt}(0) = u_1, \quad u_0, u_1 \in X\,. \end{cases}$$

For a given two linear operators $A, B : X \to X$, we will use the following four assumptions:

($\mathbf{Z}_1$) $B : X \supsetneq \mathcal{D}(B) \to X$ is a closed linear operator.

($\mathbf{Z}_2$) $\mathcal{D}(B)$ is contained in the domain $\mathcal{D}(A)$ of the operator $A : X \to X$ and A is B bounded, i.e. there exist two non negative constants a, b such that

$$\|Ax\| \le a\|Bx\| + b\|x\| \quad \text{for } x \in \mathcal{D}(B).$$

($\mathbf{Z}_3$) $0 \in \varrho(A) \cap \varrho(B)$.

($\mathbf{Z}_4$) B is a Hille–Yoshida operator of type (M,ω), i.e. there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \varrho(B)$ and

$$\|R(\lambda, B)^n\| \le \frac{M}{(\lambda-\omega)^n}, \quad \text{for } \lambda > \omega,\ n = 1, 2, \ldots$$

Definition 1 ([**3**], Def. 3.1, p. 368). A function $u : [0,T] \to X$ is said to be a classical solution of problem (2) if

(i) $u \in \mathcal{C}^2([0,T], X)$,

(ii) $u(t) \in \mathcal{D}(A)$ for $t \in [0,T]$ and the mapping $[0,T] \ni t \mapsto Au(t) \in X$ is continuous,

(iii) $u'(t) \in \mathcal{D}(B)$ for $t \in [0,T]$ and the mapping $[0,T] \ni t \mapsto Bu'(t) \in X$ is continuous,

(iv) u satisfies (2).

The second-order problem (2) can in a standard way be reduced to the first-order problem (cf. [**3**], p. 368)

$$(3)\qquad \begin{cases} \dfrac{d\mathcal{U}}{dt} = \mathcal{A}\mathcal{U} + F(t,\mathcal{U}), \quad t \in [0,T], \\ \mathcal{U}(0) = \mathcal{U}_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \end{cases}$$

where $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$, $\mathfrak{X} := X_1^B \times X$,

$$\mathcal{U}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \quad v(t) = u'(t), \quad \mathcal{A} = \begin{bmatrix} 0 & I \\ A & B \end{bmatrix}, \quad F(t,\mathcal{U}) = \begin{bmatrix} 0 \\ f(t,u(t),v(t)) \end{bmatrix},$$

$\mathcal{D}(\mathcal{A}) = \mathcal{D}(B) \times \mathcal{D}(B)$ with $\overline{\mathcal{D}(\mathcal{A})} = \mathfrak{X}_0 \neq \mathfrak{X}$.

Lemma 1. *If assumptions* $(\mathbf{Z}_1)$–$(\mathbf{Z}_4)$ *are satisfied, then* $\mathcal{A}$ *is a Hille–Yoshida operator.*

Proof. As in ([**3**], p. 370), we present the operator $\mathcal{A}$ in the form

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_1 + \mathcal{B}_2\,,$$

where

$$\mathcal{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}.$$

We first prove that $\mathcal{A}_0$ is a Hille–Yoshida operator. In fact, there is

$$\begin{aligned} \left\| \mathcal{R}^n(\lambda,\mathcal{A}_0) \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}} &= \left\| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda - B \end{bmatrix}^{-n} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}} \\ &= \left\| \left(\lambda^{-1}(\lambda - B)^{-1} \begin{bmatrix} \lambda - B & 0 \\ 0 & \lambda \end{bmatrix} \right)^n \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}} = \left\| \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & (\lambda - B)^{-1} \end{bmatrix}^n \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}} \\ &= \left\| \begin{bmatrix} \lambda^{-n} x \\ (\lambda - B)^{-n} y \end{bmatrix} \right\|_{\mathfrak{X}} \leq \frac{M}{(\lambda - \omega)^n} \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}}. \end{aligned}$$

Hence,

$$\|\mathcal{R}^n(\lambda,\mathcal{A}_0)\| \leq \frac{M}{(\lambda - \omega)^n}\,,$$

which means that $\mathcal{A}_0$ is a Hille–Yoshida operator on $\mathfrak{X}$.

