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The Chess conjecture

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Abstract We prove that the homotopy class of a Morin mapping $f\colon P^p \to Q^q$ with $p-q$ odd contains a cusp mapping. This affirmatively solves a strengthened version of the Chess conjecture [5],[3]. Also, in view of the Saeki-Sakuma theorem [10] on the Hopf invariant one problem and Morin mappings, this implies that a manifold P^p with odd Euler characteristic does not admit Morin mappings into $\mathbb{R}^{2k+1}$ for $p \geq 2k+1 \neq 1, 3, 7$.

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1 Introduction

Let P and Q be two smooth manifolds of dimensions p and q respectively and suppose that $p \geq q$. The singular points of a smooth mapping $f\colon P \to Q$ are the points of the manifold P at which the rank of the differential df of the mapping f is less than q. There is a natural stratification breaking the singular set into finitely many strata. We recall that the kernel rank $kr_x(f)$ of a smooth mapping f at a point x is the rank of the kernel of df at x. At the first stage of the stratification every stratum is indexed by a non-negative integer i_1 and defined as

$$\Sigma^{i_1}(f) = \{x \in P \mid kr_x(f) = i_1\}.$$

The further stratification proceeds by induction. Suppose that the stratum $\Sigma_{n-1}(f) = \Sigma^{i_1,\ldots,i_{n-1}}(f)$ is defined. Under assumption that $\Sigma_{n-1}(f)$ is a submanifold of P, we consider the restriction f_{n-1} of the mapping f to $\Sigma_{n-1}(f)$ and define

$$\Sigma^{i_1,\ldots,i_n}(f) = \{x \in \Sigma_{n-1}(f) \mid kr_x(f_{n-1}) = i_n\}.$$

Boardman [4] proved that every mapping f can be approximated by a mapping for which every stratum $\Sigma_n(f)$ is a manifold.

We abbreviate the sequence $(i_1, \ldots, i_n)$ of n non-negative integers by I. We say that a point of the manifold P is an *I-singular point* of a mapping f if it belongs to a singular submanifold $\Sigma^I(f)$. There is a class of in a sense the simplest singularities, which are called *Morin*. Let I_1 denote the sequence $(p-q+1,0)$ and for every integer $k>1$, the symbol I_k denote the sequence $(p-q+1,1,\dots,1,0)$ with k non-zero entries. Then Morin singularities are singularities with symbols I_k. A Morin mapping is an I_k*-mapping* if it has no singularities of type I_{k+1}. For $k=1,2$ and 3, points with the symbols I_k are called *fold, cusp* and *swallowtail singular points* respectively. In this terminology, for example, a fold mapping is a mapping which has only fold singular points.

Given two manifolds P and Q, we are interested in finding a mapping $P\to Q$ that has as simple singularities as possible. Let $f\colon P\to Q$ be an arbitrary general position mapping. For every symbol I, the $\mathbb{Z}_2$-homology class represented by the closure $\overline{\Sigma^I(f)}$ does not change under general position homotopy. Therefore the homology class $[\overline{\Sigma^I(f)}]$ gives an obstruction to elimination of I-singularities by homotopy.

In [5] Chess showed that if $p-q$ is odd and $k\geq 4$, then the homology obstruction corresponding to I_k-singularities vanishes. Chess conjectured that in this case every Morin mapping f is homotopic to a mapping without I_k-singular points.

We will show that the statement of the Chess conjecture holds. Furthermore we will prove a stronger assertion.

Theorem 1.1 *Let P and Q be two orientable manifolds, $p-q$ odd. Then the homotopy class of an arbitrary Morin mapping $f\colon P\to Q$ contains a cusp mapping.*

Remark The standard complex projective plane $\mathbb{C}P^2$ does not admit a fold mapping [9] (see also [1], [12]). This shows that the homotopy class of f may contain no mappings with only I_1-singularities.

Remark The assumption on the parity of the number $p-q$ is essential since in the case where $p-q$ is even homology obstructions may be nontrivial [5].

Remark We refer to an excellent review [11] for further comments. In particular, see Remark 4.6, where the authors indicate that Theorem 1.1 does not hold for non-orientable manifolds.

