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Skein-theoretical derivation of some formulas of Habiro

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Abstract We use skein theory to compute the coefficients of certain power series considered by Habiro in his theory of $\mathfrak{sl}_2$ invariants of integral homology 3-spheres. Habiro originally derived these formulas using the quantum group $U_q\mathfrak{sl}_2$. As an application, we give a formula for the colored Jones polynomial of twist knots, generalizing formulas of Habiro and Le for the trefoil and the figure eight knot.

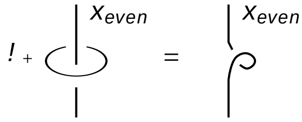
AMS Classification 57M25; 57M27

Keywords Colored Jones polynomial, skein theory, twist knots

Introduction

In a talk at the Mittag-Leffler Institute in May 1999, K. Habiro announced a new approach to computing the colored Jones polynomial of knots and quantum $\mathfrak{sl}_2$ invariants of integral homology 3-spheres. For an exposition, see his paper [H2] (some results are already announced in [H1]). His invariant for homology spheres recovers both the sl_2 Reshetikhin-Turaev invariants at roots of unity, and Ohtsuki's power series invariants. Later, Habiro and T.Q.T. Le generalized this to all quantum invariants associated to simple Lie algebras.

In the $\mathfrak{sl}_2$ case, quantum invariants can be expressed in terms of skein theory, using the Jones polynomial or the Kauffman bracket. Habiro's invariant for homology spheres can be constructed using certain skein elements $\omega = \omega_+$ and $\omega^{-1} = \omega_-$ such that circling an even number of strands with ω_+ (resp. ω_-) induces a positive (resp. negative) full twist:



(1)

More precisely, ω_+ and ω_- are not skein elements, but infinite sums (*i.e.* power series) of such. But as long as they encircle an even number of strands (corresponding to a strand colored by an integer-spin representation of $\mathfrak{sl}_2$), the result is well-defined. Also, it makes sense to consider powers of $\omega = \omega_+$, and circling an even number of strands with ω^p induces p positive full twists, where $p \in \mathbb{Z}$.

The main purpose of this paper is to give skein-theoretical proofs of Habiro's formulas for ω_+ and ω_- (they are stated already in [H1]) and for ω^p (this formula will appear in [H3]). Habiro's original proofs of these formulas use the quantum group $U_q\mathfrak{sl}_2$.

This paper is organized as follows. After stating Habiro's formula for ω_+ and ω_- in Section 1, we give a proof using orthogonal polynomials along the lines of [BHMV1] in Section 2. This proof is quite straightforward, although the computations are a little bit more involved than in [BHMV1]. Unfortunately, it seems difficult to use this approach to compute the coefficients of ω^p for $|p| \geq 2$. Therefore, in Section 3 we start afresh using the Kauffman bracket graphical calculus. A first expression for ω^p in Theorem 3.2 is easily obtained, but it is not quite good enough, as an important divisibility property of the coefficients of ω^p is not clear from this formula. This property is then shown in Section 4 by some more skein theory. The final expression for ω^p obtained in Theorem 4.5 is equivalent to Habiro's one from [H3]. (The results of Section 2 are not used here, so that this gives an independent proof in the $p = \pm 1$ case as well.)

To illustrate one use of ω^p, we conclude the paper in Section 5 by giving a formula for the colored Jones polynomial of twist knots. This generalizes formulas of Habiro [H1] (see also Le [L1, L2]) for the trefoil and the figure eight knot. (For those two knots, one only needs Habiro's original ω_+ and ω_-.)

Habiro has proved (again using quantum groups) that formulas of this type exist for all knots, but the computation of the coefficients is not easy in general. Formulas of this kind are important for at least two reasons: computing quantities related to the Kashaev-Murakami-Murakami volume conjecture [Ka, MM], and computing Habiro's invariant of the homology sphere obtained by ± 1 surgery on the knot. For more about this, see Habiro's survey article [H2].

Acknowledgements I got the idea for this work while talking to T.Q.T. Le during his visit to the University of Paris 7 in July 2002. I would like to thank both him and K. Habiro for helpful discussions, and for sending me parts of their forthcoming papers [L2] and [H3].

1 Habiro's formula for ω

We use the notations of [H1] and of [BHMV1]. In particular, we write

$$a = A^2; \quad \{n\} = a^n - a^{-n}; \quad [n] = \frac{a^n - a^{-n}}{a - a^{-1}}$$

and define $\{n\}!$ and $[n]!$ in the usual way.

Recall that the Kauffman bracket skein module, $K(M)$, of an oriented 3-manifold M is the free $\mathbb{Z}[A^{\pm 1}]$-module generated by isotopy classes of banded links (= disjointly embedded annuli) in M modulo the submodule generated by the Kauffman relations.

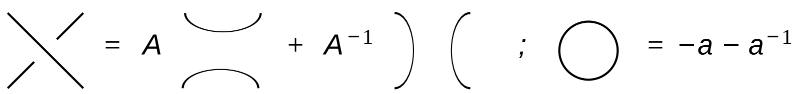


Figure 1: The Kauffman bracket relations. (Recall $a = A^2$.)

The Kauffman bracket gives an isomorphism $\langle\ \rangle : K(S^3) \longrightarrow \mathbb{Z}[A^{\pm 1}]$. It is normalized so that the bracket of the empty link is 1.

The skein module of the solid torus $S^1 \times D^2$ is $\mathbb{Z}[A^{\pm 1}][z]$. We denote it by $\mathcal{B}$. Here z is given by the banded link $S^1 \times J$, where J is a small arc in the interior of D^2, and z^n means n parallel copies of z. We define the even part $\mathcal{B}^{ev}$ of $\mathcal{B}$ to be the submodule generated by the even powers of z.