Since

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}_1^{\mathcal{A}_0}} = \|x\|_{X_1^B} + \|y\|_{X_1^B}\,,$$

there is $\mathfrak{X}_1^{\mathcal{A}_0} = X_1^B \times X_1^B$. Since $\mathcal{A}_0$ is a Hille–Yoshida operator on $\mathfrak{X}$ and $\mathcal{B}_1$ is bounded on $\mathfrak{X}_1^{\mathcal{A}_0}$, the operator $\mathcal{A}_0 + \mathcal{B}_1$ is (by virtue of ([**3**], Corollary 1.4,

p. 160) a Hille–Yoshida operator on $\mathfrak{X}_1^{\mathcal{A}_0}$ and so (by [**3**], Corollary 1.4, p. 160) $\mathcal{A}_0 + \mathcal{B}_1$ is a Hille–Yoshida operator on

$$(\mathfrak{X}_1^{\mathcal{A}_0+\mathcal{B}_1})_{-1}^{\mathcal{A}_0+\mathcal{B}_1} = \mathfrak{X}\,.$$

Since

$$\begin{aligned}\left\|\mathcal{B}_2\begin{bmatrix}x\\y\end{bmatrix}\right\|_{\mathfrak{X}} &= \|Ax\|_X \le a\|Bx\| + b\|x\| \le a\|x\|_{X_1^B} + b\|B^{-1}\|\|x\|_{X_1^B} + \|y\| \\ &\le K(\|x\|_{X_1^B} + \|y\|) = K\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|_{\mathfrak{X}}\,,\end{aligned}$$

the operator $\mathcal{B}_2$ is bounded on $\mathfrak{X}$ and so $(\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_1) + \mathcal{B}_2$ is a Hille–Yoshida operator on $\mathfrak{X}$. □

Let us denote by $\mathcal{A}_1$ the part of $\mathcal{A}$ in $\mathfrak{X}_0 := \overline{\mathcal{D}(\mathcal{A})}$. It follows from Lemma 1 that $\mathcal{A}_0$ is a generator of a $\mathcal{C}_0$ semigroup $\mathcal{T}_1(t)$ on the space $\mathfrak{X}_0$. Then, due to ([**6**], Theorems 3.1.10 and 3.1.11), the operator $\mathcal{A}_1$ can be extended to a closed densely defined operator $\mathcal{A}_{-1} : \mathfrak{X}_{-1}^{\mathcal{A}} \to \mathfrak{X}_{-1}^{\mathcal{A}}$ with the domain $\mathcal{D}(\mathcal{A}_{-1}) = \mathfrak{X}_0$. It is also known that $\mathcal{A}_{-1}$ generates the $\mathcal{C}_0$ semigroup $\mathcal{T}_{-1}(t) = (\mathcal{T}_1(t))_{-1}$.

Now, the problem (2) can be replaced by the following first order problem in the space $\mathfrak{X}_{-1}^{\mathcal{A}}$

$$\begin{cases}\dfrac{d\mathcal{U}}{dt} = \mathcal{A}_{-1}\mathcal{U} + F(t,\mathcal{U}), \quad t \in [0,T],\\ \mathcal{U}(0) = \mathcal{U}_0\end{cases} \tag{4}$$

for which the following theorem holds

Theorem 1 ([**6**], Theorem 4.3.13, p. 82). *If $F : [0,T]\times\mathfrak{X}_0 \to \mathfrak{X}$ is of class $\mathcal{C}^1$ and there exists $L > 0$ such that*

$$\|F(t,\mathcal{U}_1) - F(t,\mathcal{U}_2\|_{\mathfrak{X}} \le L\|\mathcal{U}_1 - \mathcal{U}_2\|_{\mathfrak{X}} \tag{5}$$

then problem (4) has exactly one classical solution if and only if

$$\mathcal{U}_0 \in \mathcal{D}(\mathcal{A}) \text{ and } \mathcal{A}\mathcal{U}_0 + F(0,\mathcal{U}_0) \in \mathfrak{X}_0\,, \tag{6}$$

and it is the unique solution of the following integral equation

$$\mathcal{U}(t) = \mathcal{T}_1(t)\mathcal{U}_0 + \int_0^t \mathcal{T}_{-1}(t-s)F(s,\mathcal{U}(s))ds\,. \tag{7}$$

The following theorem on existence and uniqueness of the classical solution of problem (2) is an immediate consequence of Theorem 1.