In [10] (see also [7]) Saeki and Sakuma describe a remarkable relation between the problem of the existence of certain Morin mappings and the Hopf invariant one problem. Using this relation the authors show that if the Euler characteristic of P is odd, Q is almost parallelizable, and there exists a cusp mapping $f: P \to Q$, then the dimension of Q is $1, 2, 3, 4, 7$ or 8.

Note that if the Euler characteristic of P is odd, then the dimension of P is even. We obtain the following corollary.

Corollary 1.2 *Suppose the Euler characteristic of P is odd and the dimension of an almost parallelizable manifold Q is odd and different from $1, 3, 7$. Then there exist no Morin mappings from P into Q.*

2 Jet bundles and suspension bundles

Let P and Q be two smooth manifolds of dimensions p and q respectively. *A germ at a point $x \in P$* is a mapping from some neighborhood about x in P into Q. Two germs are *equivalent* if they coincide on some neighborhood of x. The class of equivalence of germs (or simply the germ) at x represented by a mapping f is denoted by $[f]_x$.

Let U be a neighborhood of x in P and V be a neighborhood of $y = f(x)$ in Q. Let

$$\varphi_U: (U, x) \to (\mathbb{R}^p, 0) \quad \text{and} \quad \varphi_V: (V, y) \to (\mathbb{R}^q, 0)$$

be coordinate systems. Two germs $[f]_x$ and $[g]_x$ are *k-equivalent* if the mappings $\varphi_V \circ f \circ \varphi_U^{-1}$ and $\varphi_V \circ g \circ \varphi_U^{-1}$, which are defined in a neighborhood of $0 \in \mathbb{R}^p$, have the same derivatives at $0 \in \mathbb{R}^p$ of order $\leq k$. The notion of k-equivalence is well-defined, i.e. it does not depend on choice of representatives of germs and on choice of coordinate systems. A class of k-equivalent germs at x is called *a k-jet*. The set of all k-jets constitute a set $J^k(P, Q)$. The projection $J^k(P, Q) \to P \times Q$ that takes a germ $[f]_x$ into a point $x \times f(x)$ turns $J^k(P, Q)$ into a bundle (for details see [4]), which is called *the k-jet bundle over $P \times Q$*.

Let y be a point of a manifold and V a neighborhood of y. We say that two functions on V lead to the same local function at y, if at the point y their partial derivatives agree. Thus a local function is an equivalence class of functions defined on a neighborhood of y. The set of all local functions at the point y constitutes an algebra of jets $F(y)$. Every smooth mapping $f: (U, x) \to (V, y)$ defines a homomorphism of algebras $f^*: F(y) \to F(x)$. The maximal ideal m_y of $F(y)$ maps under the homomorphism f^* to the maximal ideal $m_x \subset F(x)$.

The restriction of f^* to m_y and the projection of $f^*(m_y) \subset m_x$ onto m_x/m_x^{k+1} lead to a homomorphism

$$f_{k,x}\colon m_y \to m_x/m_x^{k+1}.$$

It is easy to verify that k-jets of mappings $(U,x) \to (V,y)$ are in bijective correspondence with algebra homomorphisms $m_y \to m_x/m_x^{k+1}$. That is why we will identify a k-jet with the corresponding homomorphism.

The projections of $P \times Q$ onto the factors induce from the tangent bundles TP and TQ two vector bundles ξ and η over $P \times Q$. The latter bundles determine a bundle $HOM(\xi,\eta)$ over $P \times Q$. The fiber of $HOM(\xi,\eta)$ over a point $x \times y$ is the set of homomorphisms $Hom(\xi_x,\eta_y)$ between the fibers of the bundles ξ and η. The bundle ξ determines the k-th symmetric tensor product bundle $\odot^k\xi$ over $P \times Q$, which together with η leads to a bundle $HOM(\odot^k\xi,\eta)$.