Let $t : \mathcal{B} \to \mathcal{B}$ denote the twist map induced by a full right handed twist on the solid torus. It is well known (see *e.g.* [BHMV1]) that there is a basis $\{e_i\}_{i \geq 0}$ of eigenvectors for the twist map. It is defined recursively by

$$e_0 = 1, \quad e_1 = z, \quad e_i = ze_{i-1} - e_{i-2}. \tag{2}$$

The eigenvalues are given by

$$t(e_i) = \mu_i e_i, \text{ where } \mu_i = (-1)^i A^{i^2+2i}. \tag{3}$$

Let $\langle\ ,\ \rangle$ be the $\mathbb{Z}[A^{\pm 1}]$-valued bilinear form on $\mathcal{B}$ given by cabling the zero-framed Hopf link and taking the bracket. For $x \in \mathcal{B}$, put $\langle x \rangle = \langle x, 1 \rangle$. One has $\langle e_i \rangle = (-1)^i [i+1]$.

Moreover, for every $f(z) \in \mathcal{B}$, one has

$$\langle f(z), e_i \rangle = f(\lambda_i) \langle e_i \rangle, \text{ where } \lambda_i = -a^{i+1} - a^{-i-1}. \tag{4}$$

Following Habiro [H1], define

$$R_n = \prod_{i=0}^{n-1} (z - \lambda_{2i}), \quad S_n = \prod_{i=0}^{n-1} (z^2 - \lambda_i^2)\ .$$

The R_n form a basis of $\mathcal{B}$, and the S_n form a basis of the even part $\mathcal{B}^{ev}$ of $\mathcal{B}$. By construction, one has $\langle R_n, e_{2i}\rangle = 0$ for $i < n$, and therefore also $\langle R_n, z^{2k}\rangle = 0$ for $k < n$. Similarly, one has $\langle S_n, e_i\rangle = 0$ for $i < n$, and hence also $\langle S_n, z^k\rangle = 0$ for $k < n$. It follows that $\langle R_n, S_m\rangle = 0$ for $n \neq m$, and for $n = m$ one computes

$$\langle R_n, S_n\rangle = \langle R_n, e_{2n}\rangle = \langle e_{2n}\rangle \prod_{i=0}^{n-1} (\lambda_{2n} - \lambda_{2i}) = (-1)^n \frac{\{2n+1\}!}{\{1\}}\ . \tag{5}$$

We are looking for

$$\omega_+ = \sum_{n=0}^{\infty} c_{n,+} R_n$$

satisfying (1) for every even x, which is equivalent to requiring that

$$\langle \omega_+, x\rangle = \langle t(x)\rangle \tag{6}$$

for every $x \in \mathcal{B}^{ev}$. Note that the left hand side of (6) is a finite sum for every $x \in \mathcal{B}^{ev}$.

Theorem 1.1 (Habiro[H1]) *Eq. (6) holds for*

$$c_{n,+} = (-1)^n \frac{a^{n(n+3)/2}}{\{n\}!}\ . \tag{7}$$

Let us define ω_- to be the conjugate of ω_+, where conjugation is defined, as usual, by $\overline{A} = A^{-1}$ and $\overline{z} = z$. Since conjugation corresponds to taking mirror images, we have that

$$\langle \omega_-, x\rangle = \langle t^{-1}(x)\rangle \tag{8}$$

for every even x.

Note that $\omega_- = \sum_{n=0}^{\infty} c_{n,-} R_n$, where

$$c_{n,-} = \frac{a^{-n(n+3)/2}}{\{n\}!}\ . \tag{9}$$

This follows from (7) since $\overline{R_n} = R_n$ and $\overline{\{n\}} = -\{n\}$.

Remark 1.2 The skein element $\omega = \omega_+$ is related to, but different from, the element often called ω appearing in the surgery axiom of Topological Quantum Field Theory (see for example [BHMV2]). If we call the latter ω^{TQFT}, then Equations (6) and (8) would be satisfied by appropriate scalar multiples of $t^{-1}(\omega^{TQFT})$ and $t(\omega^{TQFT})$, respectively; moreover, they would now hold not just for even x, but for all x. This applies in particular to the ω of [BHMV1], which was constructed in a similar way as Habiro's ω_- (but using polynomials $Q_n = \prod_{i=0}^{n-1}(z - \lambda_i)$ in place of the polynomials R_n).

2 A proof using orthogonal polynomials

Habiro's proof of Theorem 1.1 uses the relationship with the quantum group $U_q\mathfrak{sl}_2$. Here is another proof, using the method of orthogonal polynomials as in [BHMV1].

Testing with the S_n-basis, we see that (8) holds if and only if

$$c_{n,-} = \frac{\langle t^{-1}S_n\rangle}{\langle R_n, S_n\rangle} .$$

Thus, it is clear that an ω_- satisfying (8) exists, and to compute its coefficients, we just need to compute $\langle t^{-1}S_n\rangle$.

As in [BHMV1], define another bilinear form $\langle\ ,\ \rangle_1$ by

$$\langle x, y\rangle_1 = \langle t(x), t(y)\rangle .$$

Define polynomials $\mathscr{R}_n$ and $\mathscr{S}_n$ by

$$t(\mathscr{R}_n) = \alpha_n R_n, \quad t(\mathscr{S}_n) = \alpha_{2n} S_n .$$

(The factors α_n and α_{2n} are included so that $\mathscr{R}_n$ and $\mathscr{S}_n$ are monic, *i.e.* have leading coefficient equal to one.)

Again, the $\mathscr{R}_n$ form a basis of B, and the $\mathscr{S}_n$ form a basis of the even part B^{ev} of B, since the twist map t preserves B^{ev}. We have $\langle \mathscr{R}_n, \mathscr{S}_m\rangle_1 = 0$ for $n \neq m$, and

$$\langle \mathscr{R}_n, \mathscr{S}_n\rangle_1 = \alpha_n \alpha_{2n} \langle R_n, S_n\rangle . \tag{10}$$

Note that $\langle t^{-1}S_n\rangle = \alpha_{2n}^{-1}\langle \mathscr{S}_n\rangle$. Thus, we just need to compute $\langle \mathscr{S}_n\rangle$.