THEOREM 2. *If assumptions* ($\mathbf{Z}_1$)–($\mathbf{Z}_4$) *are satisfied and*

(i) $u_0, u_1 \in \mathcal{D}(B)$ *and* $Au_0 + Bu_1 + f(0, u_0, u_1) \in X_0 := \overline{\mathcal{D}(B)}$,

(ii) $f : [0,T] \times X_0 \times X_0 \to X$ *is of class* $\mathcal{C}^1$,

(iii) *there exists* $L > 0$ *such that*

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|) \tag{8}$$

for $t \in [0,T]$ *and* $x_1, x_2, y_1, y_2 \in X_0$,

then problem (1) *has exactly one classical solution.*

3. Nonautonomous Cauchy problem. In this part we will study problem (1) with operators $A(t), B(t)$ dependent on t. We will assume that the operators $A(t), B(t)$ satisfy the following assumptions

($\mathbf{Z}'_1$) The domain $\mathcal{D}(B(t)) = \mathcal{D}_B$ is independent of $t \in [0,T]$, $\mathcal{D}_B$ is dense in X and $\mathcal{D}_B \subset \mathcal{D}(A(t))$ for $t \in [0,T]$.

($\mathbf{Z}'_2$) The operators $A(t)$ are uniformly $B(t)$ bounded, i.e. there exist non negative constants a, b such that

$$\|A(t)x\| \le a\|B(t)x\| + b\|x\| \quad \text{for } t \in [0,T],\ x \in \mathcal{D}_B.$$

($\mathbf{Z}'_3$) $0 \in \varrho(A(t)) \cap \varrho(B(t))$ for $t \in [0,T]$.

($\mathbf{Z}'_4$) The family $(B(t))_{t\in[0,T]}$ is a stable family of generators of C_0 semigroups, i.e. there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

(i) $$(\omega, +\infty) \subset \varrho(B(t)) \quad \text{for } t \in [0,T]\,,$$

(ii) $$\left\| \prod_{j=1}^{k} R(\lambda, B(t_j)) \right\| \le \frac{M}{(\lambda - \omega)^k} \quad \text{for } 0 \le t_1 \le \ldots \le t_k = T, \quad k = 1, 2, \ldots,\ \lambda > \omega\,.$$

Problem (1) can in the standard way be reduced to the following first-order problem

$$\begin{cases} \dfrac{d\mathcal{U}}{dt} = \mathcal{A}(t)\mathcal{U}(t) + F(t, \mathcal{U}(t)), \quad t \in [0,T], \\ \mathcal{U}(0) = \mathcal{U}_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \end{cases} \tag{9}$$

where $\mathcal{A}(t) : \mathfrak{X} \to \mathfrak{X}$, $\mathfrak{X} := X_1^B \times X$ with $B = B(0)$,

$$\mathcal{U}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \quad \mathcal{A}(t) = \begin{bmatrix} 0 & I \\ A(t) & B(t) \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}(t)) = \mathcal{D}_B \times \mathcal{D}_B \subset \mathfrak{X}.$$

$$F(t, \mathcal{U}(t)) = \begin{bmatrix} 0 \\ f(t, u(t), v(t)) \end{bmatrix}, \quad v(t) = u'(t)\,.$$

Similarly to the case of t-independent operators there is

$$\mathcal{A}(t) = \mathcal{A}_0(t) + \mathcal{B}_1(t) + \mathcal{B}_2(t)\,, \tag{10}$$

where

$$\mathcal{A}_0(t) = \begin{bmatrix} 0 & 0 \\ 0 & B(t) \end{bmatrix}, \quad \mathcal{B}_1(t) = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_2(t) = \begin{bmatrix} 0 & 0 \\ A(t) & 0 \end{bmatrix}. \tag{11}$$