Lemma 2.1 *The k-jet bundle contains a vector subbundle C^k isomorphic to $HOM(\odot^k\xi,\eta)$.*

Proof Define C^k as the union of those k-jets $f_{k,x}$ which take m_y to m_x^k. With each $f_{k,x} \in C^k$ we associate a homomorphism (for details, see [4, Theorem 4.1])

$$\underbrace{\xi_x \otimes \cdots \otimes \xi_x}_{k} \otimes m_y/m_y^2 \to \mathbb{R} \tag{1}$$

which sends $v_1 \otimes \cdots \otimes v_k \otimes \varphi$ into the value of $v_1 \circ \cdots \circ v_k$ at a function representing $f_{k,x}(\varphi).$ In view of the isomorphism $m_y/m_y^2 \approx Hom(\eta_y,\mathbb{R})$, the homomorphism (1) is an element of $Hom(\odot^k\xi_x,\eta_y)$. It is easy to verify that the obtained correspondence $C^k \to HOM(\odot^k\xi_x,\eta_y)$ is an isomorphism of vector bundles. □

Corollary 2.2 *There is an isomorphism $J^{k-1}(P,Q) \oplus C^k \approx J^k(P,Q)$.*

Proof Though the sum of two algebra homomorphisms may not be an algebra homomorphism, the sum of a homomorphism $f_{k,x} \in J^k(P,Q)$ and a homomorphism $h \in C^k$ is a well defined homomorphism of algebras $(f_{k,x}+h) \in J^k(P,Q)$. This defines an action of C^k on $J^k(P,Q)$. Two k-jets α and β map under the canonical projection

$$J^k(P,Q) \longrightarrow J^k(P,Q)/C^k$$

onto one point if and only if α and β have the same $(k-1)$-jet. Therefore $J^k(P,Q)/C^k$ is canonically isomorphic to $J^{k-1}(P,Q)$. □

Remark The isomorphism $J^{k-1}(P;Q) \quad C^k \quad J^k(P;Q)$ constructed in Corollary 2.2 is not canonical, since there is no canonical projection of the k-jet bundle onto C^k.

In [8] Ronga introduced the bundle

$$S^k(\;;\;) = HOM(\;;\;) \quad HOM(\quad ;\;) \quad ::: \quad HOM(\; ^k \;;\;);$$

which we will call the k-suspension bundle over $P \quad Q$.

Corollary 2.3 *The k-jet bundle is isomorphic to the k-suspension bundle.*

3 Submanifolds of singularities

There are canonical projections $J^{k+1}(P;Q) \; ! \; J^k(P;Q)$, which lead to the infinite dimensional *jet bundle* $J(P;Q) := \varprojlim J^k(P;Q)$. Let $f: P \; ! \; Q$ be a smooth mapping. Then at every point $x \quad f(x)$ of the manifold $P \quad Q$, the mapping f determines a k-jet. The k-jets defined by f lead to a mapping $j^k f$ of P to the k-jet bundle. These mappings agree with projections of $\varprojlim J^k(P;Q)$ and therefore define a mapping $jf: P \; ! \; J(P;Q)$, which is called the jet extension of f. We will call a subset of $J(P;Q)$ *a submanifold of the jet bundle* if it is the inverse image of a submanifold of some k-jet bundle. A function on the jet bundle is said to be *smooth* if locally is the composition of the projection onto some k-jet bundle and a smooth function on $J^k(P;Q)$. In particular, the composition $\quad jf$ of a smooth function on $J(P;Q)$ and a jet extension jf is smooth. *A tangent to the jet bundle vector* is a differential operator. *A tangent to $J(P;Q)$ bundle* is defined as a union of all vectors tangent to the jet bundle.

Suppose that at a point $x \; 2 \; P$ the mapping f determines a jet z. Then the differential of jf sends differential operators at x to differential operators at z, that is $d(jf)$ maps T_xP into some space D_z tangent to the jet bundle. In fact, the space D_z and the isomorphism $T_xP \; ! \; D_z$ do not depend on representative f of the jet z. Let denote the composition of the jet bundle projection and the projection of $P \quad Q$ onto the first factor. Then the tangent bundle of the jet space contains a subbundle D, called *the total tangent bundle*, which can be identified with the induced bundle $\quad TP$ by the property: for any vector field v on an open set U of P, any jet extension jf and any smooth function on $J(P;Q)$, the section V of D over $\;^{-1}(U)$ corresponding to v satisfies the equation