Proposition 2.1 *The polynomials $\mathscr{S}_n$ satisfy a four-term recursion formula*

$$\mathscr{S}_{n+1} = (z^2 - \beta_n)\mathscr{S}_n - \delta_{n-1}\mathscr{S}_{n-1} - \gamma_{n-2}\mathscr{S}_{n-2} \tag{11}$$

for certain $\beta_n, \delta_{n-1}, \gamma_{n-2} \in \mathbb{Z}[A^{\pm}]$.

Proof Since $\mathscr{S}_n$ is monic of degree $2n$, we have that $z^2\mathscr{S}_n - \mathscr{S}_{n+1}$ is a linear combination of the $\mathscr{S}_k$ with $k \leq n$. The coefficients can be computed by taking the scalar product with $\mathscr{R}_k$. So we just need to show that $\langle z^2\mathscr{S}_n, \mathscr{R}_k\rangle_1 = 0$ if $k < n-2$.

The point is that multiplication by z is a self-adjoint operator with respect to the bilinear form $\langle\ ,\ \rangle_1$. In other words, one has

$$\langle zx, y\rangle_1 = \langle x, zy\rangle_1$$

for all $x, y \in B$. (This is because $\langle x, y\rangle_1 = \langle t(xy)\rangle$.) It follows that

$$\langle z^2\mathscr{S}_n, \mathscr{R}_k\rangle_1 = \langle \mathscr{S}_n, z^2\mathscr{R}_k\rangle_1 = 0 \quad \text{if} \quad k < n-2,$$

since $\mathscr{R}_k$ has degree k, and $\mathscr{S}_n$ annihilates all polynomials of degree $< n$. □

Note that the coefficients in the recursion formula (11) are given by

$$\alpha_n = \frac{\langle z^2\mathscr{S}_n, \mathscr{R}_n\rangle_1}{\langle \mathscr{S}_n, \mathscr{R}_n\rangle_1}, \quad \beta_{n-1} = \frac{\langle z^2\mathscr{S}_n, \mathscr{R}_{n-1}\rangle_1}{\langle \mathscr{S}_{n-1}, \mathscr{R}_{n-1}\rangle_1}, \ \gamma_{n-2} = \frac{\langle z^2\mathscr{S}_n, \mathscr{R}_{n-2}\rangle_1}{\langle \mathscr{S}_{n-2}, \mathscr{R}_{n-2}\rangle_1}. \quad (12)$$

By convention, if $n < 0$ then $\mathscr{R}_n, \mathscr{S}_n, \alpha_n, \beta_n, \gamma_n$ are all zero.

Proposition 2.2 *One has*

$$\alpha_n = 2 + a^{6n+4}[3] - a^{2n} \quad (13)$$
$$\beta_{n-1} = (a^{4n+1} + a^{8n+1}[3])\{2n\}\{2n+1\} \quad (14)$$
$$\gamma_{n-2} = a^{10n-4}\{2n-2\}\{2n-1\}\{2n\}\{2n+1\} \quad (15)$$

Proof The formula for γ_{n-2} is the easiest. Let us use the notation $\mathrm{o}_{\leq n}$ for terms of degree $\leq n$. Since $z^2\mathscr{R}_{n-2} = \mathscr{R}_n + \mathrm{o}_{\leq n-1}$, we have

$$\langle z^2\mathscr{S}_n, \mathscr{R}_{n-2}\rangle_1 = \langle \mathscr{S}_n, z^2\mathscr{R}_{n-2}\rangle_1 = \langle \mathscr{S}_n, \mathscr{R}_n\rangle_1,$$

and hence formula (15) follows from (12), (10), and (5).

For β_{n-1}, we need to compute

$$\langle z^2\mathscr{S}_n, \mathscr{R}_{n-1}\rangle_1 = \langle \mathscr{S}_n, z^2\mathscr{R}_{n-1}\rangle_1 = \ _{2n}\ _{n-1}\langle S_n, tz^2t^{-1}R_{n-1}\rangle. \quad (16)$$

This amounts to computing the coefficient of R_n in the expression of $tz^2t^{-1}R_{n-1}$ in the R_k-basis. This coefficient can be computed as follows.

For $n \geq 1$, one has

$$z^n = e_n + (n-1)e_{n-2} + \mathrm{o}_{\leq n-4}.$$

(This follows by induction from (2).) Thus, for $'' = \ 1$, one has

$$t^{''} z^n = \ ^{''}_{n} z^n + (n-1)(\ ^{''}_{n-2} - \ ^{''}_{n}) z^{n-2} + o \ _{n-4} :$$

It follows that

$$tz^2 t^{-1} z^n = \frac{\ _{n+2}}{\ _n} z^{n+2} + (2 - (n+1)\frac{\ _{n+2}}{\ _n} + (n-1)\frac{\ _n}{\ _{n-2}}) z^n + o \ _{n-2} : \quad (17)$$

Now write

$$R_n = \prod_{i=0}^{n-1} (z - \ _{2i}) = z^n - x_{n-1} z^{n-1} + o \ _{n-2} ;$$

where $x_{n-1} = \sum_{i=0}^{n-1} \ _{2i}$. Then (17) gives

$$tz^2 t^{-1} R_{n-1} = \frac{\ _{n+1}}{\ _{n-1}} R_{n+1} + (x_n \frac{\ _{n+1}}{\ _{n-1}} - x_{n-2} \frac{\ _n}{\ _{n-2}}) R_n + o \ _{n-1} \quad (18)$$

and hence

$$\langle S_n ; tz^2 t^{-1} R_{n-1} \rangle = (x_n \frac{\ _{n+1}}{\ _{n-1}} - x_{n-2} \frac{\ _n}{\ _{n-2}}) \langle S_n ; R_n \rangle :$$

Plugging this into (16), we have

$$\begin{aligned} \langle z^2 \mathscr{S}_n ; \mathscr{R}_{n-1} \rangle_1 &= \ _{2n} \ _{n-1} \langle S_n ; tz^2 t^{-1} R_{n-1} \rangle \\ &= (x_n \frac{\ _{n+1}}{\ _n} - x_{n-2} \frac{\ _{n-1}}{\ _{n-2}}) \langle \mathscr{S}_n ; \mathscr{R}_n \rangle_1 \\ &= (A^{6n+1}[3] + A^{-2n+1}) \langle \mathscr{S}_n ; \mathscr{R}_n \rangle_1 : \end{aligned}$$

Using (12), (10), and (5) as before, this implies formula (14) for $\ _{n-1}$.

