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Skein relations for Milnor's μ-invariants

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Abstract The theory of link-homotopy, introduced by Milnor, is an important part of the knot theory, with Milnor's $\bar{\mu}$-invariants being the basic set of link-homotopy invariants. Skein relations for knot and link invariants played a crucial role in the recent developments of knot theory. However, while skein relations for Alexander and Jones invariants are known for quite a while, a similar treatment of Milnor's $\bar{\mu}$-invariants was missing. We fill this gap by deducing simple skein relations for link-homotopy μ-invariants of string links.

AMS Classification 57M25; 57M27

Keywords String links, link homotopy, Milnor's μ-invariants, skein relations

1 Introduction

1.1 Some history

Skein relations for various invariants played a significant role in knot theory and related areas in the last decade. Probably the first widely-used skein relation, discovered by Conway, was the skein relation for the Conway-Alexander polynomial. Apart from multiple technical consequences, it led to the discovery of similar skein relations for other knot invariants, notably for the Jones and HOMFLY polynomials. In the theory of 3-manifolds, the skein relation for the Casson invariant of homology spheres was also fruitfully used in different contexts.

An important part of the knot theory is the theory of link-homotopy, initiated by Milnor. Link-homotopy is a useful notion to isolate the linking fenomena from the self-knotting ones and to study it separately. A celebrated example of link-homotopy invariants is given by Milnor's $\bar{\mu}_{i_1\dots i_r,j}$ invariants [4, 5] with non-repeating indices $1 \leq i_1, \dots i_r, j \leq n$. Roughly speaking, these describe the dependence of j-th parallel on the meridians of i_1-th, …, i_r-th components.

The simplest invariant $\bar{\mu}_{i,j}$ is just the linking number of the corresponding components. The next one, $\bar{\mu}_{i_1 i_2,j}$, detects the Borromean-type linking of the corresponding 3 components and, together with the linking numbers, classify 3-component links up to link-homotopy.

Unfortunately, a complicated self-recurrent indeterminacy in the definition of $\bar{\mu}$-invariants (reflected in the use of notation $\bar{\mu}$, rather than μ) for a long time slowed down their study. The introduction of string links [2] considerably improved the situation, since a version of $\bar{\mu}$-invariants modified for string links is free of this indeterminacy; thus (and to stress a special role of the j-th component) we will further use the notation $\mu_{i_1\dots i_r}(L, L_j)$ for these invariants. Milnor's invariants classify string links up to link-homotopy [2]. Surprisingly, up to now μ-invariants remained aside from the well-developed scheme of skein relations.

1.2 Brief statement of results

We deduce new skein relations for μ-invariants. Usually in the knot theory skein relations involve links, obtained by different splittings of a diagram in a crossing. In the context of string links this leads to an appearance of tangles which are not pure any more, but contain a new "loose" component. Fortunately, it is easy to extend the definition of μ-invariants to such tangles. We next note that $\mu_{i_1\dots i_r}(L, l) = 0$ for any string link L whose loose component l passes everywhere in front of the other strings. Thus it suffices to study the jump of $\mu_{i_1\dots i_r}$ under a crossing change of l with any other, say, i_k-th, component. It turns out, that this jump can be expressed via the invariants $\mu_{i_1\dots i_{k-1}}(L - L_{i_k}, l_0)$ and $\mu_{i_{k+1}\dots i_r}(L - L_{i_k}, l_\infty)$ of string links with new loose components l_0 and l_∞ obtained by splitting the crossing in two possible ways, see Section 3.1.

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2 Preliminaries

2.1 String links and link-homotopy

Let D^2 be a disc in the xy-plane which intersects the x-axis. An *n-component string link* (see [3], [2]) L is an ordered collection of n disjoint arcs properly embedded in $D^2 \times [0,1]$ in such a way, that the i-th arc ends in the points $p_i \times \{0,1\}$, where $p_i = (x_i, 0)$ are some prescribed points on the x-axis, enumerated in the natural order $x_1 < x_2 < \cdots < x_n$. We assume that all arcs of L are oriented downwards. By the *closure* $\overline{L}$ of a string link L we mean the braid closure of L. It is an n-component link obtained from L by an addition of n disjoint arcs in the plane $\{y = 0\}$, each of which meets $D^2 \times [0,1]$ only at the endpoints $p_i \times \{0,1\}$ of L, as illustrated in Figure 1. The linking number lk of two components of L is their linking number in $\overline{L}$. Two string links are *link-*