$$V \quad jf = v(\quad jf):$$

We recall that the projections $P \times Q$ onto the factors induce two vector bundles ξ and η over $P \times Q$ which determine a bundle $HOM(\xi, \eta)$. There is a canonical isomorphism between the 1-jet bundle and the bundle $HOM(\xi, \eta)$. Consequently 1-jet component of a k-jet z at a point $x \in P$ defines a homomorphism $h\colon T_xP \to T_yQ$, $y = z(x)$. We denote the kernel of the homomorphism h by $K_{1,z}$. Identifying the space T_xP with the fiber D_z of D, we may assume that $K_{1,z}$ is a subspace of D_z. Hence at every point $z \in J(P, Q)$ we have a space $K_{1,z}$. Boardman showed that the union $\Sigma^i = \Sigma^i(P, Q)$ of jets z with $dim\, K_{1,z} = i$ is a submanifold of $J(P, Q)$.

Suppose that we have already defined a submanifold $\Sigma_{n-1} = \Sigma^{i_1,...,i_{n-1}}$ of the jet space. Suppose also that at every point $z \in \Sigma_{n-1}$ we have already defined a space $K_{n-1,z}$. Then the space $K_{n,z}$ is defined as $K_{n-1,z} \cap T_z\Sigma_{n-1}$ and Σ_n is defined as the set of points $z \in \Sigma_{n-1}$ such that $dim\, K_{n,z} = i_n$. Boardman proved that the sets Σ_n are submanifolds of $J(P, Q)$. In particular every submanifold Σ_n comes from a submanifold of an appropriate finite dimensional k-jet space. In fact the submanifold with symbol I_n is the inverse image of the projection of the jet space onto n-jet bundle. To simplify notation, we denote the projections of Σ_n to the k-jet bundles with $k \geq n$ by the same symbol Σ_n.

Let us now turn to the k-suspension bundle. Following the paper [4], we will define submanifolds $\tilde{\Sigma}^I$ of the k-suspension bundle.

A point of the k-suspension bundle over a point $x \times y \in P \times Q$ is the set of homomorphisms $h = (h_1, ..., h_k)$, where $h_i \in Hom(\circ^i \xi_x, \eta_y)$. For every k-suspension h we will define a sequence of subspaces $T_xP = K_0 \supset K_1 \supset ... \supset K_k$. Then we will define the singular set $\tilde{\Sigma}^{i_1,...,i_n}$ as

$$\tilde{\Sigma}^{i_1,...,i_n} = \{\, h \mid dim\, K_j = i_j \text{ for } j = 1, ..., n \,\}.$$

We start with definition of a space $K_1 \subset K_0$ and a projection of $P_0 = T_yQ$ onto a factor space Q_1. The h_1-component of h is a homomorphism of K_0 into P_0. We define K_1 and Q_1 as the kernel and the cokernel of h_1:

$$0 \longrightarrow K_1 \longrightarrow K_0 \xrightarrow{h_1} P_0 \longrightarrow Q_1 \longrightarrow 0.$$

The cokernel homomorphism of this exact sequence gives rise to a homomorphism $Hom(K_1, P_0) \to Hom(K_1, Q_1)$, coimage of which is denoted by P_1. The sequence of the homomorphisms

$$Hom(K_1 \circ K_1, P_0) \to Hom(K_1, Hom(K_1, P_0)) \to Hom(K_1, P_1)$$

takes the restriction of h_2 on $K_1 \circ K_1$ to a homomorphism $\phi(h_2)\colon K_1 \to P_1$. Again the spaces K_2 and Q_2 are respectively defined as the kernel and the cokernel of the homomorphism $\phi(h_2)$.

The definition continues by induction. In the n-th step we are given some spaces K_i, Q_i for $i \le n$, spaces P_i for $i \le n-1$ and projections

$$Hom(K^{n-1}, P_0) \to P_{n-1},$$

$$P_{n-1} \to Q_n,$$

where K^{n-1} abbreviates the product $K_{n-1} \otimes \dots \otimes K_1$.