Lemma 2. *If, for any $x \in \mathcal{D}_B$, the mapping*

$$[0,T] \ni t \mapsto B(t)x \in X \tag{12}$$

is of class $\mathcal{C}^1$ and assumptions $(\mathbf{Z'}_1)$, $(\mathbf{Z'}_3)$, $(\mathbf{Z'}_4)$ are satisfied, then

(i) *$\mathcal{A}_0(t)$ is a generator of a C_0 semigroup on $\mathfrak{X}$, for each $t \in [0,T]$,*
(ii) *the family $(\mathcal{A}_0(t))_{t\in[0,T]}$ is stable in $\mathfrak{X}$,*
(iii) *the mapping*

$$[0,T] \ni t \mapsto \mathcal{A}_0(t) \begin{bmatrix} x \\ y \end{bmatrix} \in \mathfrak{X}$$

is of class $\mathcal{C}^1$ for $x, y \in \mathcal{D}_B$.

Proof. (i) For $(x,y) \in \mathfrak{X}$ and $\lambda \in \varrho(B(t))$ there is

$$\left\| (\lambda I - \mathcal{A}_0(t))^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}} = \left\| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda - B(t) \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}}$$
$$= \left\| \lambda^{-1}(\lambda - B(t))^{-1} \begin{bmatrix} (\lambda - B(t))x \\ \lambda y \end{bmatrix} \right\|_{\mathfrak{X}} = \left\| \begin{bmatrix} \lambda^{-1}x \\ (\lambda - B(t))^{-1}y \end{bmatrix} \right\|_{\mathfrak{X}}$$
$$= \left\| \lambda^{-1}x \right\|_{X_1^B} + \left\| (\lambda - B(t))^{-1}y \right\| = \left\| \lambda^{-1}B(0)x \right\| + \left\| (\lambda - B(t))^{-1}y \right\| .$$

It follows from $(\mathbf{Z'}_4)$ that

$$\left\| (\lambda - B(t))^{-1}y \right\| \le \frac{M}{\lambda - \omega} \left\| y \right\| .$$

Thus,

$$\left\| (\lambda I - \mathcal{A}_0(t))^{-n} \begin{bmatrix} x \\ y \end{bmatrix} \right\| \le \max\left(\frac{1}{|\lambda|^n}, \frac{M}{(\lambda - \omega)^n} \right) \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}} \le \frac{M}{(\lambda - \omega)^n} \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}} .$$

Hence,

$$\left\| \mathcal{R}^n(\lambda, \mathcal{A}_0(t)) \right\| \le \frac{M}{(\lambda - \omega)^n} \quad \text{for } t \in [0,T],\ n = 1, 2, \ldots,$$

the operator $\mathcal{A}_0(t)$ is a generator of a C_0 semigroup in $\mathfrak{X}$, which ends the proof of (i).

(ii) Now it can be immediately verified that

$$\left\| \prod_{j=1}^{k} \mathcal{R}(\lambda, \mathcal{A}_0(t_j)) \right\|_{\mathfrak{X}} = \left\| \begin{bmatrix} \frac{1}{\lambda^k} & 0 \\ 0 & \prod_{j=1}^{k} R(\lambda, B(t_j)) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}}$$

$$\leq \frac{1}{\lambda^k} \|x\|_{X_1^B} + \frac{M}{(\lambda-\omega)^k} \|y\| \leq \max\left(\frac{1}{\lambda^k}, \frac{M}{(\lambda-\omega)^k} \right) \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\mathfrak{X}}.$$

This completes the proof of (ii).

(iii) is an immediate consequence of the definition of $\mathcal{A}_o$ and the assumed class of the mapping (12). □

LEMMA 3. *Under assumptions of Lemma 2, there is*

(i) $\mathcal{A}_0(t) + \mathcal{B}_1(t)$ *is a generator of* C_0 *semigroup on* $\mathfrak{X}$, *for each* $t \in [0, T]$,

(ii) *the family* $(\mathcal{A}_0(t) + \mathcal{B}_1(t))_{t \in [0,T]}$ *is stable in* $\mathfrak{X}_1^{\mathcal{A}_0}$.