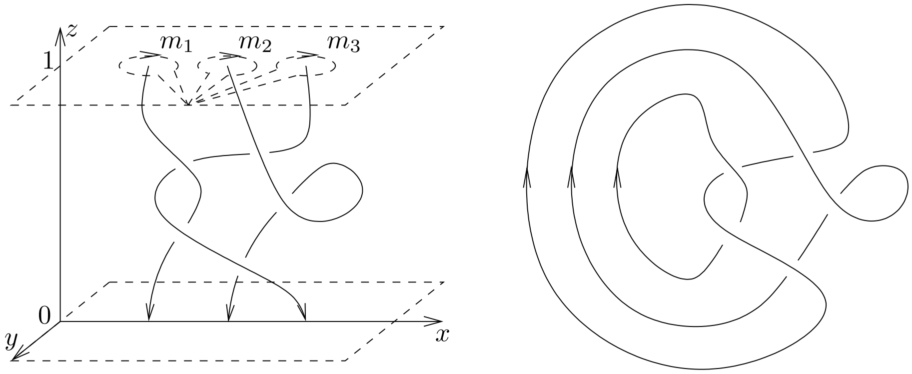

Figure 1: String link and its closure

homotopic, if one can be transformed into the other by homotopy, which fails to be isotopy only in a finite number of instants, when a (generic) self-intersection point appears on one of the arcs.

2.2 String links with a loose component

Further we will consider a more general class of tangles. A *n-component string link L with an additional loose component l* is a pair (L, l) which consists of an n-string link L together with an additional oriented arc l properly embedded in $D^2 \times [0,1]$ in such a way, that $l \cap L = \emptyset$ and l starts on $D^2 \times \{1\}$ and ends on $D^2 \times \{0,1\}$. See Figure 2, where loose components are depicted in bold. By the *closure* $\overline{(L,l)}$ of a string link L with a loose component l we mean the closure of L, together with a closure of l by a standard arc in $\mathbb{R}^3 - D^2 \times (0,1)$, which meets l only at its endpoints and passes "in front" of L i.e. lies in the half-space $y \geq 0$, as illustrated in Figure 2.

In particular, for any n-string link $L = \cup_{i=1}^n L_i$ and $1 \geq j \geq n$ one may consider $(L - L_j, L_j)$ as an $(n-1)$-string link with an additional loose component L_j.

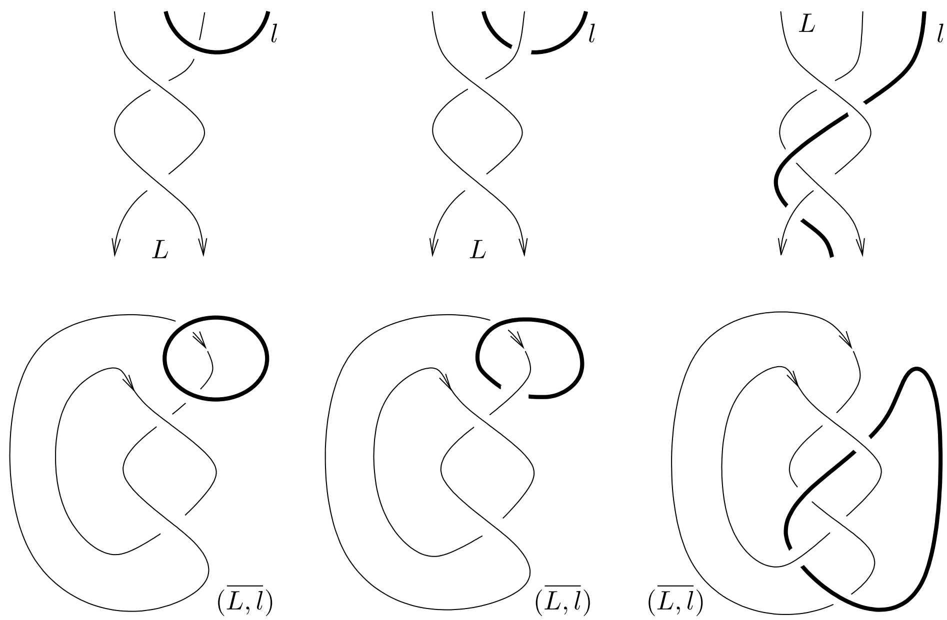

Figure 2: 2-component string links with a loose component and their closures

In this case it does not matter whether to close L_j in front of L or in the plane $\{y=0\}$ and closures $\overline{L}$ and $\overline{(L-L_j, L_j)}$ give isotopic links.