First we define P_n as the coimage of the composition

$$Hom(K^n, P_0) \to Hom(K_n, Hom(K^{n-1}, P_0)) \to Hom(K_n, Q_n),$$

where the latter homomorphism is determined by the two given projections. Then we transfer the restriction of the homomorphism h_{n+1} on $K_n \otimes K^n$ to a homomorphism $\varphi(h_{n+1})\colon K_n \to P_n$ using the composition

$$Hom(K_n \otimes K^n, P_0) \to Hom(K_n, Hom(K^n, P_0)) \to Hom(K_n, P_n).$$

Finally we define K_{n+1} and Q_{n+1} by the exact sequence

$$0 \longrightarrow K_{n+1} \longrightarrow K_n \xrightarrow{\varphi(h_{n+1})} P_n \longrightarrow Q_{n+1} \longrightarrow 0.$$

In the previous section we established a homeomorphism between the fibers of the k-jet bundle and k-suspension bundle. Suppose that neighborhoods of points $x \in P$ and $y \in Q$ are equipped with coordinate systems. Then every k-jet g which takes x to y has the canonical decomposition into the sum of k-jets g_i, $i = 1, \dots, k$, such that in the selected coordinates the partial derivatives of the jet g_i at x of order $\neq i$ and $\le k$ are trivial. In other words the choice of local coordinates determines a homeomorphism

$$J^k(P,Q)|_{x \times y} \to C^1|_{x \times y} \oplus \dots \oplus C^k|_{x \times y}. \quad (2)$$

Since $C^i|_{x \times y}$ is isomorphic to $Hom(\odot^i \tau_x, \nu_y)$, we obtain a homeomorphism between the fibers of the k-jet bundle and k-suspension bundle.

Remark From [4] we deduce that this homeomorphism takes the singular submanifolds Σ^I to $\tilde{\Sigma}^I$. Suppose that a k-jet z maps onto a k-suspension $h = (h_1, \dots, h_k)$. The homomorphisms $\{h_i\}$ depends not only on z but also on choice of coordinates in U_i. However Boardman [4] showed that the spaces K_i, Q_i, P_i and the homomorphisms $\varphi(h_i)$ defined by h are independent from the choice of coordinates.

Lemma 3.1 *For every integer $k \ge 1$, there is a homeomorphism of bundles $r_k\colon J^k(P,Q) \to S^k(\tau, \nu)$ which takes the singular sets Σ^I to $\tilde{\Sigma}^I$.*

Proof Choose covers of P and Q by closed discs. Let $U_1, \ldots, U_t$ be the closed discs of the product cover of $P \times Q$. For each disc U_i, choose a coordinate system which comes from some coordinate systems of the two disc factors of U_i. We will write J^k for the k-jet bundle and $J^k|_{U_i}$ for its restriction on U_i. We adopt similar notations for the k-suspension bundle. The choice of coordinates in U_i leads to a homeomorphism

$$ {}_i\colon J^k|_{U_i} \to S^k|_{U_i}. $$

Let $\{\varphi_i\}$ be a partition of unity for the cover $\{U_i\}$ of $P \times Q$. We define $r_k\colon J^k \to S^k$ by

$$ r_k = \varphi_1 {}_1 + \varphi_2 {}_2 + \ldots + \varphi_k {}_k. $$

Suppose that $U_i \cap U_j$ is nonempty and z is a k-jet at a point of $U_i \cap U_j$. Suppose

$$ {}_i(z) = (h^i_1, \ldots, h^i_k) \quad \text{and} \quad {}_j(z) = (h^j_1, \ldots, h^j_k). $$

Then by the remark preceding the lemma, the homomorphisms (h^i_s) and (h^j_s) coincide for all $s = 1, \ldots, k$. Consequently, r_k takes ^{I} to $\tilde{}^I$.

The mapping r_k is continuous and open. Hence to prove that r_k is a homeomorphism it suffices to show that r_k is one-to-one.

For $k = 1$, the mapping r_k is the canonical isomorphism. Suppose that r_{k-1} is one-to-one and for some different k-jets z_1 and z_2, we have $r_k(z_1) = r_k(z_2)$. Since r_{k-1} is one-to-one, the k-jets z_1 and z_2 have the same $(k-1)$-jet components. Hence there is $v \in C^k$ for which $z_1 = z_2 + v$. Here we invoke the fact that C^k has a canonical action on J^k.