PROOF. (i) is an immediate consequence of ([**3**], Corollary 1.4, p. 160).

(ii) The operator $\mathcal{B}_1(t)$ (defined by (11)) is not bounded in $\mathfrak{X}$. To have it bounded we will consider it as an operator defined on

$$\mathfrak{X}_1^{\mathcal{A}_0} := \mathfrak{X}_1^{\mathcal{A}_0(0)} = X_1^B \times X_1^B.$$

By Lemma 1 and ([**7**], Theorem 4.8, p. 145) the family $(\mathcal{A}_0(t))_{t \in [0,T]}$ is stable in $\mathfrak{X}_1^{\mathcal{A}_0}$. Hence and by ([**7**], Theorem 2.3, p. 132), the family $(\mathcal{A}_0(t) + \mathcal{B}_1(t))_{t \in [0,T]}$ is stable in $\mathfrak{X}_1^{\mathcal{A}_0}$. □

LEMMA 4. *If the assumptions of Lemma 2 are satisfied and for any* $x \in \mathcal{D}_B$ *the mapping* $[0, T] \ni t \mapsto A(t)x \in X$ *is of class* $\mathcal{C}^1$, *then*

(i) $\mathcal{A}(t)$ *is a generator of a* C_0 *semigroup on* $\mathfrak{X}$, *for each* $t \in [0, T]$.

(ii) *The family* $(\mathcal{A}(t))_{t \in [0,T]}$ *(defined by* (10)*) is stable in* $\mathfrak{X}$.

(iii) *For any* $(x, y) \in \mathcal{D}(\mathcal{A})$ *the mapping*

$$[0, T] \ni t \mapsto \mathcal{A}(t) \begin{bmatrix} x \\ y \end{bmatrix} \in \mathfrak{X}$$

is of class $\mathcal{C}^1$.

PROOF. (i) By Lemma 3, for any fixed $t \in [0, T]$, the operator $\mathcal{A}_0(t) + \mathcal{B}_1(t)$ is a generator of a C_0 semigroup in $\mathfrak{X}_1^{\mathcal{A}_0}$. Thus, by ([**3**], Corollary 1.4, p. 160), $\mathcal{A}_0(t) + \mathcal{B}_1(t)$ is a generator of a C_0 semigroup in $\mathfrak{X}_1^{\mathcal{A}_0 + \mathcal{B}_1}$, for each $t \in [0, T]$, and in the extrapolation space

$$(\mathfrak{X}_1^{\mathcal{A}_0 + \mathcal{B}_1})_{-1}^{\mathcal{A}_0 + \mathcal{B}_1} = \mathfrak{X}.$$

By assumption ($\mathbf{Z}_1'$) there is

$$\left\|\mathcal{B}_2(t)\begin{bmatrix}x\\y\end{bmatrix}\right\|_{\mathfrak{X}} = \left\|\begin{bmatrix}0 & 0\\A(t) & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\right\|_{\mathfrak{X}} = \|A(t)x\| \le a\,\|B(t)x\| + b\,\|x\|$$
$$\le a\,\|B(t)B^{-1}(0)\|\,\|B(0)x\| + b\,\|x\| \le M_0\,\|x\|_{X_1^B} + b\,\|B^{-1}(0)\|\,\|x\|_{X_1^B} + \|y\|$$
$$\le \tilde{M}(\|x\|_{X_1^B} + \|y\|) = \tilde{M}\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|_{\mathfrak{X}}.$$

Hence, the operators $\mathcal{B}_2(t)$ are uniformly bounded on $\mathfrak{X}$. Therefore, for any fixed $t \in [0,T]$, $\mathcal{A}(t)$ is a generator of a C_0 semigroup in $\mathfrak{X}$.