2.3 Magnus expansion

Milnor's link-homotopy μ-invariants [4] can be defined in several ways. We choose a construction most suitable for our purposes and refer the reader to Milnor's work [4, 5] and its adaptations to string links (e.g. [3], [2]) for the general case.

Let $L = \cup_{i=1}^n L_i$ be an n-component string link. Consider the link group $\pi = \pi_1(D^2 \times [0,1] - L)$ (with the base point on the upper boundary disc $D^2 \times \{1\}$). Denote by $m_i \in \pi$, $i = 1, \dots, n$ the *canonical meridians* represented by the standard non-intersecting curves in $D^2 \times \{1\}$ with $\mathrm{lk}(m_i, L_i) = +1$, as shown in Figure 1. If L would be a braid, these meridians would freely generate π, with any other meridian of L_i in π being a conjugate of m_i. For the string links, similar results hold for the reduced link group $\tilde{\pi}$.

For any group G with a finite set of generators $x_1, \dots x_n$, the *reduced group* $\tilde{G}$ is the factor group of G by relations $[x_i', x_i''] = 1$, $i = 1, \dots, n$ for any two elements x_i' and x_i'' in the conjugacy class of x_i.

Proceeding similarly to the usual construction of Wirtinger's presentation, one can show (see [2]) that $\tilde{\pi}$ is generated by m_i. Let F be the free group on n generators $x_1, \dots x_n$. The map $F \to \pi$ defined by $x_i \mapsto m_i$ induces the isomorphism $\widetilde{F} \cong \tilde{\pi}$ of the reduced groups [2]. We will use the same notation for the elements of π and their images in $\tilde{\pi} \cong \widetilde{F}$.

Now, let $\mathbb{Z}[[X_1, \dots, X_n]]$ be the ring of power series in n non-commuting variables X_i and denote by $\widetilde{Z}$ its factor by all the monomials, where at least one of the generators appears more than once. The *Magnus expansion* is a ring homomorphism of the group ring $\mathbb{Z}F$ into $\mathbb{Z}[[X_1, \dots, X_n]]$, defined by $x_i \mapsto 1 + X_i$. It induces the homomorphisms $\theta : \mathbb{Z}\widetilde{F} \to \widetilde{Z}$ and $\theta_L : \mathbb{Z}\tilde{\pi} \to \widetilde{Z}$ of the corresponding reduced group rings.

2.4 Milnor's μ-invariants

Let l an immersed arc in $D^2 \times [0,1] - L$ with $\partial l \in R^1 \times \{0\} \times \{0,1\}$. Its closure by an arc on the boundary of the cylinder $D^2 \times [0,1]$ in front of L (i.e. in the half-space $y \geq 0$) gives a well-defined loop $\bar{l}$ in $\tilde{\pi}$. Thus one may define the *Milnor's invariants* $\mu_{i_1 \dots i_r}(L, l)$ of a string link with a loose component (L, l) as coefficients of the Magnus expansion $\theta_L(\bar{l})$ of $\bar{l}$:

$$\theta_L(l) = \sum \mu_{i_1 \dots i_r}(L, l) X_{i_1} X_{i_2} \dots X_{i_r} \, .$$

Note that if l goes in front of L, i.e. overpasses all other components on the xz-plane diagram of $L \cup l$, then all invariants $\mu_{i_1 \dots i_r}(L, l)$ vanish. Indeed, we choose to close l in the front half-space $\{y \geq 0\} \cap \partial(D^2 \times [0,1])$, so the loop $\bar{l}$ is trivial in π (and hence trivial in $\tilde{\pi}$).

For string links, Milnor's invariants $\mu_{i_1 \dots i_r}(L - L_j, L_j)$ are usually denoted by $\mu_{i_1 \dots i_r, j}(L)$ (see e.g. [3], [2]). Modulo lower degree invariants $\mu_{i_1 \dots i_r}(L - L_j, L_j) \equiv \bar{\mu}_{i_1 \dots i_r, j}(\overline{L})$, where $\bar{\mu}_{i_1 \dots i_r, j}(\overline{L})$ are the original Milnor's link invariants [4, 5].