Finally, for $\ _n$, let us compute

$$\langle z^2 \mathscr{S}_n ; \mathscr{R}_n \rangle_1 = \ _{2n} \ _n \langle tz^2 t^{-1} S_n ; R_n \rangle : \quad (19)$$

This amounts to computing the coe cient of S_n in the expression of $tz^2 t^{-1} S_n$ in the S_k-basis.[1]

The computation is similar to the one above. We write

$$S_n = \prod_{i=0}^{n-1} (z^2 - \ ^2_i) = z^{2n} - y_{n-1} z^{2n-2} + o \ _{2n-4} ;$$

[1] This is easier than computing $\langle S_n ; tz^2 t^{-1} R_n \rangle$ by expanding $tz^2 t^{-1} R_n$ in the R_k-basis, because the latter would require computing the rst *three* terms, and not just the rst two terms as in (18) above and also in (20) below.

where $y_{n-1} = \sum_{i=0}^{n-1} {}_i^2$. Then (17) gives

$$tz^2t^{-1}S_n = \tag{20}$$

$$\frac{{}_{2n+2}}{{}_{2n}}S_{n+1} + \quad 2 + (y_n - 2n - 1)\frac{{}_{2n+2}}{{}_{2n}} - (y_{n-1} - 2n + 1)\frac{{}_{2n}}{{}_{2n-2}} \quad S_n + o \; {}_{2n-2}$$

and hence

$$\langle tz^2t^{-1}S_n, R_n\rangle = \quad 2 + (y_n - 2n - 1)\frac{{}_{2n+2}}{{}_{2n}} - (y_{n-1} - 2n + 1)\frac{{}_{2n}}{{}_{2n-2}} \quad \langle S_n, R_n\rangle$$
$$= (2 + a^{6n+4}[3] - a^{2n})\langle S_n, R_n\rangle .$$

Plugging this into (19), we get

$$\langle z^2\mathscr{S}_n, \mathscr{R}_n\rangle_1 = (2 + a^{6n+4}[3] - a^{2n})\langle\mathscr{S}_n, \mathscr{R}_n\rangle_1 ,$$

proving formula (13) for ${}_n$. □

Proof of Habiro's Theorem 1.1 As already observed, we have

$$c_{n,-} = \frac{\langle t^{-1}S_n\rangle}{\langle R_n, S_n\rangle} = {}^{-1}_{2n}\frac{\langle\mathscr{S}_n\rangle}{\langle R_n, S_n\rangle} .$$

But $\langle\mathscr{S}_n\rangle$ satisfies the recursion relation

$$\langle\mathscr{S}_{n+1}\rangle = ({}_0^2 - {}_n)\langle\mathscr{S}_n\rangle - {}_{n-1}\langle\mathscr{S}_{n-1}\rangle - y_{n-2}\langle\mathscr{S}_{n-2}\rangle \tag{21}$$

(since $\langle z\rangle = {}_0$). It follows that

$$\langle\mathscr{S}_n\rangle = (-1)^n a^{(3n^2+n)/2}\frac{\{n+1\}\{n+2\} \quad \{2n+1\}}{\{1\}} , \tag{22}$$

since one can check that (22) is true for $n = 0, 1, 2$ and that it solves the recursion (21). This implies Habiro's formula (9) for $c_{n,-}$. Taking conjugates, one then also obtains formula (7) for $c_{n,+}$. □

Remark 2.3 Although it might be hard to guess formula (22), once one knows it the recursion relation (21) is easily checked. Observe that ${}_0^2 = a^2 + a^{-2} - 2$. Put $q(n) = (3n^2 + n)/2$. Then (21) is equivalent to

$$(a^2 + a^{-2} - a^{6n+4}[3] + a^{2n})a^{q(n)} + (a^{4n+1} + a^{8n+1}[3])\{n\}a^{q(n-1)}$$
$$- a^{10n-4}\{n-1\}\{n\}a^{q(n-2)} = -\frac{\{2n+2\}\{2n+3\}}{\{n+1\}}a^{q(n+1)}$$

which is a straightforward computation.

3 Graphical calculus and a formula for ω^p

Let us write $\omega = \omega_+$ and put

$$\omega^p = \sum_{n=0}^{\infty} c_{n,p} R_n . \tag{23}$$

Note that the coefficients $c_{n,p}$ are well-defined (because R_n divides R_{n+1} and therefore the coefficients $C^k_{n,m}$ in the product expansion $R_n R_m = \sum_k C^k_{n,m} R_k$ are zero if n or m is bigger than k.) We have

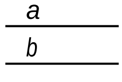
$$\langle \omega^p, x \rangle = \langle t^p(x) \rangle \tag{24}$$

for every even x. (This follows from (6) since circling with ω^p is the same as circling with p parallel copies of ω.) Of course, $c_{n,1} = c_{n,+}$ and $c_{n,-1} = c_{n,-}$ (it follows from the uniqueness of ω_- that $\omega_- = \omega^{-1}$). The aim of this section is to give a formula for the coefficients $c_{n,p}$ (see Theorem 3.2 below).