(ii) By virtue of Lemma 3, the family $\mathcal{A}_0(t) + \mathcal{B}_1(t)$ is stable in $\mathfrak{X}_1^{\mathcal{A}_0}$. Since the norms $\|\cdot\|_{\mathfrak{X}_1^{\mathcal{A}_0}}$ and $\|\cdot\|_{\mathfrak{X}_1^{\mathcal{A}_0+\mathcal{B}_1}}$ are equivalent (cf. [**3**], p. 160), the spaces $\mathfrak{X}_1^{\mathcal{A}_0}$ and $\mathfrak{X}_1^{\mathcal{A}_0+\mathcal{B}_1}$ can be identified. Hence, the family $\mathcal{A}_0(t) + \mathcal{B}_1(t)$ is stable in the space $\mathfrak{X}_1^{\mathcal{A}_0+\mathcal{B}_1}$. By ([**8**], Theorem 5), the family is stable in $\mathfrak{X}$. It follows from ($\mathbf{Z}'_2$) that the family $\mathcal{B}_2(t)$ is uniformly bounded in $\mathfrak{X}$. Thus, by ([**7**], Theorem 2.3, p. 132), the family $\mathcal{A}(t)$ is stable in $\mathfrak{X}$.

(iii) follows immediately from the assumptions. □

Since, by Lemma 4, all the assumptions of ([**7**], Theorem 4.8, p. 145) are satisfied, there exists a fundamental solution

$$\mathcal{V}(t,s) = \begin{bmatrix}v_1(t,s) & v_2(t,s)\\ v_3(ts) & v_4(t,s)\end{bmatrix} \tag{13}$$

to problem (9). Thus, $\mathcal{U}(t) = \mathcal{V}(t,0)\mathcal{U}_0$ is a solution of the homogeneous problem corresponding to the problem (9).

To study semilinear problem (9) we will restrict ourselves to a smaller class of the spaces X, because we shall use the following version of ([**2**], Theorem 4, p. 20).

Theorem 3 ([**2**], Theorem 4). *Let $\mathfrak{X}$ be a reflexive space. If*

(i) *for any $t \in [0,T]$ the operator $\mathcal{A}(t)$ is a generator of a C_0 semigroup in $\mathfrak{X}$,*
(ii) *the domain $\mathcal{D}(\mathcal{A}(t))$ is independent of t and dense in $\mathfrak{X}$,*
(iii) *the family $(\mathcal{A}(t))_{t\in[0,T]}$ is stable,*
(iv) *for any $x, y \in \mathcal{D}(\mathcal{A}(t))$ the mapping $[0,T] \ni t \mapsto \mathcal{A}(t)\begin{bmatrix}x\\y\end{bmatrix} \in \mathfrak{X}$ is of class $\mathcal{C}^1$,*
(v) $0 \in \varrho(\mathcal{A}(t))$ *for $t \in [0,T]$,*
(vi) *the mapping F satisfies the Lipschitz condition with a constant $L > 0$,*

then problem (9) *has exactly one classical solution which is also a solution of the integral equation*

$$\mathcal{U}(t) = \mathcal{V}(t,0)\mathcal{U}_0 + \int_0^t \mathcal{V}(t,s)F(s,\mathcal{U}(s))ds\,,$$

where $\mathcal{V}(t,s)$ *is a fundamental solution to problem* (9)*.*

Now we will pass to the semilinear problem

$$\text{(14)} \qquad \begin{cases} \dfrac{d\mathcal{U}}{dt} = \mathcal{A}(t)\mathcal{U} + F(t,\mathcal{U}), \\ \mathcal{U}(0) = \mathcal{U}_0. \end{cases}$$

Since now, we shall be assuming that X is a reflexive Banach space. Then X_1^B is also a reflexive space (cf. [**1**], Theorem 1.4.9, p. 272). Thus $\mathfrak{X} = X_1^B \times X$ is reflexive too (cf. [**4**], p. 164).