2.5 Fox's free calculus

Instead of the Magnus expansion, μ-invariants may be defined via Fox free calculus.

Fox's *free derivatives* $\delta_i : \mathbb{Z}F \to \mathbb{Z}F$, $i = 1, \dots, n$, are defined by putting $\delta_i 1 = 0$, $\delta_i x_i = 1$, $\delta_i x_j = 0$, $j \neq i$ and extending the function δ_i to $\mathbb{Z}F$ by linearity and the rule $\delta_i(uv) = \delta_i u + u \cdot \delta_i v$, $u, v \in ZF$. In particular, it is easy to see that $\delta_i(x_i^{-1}) = -x_i$. For $u \in \mathbb{Z}F$, denote by $\delta_{i_1} \dots \delta_{i_r} u(1) \in \mathbb{Z}$ the value of $\delta_{i_1} \dots \delta_{i_r} u$ in $x_1 = \dots = x_n = 1$.

Free derivatives in $\mathbb{Z}F$ induce similar Fox derivatives $\delta_i : \mathbb{Z}\widetilde{F} \to \mathbb{Z}\widetilde{F}$ and $\delta_i : \mathbb{Z}\tilde{\pi} \to \mathbb{Z}\tilde{\pi}$ in the reduced group rings. From the definition of μ-invariants we conclude that $\mu_{i_1 \dots i_r}(L, l) = \delta_{i_1} \dots \delta_{i_r} l(1)$.

3 Skein relations for μ-invariants

3.1 Main Theorem

Consider two n-string links (L, l_+) and (L, l_-) with loose components such that their diagrams coincide everywhere, except for a crossing d, where l_+ has the positive crossing and l_- the negative crossing with i_k-th component L_{i_k} of L, see Figure 3. We define two new $(n-1)$-string links with loose components

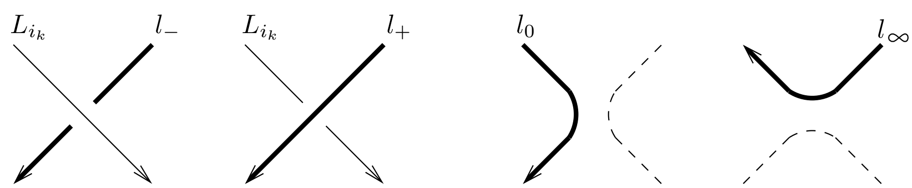

Figure 3: Splitting the link in a crossing

by splitting the diagram in d in two possible ways. Let l_∞ be the component of the splitting going along l, respecting the orientation, and then switching to L_{i_k} against the orientation. Let also l_0 be the component going first along L_{i_k} and then along l, respecting the orientation. Each of $(L - L_{i_k}, l_0)$ and $(L - L_{i_k}, l_\infty)$ is a $(n-1)$-string link with a loose component, see Figure 3.

Theorem 3.1 *Let L, $l_\pm$, l_0 and l_∞ be as above. Then*

$$\mu_{i_1 \dots i_k \dots i_r}(L, l_+) - \mu_{i_1 \dots i_k \dots i_r}(L, l_-) = \mu_{i_1 \dots i_{k-1}}(L - L_{i_k}, l_\infty) \cdot \mu_{i_{k+1} \dots i_r}(L - L_{i_k}, l_0).$$

Here it is understood that in particular cases $k = 1$ or $k = r$ we have

$$\mu_{i_k}(L, l_+) - \mu_{i_k}(L, l_-) = 1;$$

$$\mu_{i_1 \dots i_k}(L, l_+) - \mu_{i_1 \dots i_k}(L, l_-) = \mu_{i_1 \dots i_{k-1}}(L - L_{i_k}, l_\infty);$$

$$\mu_{i_k \dots i_r}(L, l_+) - \mu_{i_k \dots i_r}(L, l_-) = \mu_{i_{k+1} \dots i_r}(L - L_{i_k}, l_0).$$

Example 3.2 Consider the string link (L, l) depicted in Figure 4 and let us compute $\mu_{12}(L, l)$. Notice that if we switch the crossing d to the positive one, we get a link (L, l_+) with L_1 unlinked from L_2 and l_+, so $\mu_{12}(L, l_+) = 0$. Thus $\mu_{12}(L, l) = \mu_{12}(L, l) - \mu_{12}(L, l_+) = -\mu_1(L_1, l_\infty) = -1$, in agreement with the fact that the closure of (L, l) is the (negative) Borromean link.