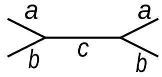
We use the extension of the Kauffman bracket to admissibly colored banded trivalent graphs as in [MV]. (Such graphs are sometimes called spin networks; for more background see *e.g.* [KL] and references therein.) A color is just an integer ≥ 0. A triple of colors (a,b,c) is *admissible* if $a+b+c \equiv 0 \pmod 2$ and $|a-b| \leq c \leq a+b$. Let D be a planar diagram of a banded trivalent graph. An admissible coloring of D is an assignment of colors to the edges of D so that at each vertex, the three colors meeting there form an admissible triple. The Kauffman bracket of D is defined to be the bracket of the *expansion* of D obtained as follows. The expansion of an edge colored n consists of n parallel strands with a copy of the Jones-Wenzl idempotent f_n inserted. (The idempotent f_n is characterized by the fact that $x f_n = f_n x = 0$ for every element x of the standard basis of the Temperley-Lieb algebra other than the identity element; here, the standard basis consists of the (n,n)-tangle diagrams without crossings and without closed loops.) The expansion of a vertex is defined as in Fig. 2, where the *internal colors* i, j, k are defined by

$$i = (b+c-a)/2, \quad j = (c+a-b)/2, \quad k = (a+b-c)/2 . \tag{25}$$

We have the *fusion* equation



$$= \sum_c \frac{\langle c \rangle}{\langle a,b,c \rangle} \tag{26}$$



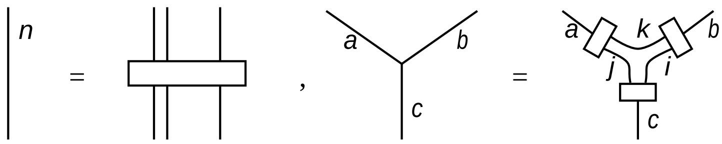
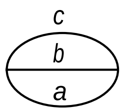
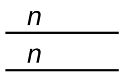
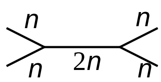


Figure 2: How to expand colored edges and vertices. The boxes stand for appropriate Jones-Wenzl idempotents.

Here the sum is over those colors c so that the triple (a,b,c) is admissible, we have $\langle c\rangle = \langle e_c\rangle = (-1)^c[c+1]$, and the trihedron coefficient $\langle a,b,c\rangle$ is (see [MV, Thm. 1]):

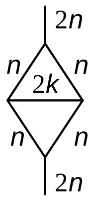
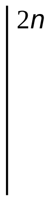
$\langle a,b,c\rangle =$  $= (-1)^{i+j+k}\dfrac{[i+j+k+1]!\,[i]!\,[j]!\,[k]!}{[a]!\,[b]!\,[c]!}$ (27)

(here i,j,k are the internal colors as defined in (25)). Note that $\langle n,n,2n\rangle = \langle 2n\rangle$ so that

 $= $  $+ \;\ldots$ (28)

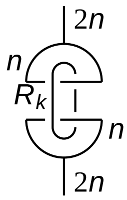
We will need the following lemma.

Lemma 3.1 *For* $0 \le k \le n$*, one has*

 $= \dfrac{([k]!)^2}{[2k]!}$ 

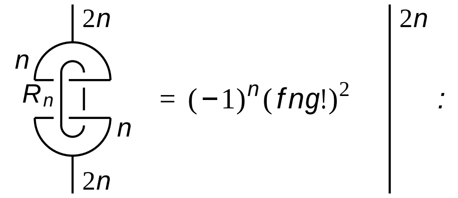
Proof This follows from the formula for the tetrahedron coefficient given in [MV, Thm. 2]. (The sum over in that formula reduces to just one term.) □

The key observation is that

 $= 0 \quad \text{for } k \neq n\,.$ (29)

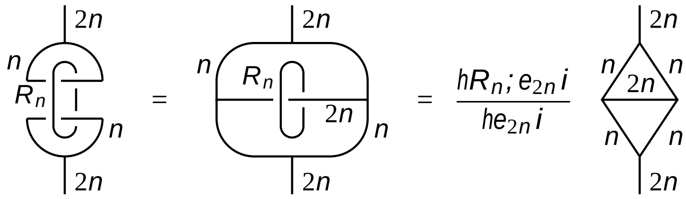
Indeed, if $k < n$, there are at most $2k$ vertical strands in the middle and so the result is zero because of the Jones-Wenzl idempotents f_{2n} at the top and bottom. On the other hand, if $k > n$, the result is zero because R_k annihilates all even polynomials in z of degree $< 2k$.

If $k = n$ one finds



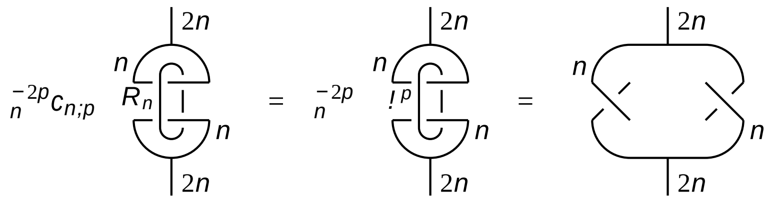
(30)

Indeed, applying (28), one has



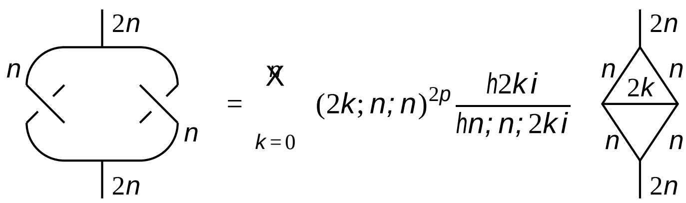
hence (30) follows from (5) and Lemma 3.1.