Theorem 4. *If*

(a) *assumptions* $(\mathbf{Z}'_1)$–$(\mathbf{Z}'_4)$ *are satisfied,*
(b) *for any fixed* $x \in \mathcal{D}_B$*, the mappings* $[0,T] \ni t \mapsto A(t)x \in X$*,* $[0,T] \ni t \mapsto B(t)x \in X$ *are of class* $\mathcal{C}^1$ *,*
(c) $u_0, v_0 \in \mathcal{D}_B$*,*
(d) *the mapping* $F : [0,T] \times X \times X \to X$ *satisfies the Lipschitz condition with a constant* $L > 0$*,*

then problem (14) *has exactly one classical solution.*

Proof. We will show that the theorem results from Theorem 3. Indeed, because of Lemma 4, we must only prove that $0 \in \varrho(\mathcal{A}(t))$ for every $t \in [0,T]$. We easily see that $(\mathcal{A}(t))_{t\in[0,T]}$ is invertible but with not necessarily bounded inverse operator. Since the family $\mathcal{A}(t)$ is stable, there exists $\lambda > 0$ such that new operators $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \lambda\mathcal{I}$ form a family of closed operators with bounded inverses, where $\mathcal{I}$ is the identity map on $\mathfrak{X}$. Let us set $\tilde{F}(t,\mathcal{U}) = F(t,\mathcal{U}) + \lambda\mathcal{U}$. Then problem (14) is equivalent to the problem

$$\text{(15)} \qquad \begin{cases} \dfrac{d\mathcal{U}}{dt} = \tilde{\mathcal{A}}(t)\mathcal{U} + \tilde{F}(t,\mathcal{U}), \\ \mathcal{U}(0) = \mathcal{U}_0 \end{cases}$$

and to use Theorem 3 we must only verify that $\tilde{F}$ satisfies the Lipschitz condition with a constant $\tilde{L}$. To do it let us observe that for

$$t_1, t_2 \in [0,T],\ \mathcal{U}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in \mathfrak{X},\ \mathcal{U}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathfrak{X}$$

there is

$$\left\|\tilde{F}(t_1,\mathcal{U}_1)-\tilde{F}(t_2,\mathcal{U}_2)\right\|_{\mathfrak{X}} \le \left\|\begin{bmatrix}\lambda x_1\\ f(t_1,x_1,y_1)+\lambda y_1\end{bmatrix}-\begin{bmatrix}\lambda x_2\\ f(t_2,x_2,y_2)+\lambda y_2\end{bmatrix}\right\|$$
$$\le \lambda\left\|x_1-x_2\right\|_{X_1^B}+L(|t_1-t_2|+\|x_1-x_2\|_X+\|y_1-y_2\|)+\lambda\left\|y_1-y_2\right\|$$
$$\le \tilde{L}(|t_1-t_2|+\|\mathcal{U}_1-\mathcal{U}_2\|_{\mathfrak{X}}) \quad \text{with } \tilde{L}=L+\lambda\,.$$

By Theorem 3, there exists exactly one classical solution of problem (14) and it is the only solution of the integral equation

$$\mathcal{U}(t)=\mathcal{V}(t)\mathcal{U}_0+\int_0^t \mathcal{V}(t,s)F(s,\mathcal{U}(s))ds\,.$$

Since the fundamental solution $\mathcal{V}(t,s)$ is of form (13), it follows from Theorem 3 that the equation

$$u(t)=v_1(t,0)u_0+v_2(t,0)u_1+\int_o^t v_2(t,s)f(s,u(s),u'(s))ds$$

has exactly one solution which is also the unique classical solution of problem (1). □

References

1. Amann H., *Linear and Quasilinear Parabolic Problems*, Birkhäuser, 1995.
2. Bochenek J., *An abstract semilinear first order differential equation in the hyperbolic case*, Ann. Polon. Math., **61.1** (1995), 13–23.
3. Engel K.-J. and Nagel R., *One-Parameter Semigroups for Linear Evolution Equations*, Springer, 2000.
4. Kato T., *Perturbation Theory for Linear Operators*, Springer, 1980.
5. Krein S.G., *Linear Differential Equations i Banach Spaces*, Amer. Math. Soc., 1972.
6. van Neerven J., *The Adjoint of a Semigroup of Linear Operators*, Springer, 1992.
7. Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.
8. Winiarska T., *Quasilinear evolution equations with operators dependent on* t, to appear.

Received June 8, 2005

Cracow University of Technology
Institute of Mathematics
ul. Warszawska 24
31-155 Kraków
Poland
e-mail: `twiniars@usk.pk.edu.pl`