For every i, we have ${}_i(z_1) = {}_i(z_2) + {}_i(v)$. Therefore

$$ r_k(z_1) = r_k(z_2) + r_k(v). \tag{3} $$

The restriction of the mapping r_k to C^k is a canonical identification of C^k with $HOM({}^k {}_k,)$. Hence $r_k(v) \neq 0$. Then (3) implies that $r_k(z_1) \neq r_k(z_2)$. □

Corollary 3.2 *There is an isomorphism of bundles $r\colon J(P, Q) \to S(,)$ which takes every set ${}_n$ isomorphically onto $\tilde{}_n$.*

The space $J^k(P, Q)$ may be also viewed as a bundle over P with projection

$$ \colon J^k(P, Q) \to P \times Q \to P. $$

Let $f\colon P \to Q$ be a smooth mapping. Then at every point $p \in P$ the mapping f defines a k-jet. Consequently, every mapping $f\colon P \to Q$ gives rise to a section $j^k f\colon P \to J^k(P, Q)$, which is called *the k-extension of f* or *the k-jet* section *afforded by* f. The sections $\{j^k f\}_k$ determined by a smooth mapping f commute with the canonical projections $J^{k+1}(P,Q) \to J^k(P,Q)$. Therefore every smooth mapping $f\colon P \to Q$ also defines a section $jf\colon P \to J(P,Q)$, which is called the jet extension of f.

A smooth mapping f is *in general position* if its jet extension is transversal to every singular submanifold Σ^I. By the Thom Theorem every mapping has a general position approximation.

Let f be a general position mapping. Then the subsets $(jf)^{-1}(\Sigma^I)$ are submanifolds of P. Every condition $kr_x(f_{n-1}) = i_n$ in the definition of $\Sigma^I(f)$ can be substituted by the equivalent condition $\dim K_{n,x}(f) = i_n$, where the space $K_{n,x}(f)$ is the intersection of the kernel of df at x and the tangent space $T_x\Sigma_{n-1}(f)$. Hence the sets $(jf)^{-1}(\Sigma^I)$ coincide with the sets $\Sigma^I(f)$. In particular the jet extension of a mapping f without I-singularities does not intersect the set Σ^I.

Let $\Omega_r = \Omega_r(P,Q) \subset J(P,Q)$ denote the union of the regular points and the Morin singular points with indexes of length at most r.

Theorem 3.3 (Ando-Eliashberg, [2], [6]) *Let $f\colon P^p \to Q^q, p \ge q \ge 2$, be a continuous mapping. The homotopy class of the mapping f contains an I_r-mapping, $r \ge 1$, if and only if there is a section of the bundle Ω_r.*

Note that every general position mapping $f\colon P^p \to Q^q$, $q = 1$, is a fold mapping. That is why for $q = 1$, Theorem 1.1 holds and we will assume that $q \ge 2$.

Let $\tilde\Omega_r$ denote the subset of the suspension bundle corresponding to the set $\Omega_r(P,Q) \subset J(P,Q)$. Every mapping $f\colon P \to Q$ defines a section jf of $J(P,Q)$. The composition $r \circ (jf)$ is a section of $S(P,Q)$. In view of Lemma 3.1 the Ando-Eliashberg Theorem implies that to prove that the homotopy class of a mapping f contains a cusp mapping, it suffices to show that the section of the suspension bundle defined by f is homotopic to a section of the bundle $\tilde\Omega_2 \subset S(\ ,\ )$.

4 Proof of Theorem 1.1

We recall that in a neighborhood of a fold singular point x, the mapping f has the form

$$\begin{aligned} T_i &= t_i, \quad i = 1,2,\dots,q-1, \\ Z &= Q(x), \quad Q(x) = \pm k_1^2 \pm \cdots \pm k_{p-q+1}^2. \end{aligned} \tag{4}$$

If x is an I_r-singular point of f and $r > 1$, then in some neighborhood about x the mapping f has the form

$$\begin{aligned} T_i &= t_i; \quad i = 1;2;\dots;q-r; \\ L_i &= l_i; \quad i = 2;3;\dots;r; \\ Z &= Q(x) + \sum_{t=2} l_t k^{t-1} + k^{r+1}; \quad Q(x) = \quad k_1^2 \quad \dots \quad k_{p-q}^2. \end{aligned} \tag{5}$$