On the other hand, since circling with ω^p induces p full twists on even numbers of strands (see (24)), we have, using (29), that



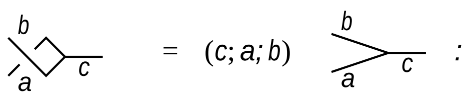
(31)

where there are $2p$ crossings in the last diagram. (See (3) for the twist eigenvalues λ_i.) Applying the fusion equation (26), we have



(32)

where $\gamma(c; a, b)$ is the *half-twist coefficient* defined by



This coe cient is computed in [MV, Thm. 3]. For us, it is enough to know that

$$(c;a;b)^2 = \frac{c}{a\ \ b}$$

which is easy to see. Using (30) and Lemma 3.1, it follows that

$$_n^{-2p} c_{n;p} (-1)^n (\{n\}!)^2 = \sum_{k=0} \frac{\binom{p}{2k}}{\binom{2p}{n}} \frac{\langle 2k\rangle}{\langle n;n;2k\rangle} \frac{([k]!)^2}{[2k]!} \, .$$

The factors of $_n^{-2p}$ cancel out, and in view of (27), this gives the following result:

Theorem 3.2 *The coe cients $c_{n;p}$ of ω^p in (23) are given by*

$$c_{n;p} = \frac{1}{(a-a^{-1})^{2n}} \sum_{k=0} \frac{(-1)^k \binom{p}{2k}[2k+1]}{[n+k+1]!\,[n-k]!} \, . \tag{33}$$

4 Another formula for the coe cients of ω^p

Following Habiro, we introduce the polynomials $R'_n = (\{n\}!)^{-1} R_n$ and write

$$\omega^p = \sum_{n=0} c'_{n;p} R'_n \, ; \tag{34}$$

where

$$c'_{n;p} = \{n\}!\, c_{n;p} \, .$$

The aim of this section is to show that $c'_{n;p}$ is a Laurent polynomial, *i.e.* that $c'_{n;p} \in \mathbb{Z}[A\ \ ]$. This fact was shown by Habiro [H3] using the quantum group $U_q\mathfrak{sl}_2$. Observe that by (9) and (7), we already know this fact for $p = \ \ 1$:

$$c'_{n;1} = (-1)^n a^{n(n+3)/2} \, ; \quad c'_{n;-1} = a^{-n(n+3)/2} \, .$$

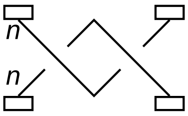
(But Formula (33) tells us only that

$$c'_{n;p} = \frac{1}{(a-a^{-1})^n} \sum_{k=0} (-1)^k \binom{p}{2k}[2k+1] \frac{[n]!}{[n+k+1]!\,[n-k]!} \, ; \tag{35}$$

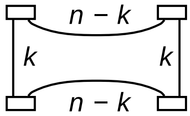
from which it is not clear that $c'_{n;p} \in \mathbb{Z}[A\ \ ]$.)

To do so, we will replace Formula (32) in the previous section by Formula (42) below. For this, we need the following two Lemmas.

Lemma 4.1 *We have*



$$= \sum_{k=0}^{n} C_{n;k}$$



;

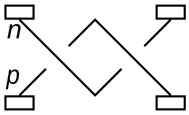
where

$$C_{n;k} = a^{n(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=n-k+1}^{n} (1 - a^{-2j}) : \qquad (36)$$

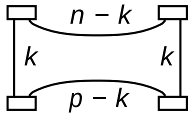
Here, as usual,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\cdots[n-k+1]}{[k]!} :$$

Proof For $0 \le p \le n$, let us write, more generally,

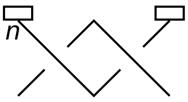


$$= \sum_{k=0}^{p} C_{n;p;k}$$

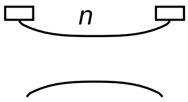


:

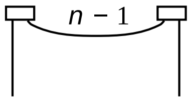
First, we consider the case $p = 1$. By induction on n, it is easy to prove that



$= a^n$



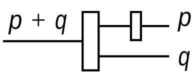
$+ a^{n-1}(1 - a^{-2n})$



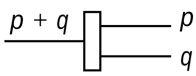
(recall $a = A^2$). Using this, we now fix n and do induction on p to obtain the recursion formula

$$C_{n;p+1;k} = a^{n-2k} C_{n;p;k} + a^{n-2k+1}(1 - a^{-2(n-k+1)}) C_{n;p;k-1} : \qquad (37)$$

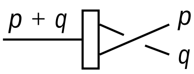
Here we have used the following two facts which follow from the defining properties of the Jones-Wenzl idempotents.[2]



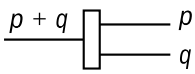
=



(38)



$= A^{-pq}$



(39)

[2]Equation (39) is a special case of the half-twist coefficient.

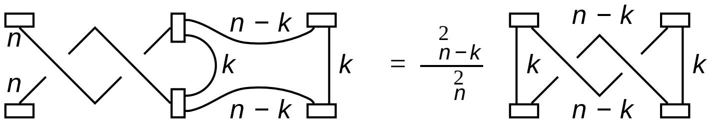
Note that the coefficients $C_{n;p;k}$ behave like the binomial coefficients $\begin{bmatrix}p\\k\end{bmatrix}$ in that $C_{n;0;0} = 1$, and $C_{n;p;k} = 0$ for $k < 0$ or $k > p$. It follows that the recursion formula (37) determines the $C_{n;p;k}$ uniquely. One finds

$$C_{n;p;k} = a^{p(n-k)} \begin{bmatrix}p\\k\end{bmatrix} \prod_{j=n-k+1}^{p} (1 - a^{-2j}) \,.$$

Specializing to the case $p = n$, this proves the Lemma. □

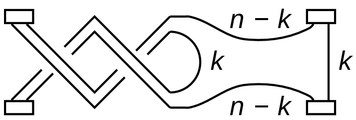
Remark 4.2 The coefficient $C_{n;n}$ was computed by a different method in [A, Prop. 4.4]. Knowing this coefficient would be enough to obtain Habiro's formula (7) for ω (see Remark 4.4 below). Unfortunately, the method of [A] does not give the coefficients $C_{n;k}$ for $k \neq n$, which we will need to obtain a formula for ω_p.

Lemma 4.3 *We have*

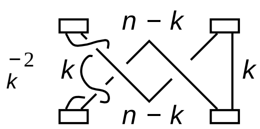


(40)

Proof The left hand side of (40) is equal to

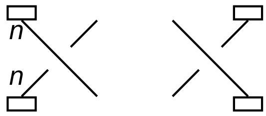


By an isotopy, this becomes

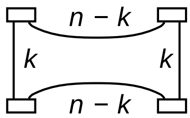


Applying the coefficients $(n; n-k; k)^{-2}$ and $(n; k; n-k)^{-2}$, which are both equal to $_k \; _{n-k} \; _n^{-1}$ (see also (39)), we see that this is equal to the right hand side of (40). □

Let $C^{(p)}_{n,k}$ be the coefficient defined by the expansion



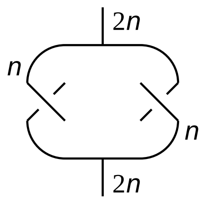
$$= \sum_{k=0} C^{(p)}_{n,k}$$



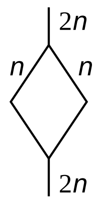
$$, \qquad (41)$$

(where the diagram on the left hand side of (41) has $2p$ crossings). Putting the two preceding Lemmas together, we may obtain a formula for this coefficient. In particular, it follows by induction on p that $C^{(p)}_{n,k}$ is a Laurent polynomial divisible by $\prod_{j=n-k+1}^{n}(1-a^{2j})$.

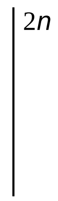
We are interested in the coefficient $C^{(p)}_{n,n}$, since we have



$$= C^{(p)}_{n,n}$$



$$= C^{(p)}_{n,n}$$



$$(42)$$

(where there are $2p$ crossings in the diagram on the left). Using (30) and (31) from Section 3, it follows that

$${}_n^{-2p} c_{n,p}(-1)^n(\{n\}!)^2 = C^{(p)}_{n,n}$$

and therefore

$$c'_{n,p} = \{n\}!\, c_{n,p} = (-1)^n \, {}^{2p}_n (\{n\}!)^{-1} C^{(p)}_{n,n} . \qquad (43)$$

As already observed, $C^{(p)}_{n,n}$ is divisible by

$$\prod_{j=1}^{n}(1-a^{2j}) = a^{-n(n+1)/2}\{n\}!$$

and hence $c'_{n,p}$ is indeed a Laurent polynomial.

Remark 4.4 In the case $p = 1$, we have

$$C^{(1)}_{n,n} = C_{n,n} = a^{-n(n+1)/2}\{n\}! . \qquad (44)$$

Plugging this into (43), we get

$$c'_{n,1} = (-1)^n a^{n^2+2n} a^{-n(n+1)/2} = (-1)^n a^{n(n+3)/2} ,$$

giving another proof of Habiro's formula (7) for $c_{n,1}$.

Here is an explicit formula for $C^{(p)}_{n,n}$ which follows from (36) and (40). The sum is over all multi-indices $\underline{k} = (k_1, \ldots, k_p)$ such that $k_i \geq 0$ for all i, and $\sum k_i = n$. For convenience, put $s_i = k_1 + \cdots + k_i$ and $r_i = n - s_i = k_{i+1} + \cdots + k_p$, and define

$$\varphi(\underline{k}) = \sum_{i=1}^{p-1} r_i(r_{i-1} + r_i + 2)\,. \tag{45}$$

$$\begin{aligned}
C^{(p)}_{n,n} &= \sum_{\underline{k}} C_{n,k_1}\frac{\mu^2_{n-k_1}}{\mu^2_n} C_{n-k_1,k_2}\frac{\mu^2_{n-k_1-k_2}}{\mu^2_n} \cdots C_{n-s_{p-2},k_{p-1}}\frac{\mu^2_{n-s_{p-1}}}{\mu^2_n} C_{n-s_{p-1},k_p} \\
&= \mu_n^{2-2p} \sum_{\underline{k}} \left(\prod_{i=1}^{p-1} \mu^2_{r_i}\right) \prod_{i=1}^{p} C_{r_{i-1},k_i} \\
&= \mu_n^{2-2p} \left(\prod_{j=1}^{n} (1 - a^{-2j})\right) \sum_{\underline{k}} \prod_{i=1}^{p-1} a^{r_i^2 + 2r_i} \left(\prod_{i=1}^{p-1} a^{r_{i-1}r_i} \begin{bmatrix} r_{i-1} \\ k_i \end{bmatrix}\right) \\
&= \mu_n^{2-2p} a^{-n(n+1)/2} \{n\}! \sum_{\underline{k}} a^{\varphi(\underline{k})} \begin{bmatrix} n \\ \underline{k} \end{bmatrix}
\end{aligned}$$

where we have put, as usual,

$$\begin{bmatrix} n \\ \underline{k} \end{bmatrix} = \frac{[n]!}{[k_1]! \cdots [k_p]!}\,.$$

In view of (43), and using $\mu^2_n = a^{n^2+2n}$, we obtain the following final result:

Theorem 4.5 (Habiro [H3]) *For $p \geq 1$, the coefficients $c'_{n,p}$ of ω^p in (34) are given by*

$$c'_{n,p} = (-1)^n a^{n(n+3)/2} \sum_{\substack{\underline{k} = (k_1,\ldots,k_p) \\ k_i \geq 0,\ \sum k_i = n}} a^{\varphi(\underline{k})} \begin{bmatrix} n \\ \underline{k} \end{bmatrix}\,. \tag{46}$$

where $\varphi(\underline{k})$ is defined in (45).

Remark 4.6 Since $c'_{n,-p} = (-1)^n \overline{c'_{n,p}}$, this also determines the coefficients of negative powers of ω.

Remark 4.7 In [H3], Habiro has obtained a similar formula using the quantum group $U_q\mathfrak{sl}_2$.

Example: Assume $p = 2$. We may write $\underline{k} = (k, n-k)$. Then

$$c'_{n,2} = (-1)^n a^{n(n+3)/2} \sum_{k=0}^{n} a^{(n-k)(n+n-k+2)} \binom{n}{k}$$
$$= (-1)^n a^{(5n^2+7n)/2} \sum_{k=0}^{n} a^{k^2-2k-3nk} \binom{n}{k}$$

5 The colored Jones polynomial of twist knots

In this last section, we illustrate one use of ω^p, namely to give a formula for the colored Jones polynomial of twist knots (see Figure 3) in terms of the coefficients $c'_{n,p}$. The results of this section are known to K. Habiro and T.Q.T. Le.