Let $f: P \to Q$ be a Morin mapping, for which the set $\ _2(f)$ is nonempty. We de ne the section $f_i: P \to Hom(\ ^i;\ )$ as the i-th component of the section $r\ (jf)$ of the suspension bundle $S(\ ;\ ) \to P$. Over $\overline{\ _2(f)}$ the components f_1 and f_2 de ned by the mapping f determine the bundles $K_i;Q_i$, $i = 1;2$ and the exact sequences

$$0 \longrightarrow K_1 \longrightarrow TP \longrightarrow TQ \longrightarrow Q_1 \longrightarrow 0;$$

$$0 \longrightarrow K_2 \longrightarrow K_1 \longrightarrow HOM(K_1;Q_1) \longrightarrow Q_2 \longrightarrow 0.$$

From the latter sequence one can deduce that the bundle Q_2 is canonically isomorphic to $HOM(K_2;Q_1)$ and that the homomorphism

$$K_1{=}K_2 \quad K_1{=}K_2 \longrightarrow Q_1; \tag{6}$$

which is de ned by the middle homomorphism of the second exact sequence, is a non-degenerate quadratic form (see Chess, [5]). Since the dimension of $K_1{=}K_2$ is odd, the quadratic form (6) determines a canonical orientation of the bundle Q_1. In particular the 1-dimensional bundle Q_1 is trivial. This observation also belongs to Chess [5].

Assume that the bundle K_2 is trivial. Then the bundle Q_2 being isomorphic to $HOM(K_2;Q_1)$ is trivial as well. Let

$$\tilde{h}: K_2 \to HOM(K_2;Q_2) \quad HOM(K_2 \quad K_2;Q_1)$$

be an isomorphism over $\overline{\ _2(f)}$ and $h: P \to HOM(\ ^3;\ )$ an arbitrary section, the restriction of which on $\ ^3K_2$ over $\overline{\ _2(f)}$ followed by the projection given by $\ \to Q_1$, induces the homomorphism $\tilde{h}$. Then the section of a suspension bundle whose rst three components are $f_1; f_2$ and h is a section of the bundle Ω_2. Since for $i > 0$ the bundle $HOM(\ ^i;\ )$ is a vector bundle, we have that the composition $r\ (jf)$ is homotopic to the section s and therefore the original mapping f is homotopic to a cusp mapping.

Now let us prove the assumption that K_2 is trivial over $\overline{\ _2(f)}$.

Lemma 4.1 *The submanifold $\overline{\ _2(f)}$ is canonically cooriented in the submanifold $\overline{\ _1(f)}$.*

Proof For non-degenerate quadratic forms of order n, we adopt the convention to identify the index λ with the index $n-\lambda$. Then the index $ind\,Q(x)$ of the quadratic form $Q(x)$ in (4) and (5) does not depend on choice of coordinates.

With every I_k-singular point x by (4) and (5) we associate a quadratic mapping of the form $Q(x)$. It is easily verified that for every cusp singular point y and a fold singular point x of a small neighborhood of y, we have $Q(x) = Q(y) \oplus k^2_{p-q+1}$. Moreover, if x_1 and x_2 are two fold singular points and there is a path joining x_1 with x_2 which intersects $\overline{\Sigma_2(f)}$ transversally and at exactly one point, then $ind\,Q(x_1) - ind\,Q(x_2) = \pm 1$. In particular, the normal bundle of $\overline{\Sigma_2(f)}$ in $\overline{\Sigma_1(f)}$ has a canonical orientation. □

Lemma 4.2 *Over every connected component of* $\Sigma_2(f)$ *the bundle* K_2 *has a canonical orientation.*

Proof At every point $x \in \overline{\Sigma_2(f)}$ there is an exact sequence

$$0 \longrightarrow K_{3,x} \longrightarrow K_{2,x} \longrightarrow HOM(K_{2,x}, Q_{2,x}) \longrightarrow Q_{3,x} \longrightarrow 0.$$

If the point x is in fact a cusp singular point, then the space $K_{3,x}$ is trivial and therefore the sequence reduces to

$$0 \longrightarrow K_{2,x} \longrightarrow HOM(K_{2,x}, Q_{2,x}) \longrightarrow 0$$

and gives rise to a quadratic form

$$K_{2,x} \otimes K_{2,x} \longrightarrow Q_{2,x} \approx HOM(K_{2,x}, Q_{1,x}).$$

This form being non-degenerate orients the space $HOM(K_{2,x}, Q_{1,x})$. Since $Q_{1,x}$ has a canonical orientation, we obtain a canonical orientation of $K_{2,x}$. □

Let $\gamma\colon [-1,1] \to \overline{\Sigma_2(f)}$ be a path which intersects the submanifold of non-cusp singular points transversally and at exactly one point.