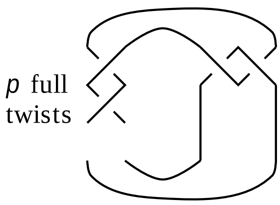


Figure 3: The twist knot K_p. (Here $p \in \mathbb{Z}$.) For $p = 1$, K_p is a left-handed trefoil, and for $p = -1$, K_p is the figure eight knot.

The colored Jones polynomial of a knot K colored with the N-dimensional irreducible representation of $\mathfrak{sl}_2$ can be expressed as the Kauffman bracket of K cabled by $(-1)^{N-1}e_{N-1}$:

$$J_K(N) = (-1)^{N-1}\langle K(e_{N-1})\rangle .$$

(The factor of $(-1)^{N-1}$ is included so that $J_{Unknot}(N) = [N]$.) We will use the normalization

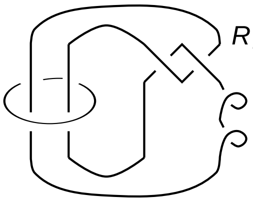
$$J'_K(N) = \frac{J_K(N)}{J_{Unknot}(N)} = \frac{\langle K(e_{N-1})\rangle}{\langle e_{N-1}\rangle} .$$

Here, we assume the knot K is equipped with the zero framing.

Let us compute $\langle K_p(e_{N-1})\rangle$. We use the surgery description given in Fig. 4. Recall that $\omega^p = \sum c'_{k,p} R'_k$. By induction, one can check that

$$e_{N-1} = \sum_{n=0}^{N-1} (-1)^{N-1-n} \binom{N+n}{N-1-n} R_n . \tag{47}$$

The key observation (which I learned from T.Q.T. Le) is that

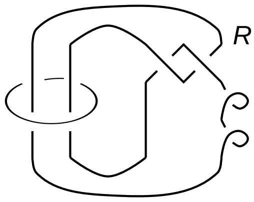


$$= 0$$

for $k \neq n$. This is because each component of this link is a zero-framed unknot having a spanning disk pierced twice by the other component, and circling with R_m annihilates all even polynomials in z of degree $< 2m$.

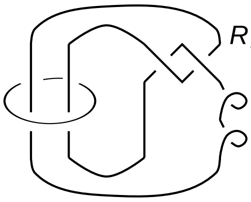
Thus only terms with $k = n$ survive, and so we have

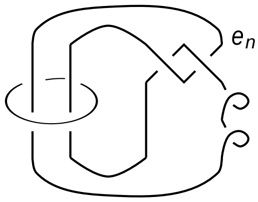
$$\langle K_p(e_{N-1})\rangle = \sum_{n=0}^{N-1} (-1)^{N-1-n} \binom{N+n}{N-1-n} c'_{n;p} \quad R'_n$$

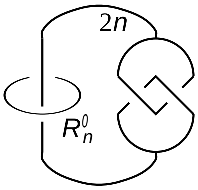


(48)

Now, using that $R_n - e_n$ has degree $< n$, we compute

R'_n 

$=$ R'_n 

$= {}^{2}_{n}$ 

$= {}^{2}_{n} C_{n;n} \langle R'_n, e_{2n}\rangle$

$$= (-1)^n a^{n(n+3)/2} \frac{\{2n+1\}!}{\{1\}}$$

where we have used (42) in the last but one equation, and $C_{n;n}$ and $\langle R_n, e_{2n}\rangle$ are given in (44) and (5), respectively (but notice we are using $R'_n = (\{n\}!)^{-1} R_n$ here).

Plugging this into (48), we obtain the following result.

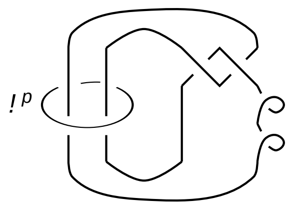


Figure 4: A surgery description of the twist knot K_p with zero framing.

Theorem 5.1 *The colored Jones polynomial of the twist knot K_p is given by*

$$J'_{K_p}(N) = \sum_{n=0}^{\infty} f_{K_p,n} \frac{\{N-n\}\{N-n+1\}\cdots\{N+n\}}{\{N\}}, \tag{49}$$

where

$$f_{K_p,n} = a^{n(n+3)/2} c'_{n,p}.$$

(The sum is actually finite, the terms with $n \geq N$ being all zero.)

Note that since $c'_{n,p}$ is a Laurent polynomial in a, so are the coefficients $f_{K_p,n}$. For example, for the figure eight knot K_{-1}, we have $f_{K_{-1},n} = 1$, and for the left-handed trefoil K_1, we have $f_{K_1,n} = (-1)^n a^{n(n+3)}$. (The right-handed trefoil is the mirror image of K_1, so one just needs to take the conjugate of $J'_{K_1}(N)$.) These formulas can be found in [H2].

Remark 5.2 Here is another expression for $J'_{K_p}(N)$. Put $q = a^2 = A^4$ and, as is customary in q-calculus,

$$(x)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}).$$

Then (49) gives

$$J'_{K_p}(N) = \sum_{n=0}^{\infty} \tilde{f}_{K_p,n}(q^{1-N})_n(q^{1+N})_n,$$

where

$$\tilde{f}_{K_p,n} = (-1)^n q^{-n(n+1)/2} f_{K_p,n}.$$

For example, for the figure eight knot K_{-1}, we have $\tilde{f}_{K_{-1},n} = (-1)^n q^{-n(n+1)/2}$, and for the left-handed trefoil K_1, we have $\tilde{f}_{K_1,n} = q^n$. These formulas can be found in [L2] (see also [L1]).

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