Lemma 4.3 *The canonical orientations of* K_2 *at* $\gamma(-1)$ *and* $\gamma(1)$ *lead to different orientations of the trivial bundle* $\gamma^* K_2$.

Proof If necessary we slightly modify the path γ so that the unique intersection point of γ and the set $\overline{\Sigma_3(f)}$ is a swallowtail singular point. Then the statement of the lemma is easily verified using the formulas (5). □

Now we are in position to prove the assumption.

Lemma 4.4 *The bundle K_2 is trivial over $\overline{{}_2(f)}$.*

Proof Assume that the statement of the lemma is wrong. Then there is a closed path $\gamma\colon S^1 \to \overline{{}_2(f)}$ which induces a non-orientable bundle $\gamma\ K_2$ over the circle S^1.

We may assume that the path γ intersects the submanifold $\overline{{}_3(f)}$ transversally. Let $t_1, \dots, t_k, t_{k+1} = t_1$ be the points of the intersection $\gamma \cap \overline{{}_3(f)}$. Over every interval (t_i, t_{i+1}) the normal bundle of $\overline{{}_2(f)}$ in $\overline{{}_1(f)}$ has two orientations. One orientation is given by Lemma 4.1 and another is given by the canonical orientation of the bundle K_2. By Lemma 4.3 if these orientations coincide over (t_{i-1}, t_i), then they differ over (t_i, t_{i+1}). Therefore the number of the intersection points is even and the bundle $\gamma\ K_2$ is trivial. Contradiction. □

Remark The statement similar to the assertion of Lemma 4.4 for the jet bundle $J(P;Q)$ is not correct. The vector bundle K_2 over $\overline{{}^{I_2}}\ J(P;Q)$ is non-orientable. This follows for example from the study of topological properties of ^{I_r} in [2, §4].

References

[1] P. Akhmetev, R. Sadykov, *A remark on elimination of singularities for mappings of 4-manifold into 3-manifold*, Top. Appl., 131 (2003), 51-55.

[2] Y. Ando, *On the elimination of Morin singularities*, J. Math. Soc. Japan, 37 (1985), 471-487; Erratum 39 (1987), 537.

[3] V. I. Arnol'd, V. A. Vasil'ev, V. V. Goryunov, O. V. Lyashenko, *Dynamical systems VI. Singularities, local and global theory*, Encyclopedia of Mathematical Sciences - Vol. 6 (Springer, Berlin, 1993).

[4] J. M. Boardman, *Singularities of differentiable maps*, Publ. Math., 33 (1967), 21-57.

[5] D. S. Chess, *A note on the classes* $[S_1^k(f)]$, Proc. Symp. Pure Math., 40 (1983), 221-224.

[6] J. M. Eliashberg, *Surgery of singularities of smooth mappings*, Math. USSR Izv., 6 (1972), 1302-1326.

[7] S. Kikuchi, O. Saeki, *Remarks on the topology of folds*, Proc. Amer. Math. Soc., 123 (1995), 905-908.

[8] F. Ronga, *Le calculus des classes duales singularites de Boardman d'ordre deux*, Comment. Math. Helv., 47 (1972), 15-35.

[9] O. Saeki, *Notes on the topology of folds*, J. Math. Soc. Japan, v.44, 3 (1992), 551-566.

[10] O. Saeki, K. Sakuma, *Maps with only Morin singularities and the Hopf invariant one problem*, Math. Proc. Camb. Phil. Soc., 124 (1998), 501-511.

[11] O. Saeki, K. Sakuma, *Elimination of Singularities: Thom Polynomials and Beyond*, London Math. Soc., Lecture Notes Ser. 263.

[12] K. Sakuma, *A note on nonremovable cusp singularities*, Hiroshima Math. J., 31 (2001), 461-465.

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