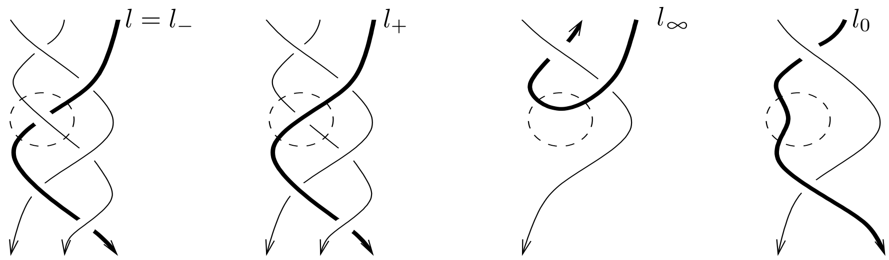

Figure 4: A computation of $\mu_{12,3}$ for Borromean rings

3.2 The proof of Theorem 3.1

We will need the following simple key fact about Fox free calculus, which we leave to the reader as an exercise:

Lemma 3.3 *Let $1 \le k \le r < j \le n$ and $\varepsilon = \pm 1$. Then for any $u, v \in F$*

$$\delta_1 \dots \delta_r(ux_k^\varepsilon v)(1) = \varepsilon \cdot \delta_1 \dots \delta_{k-1} u(1) \cdot \delta_{k+1} \dots \delta_r v(1) + \delta_1 \dots \delta_r(uv)(1). \quad (1)$$

Also, to simplify the notations in the statement of Theorem 3.1, let us reorder the components so that $i_m \to m$, $m = 1, \dots r$ (and $j \ne 1, \dots, r$).

Proof of Theorem 3.1 The component l_+ passes either over, or under L_k in the crossing d. Let us consider these cases separately. Assume first that l_+ passes under L_k in d. Then $l_+ = vu^{-1}x_k uw$, where v and w are the parts of l_+ before and after d, and u is the part of l_k before d, see Figure 5. Thus, by the definition of $\mu_{1\dots k\dots r}(L, l)$ and (1) we have

$$\mu_{1\dots r}(L, l_+) = \delta_1 \dots \delta_r(vu^{-1}x_k uw)(1) = \\ \delta_1 \dots \delta_{k-1}(vu^{-1})(1) \cdot \delta_{k+1} \dots \delta_r(uw)(1) + \delta_1 \dots \delta_r(vw)(1).$$

Now, l_- passes over L_k, so $l_- = vw$, where again v and w are the parts of l_- before and after d, see Figure 5. Thus

$$\mu_{1\dots k\dots r}(L, l_-) = \delta_1 \dots \delta_k \dots \delta_r(vw)(1).$$

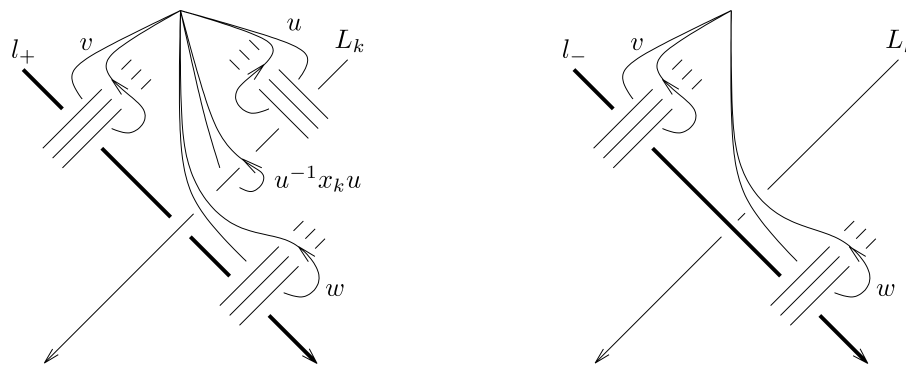

Figure 5: Computing l_+ and l_-

Therefore

$$\mu_{1\dots k\dots r}(L, l_+) - \mu_{1\dots k\dots r}(L, l_-) = \delta_1 \dots \delta_{k-1}(vu^{-1})(1) \cdot \delta_{k+1} \dots \delta_r(uw)(1).$$

But vu^{-1} and uw are exactly l_∞ and l_0, see Figure 6.

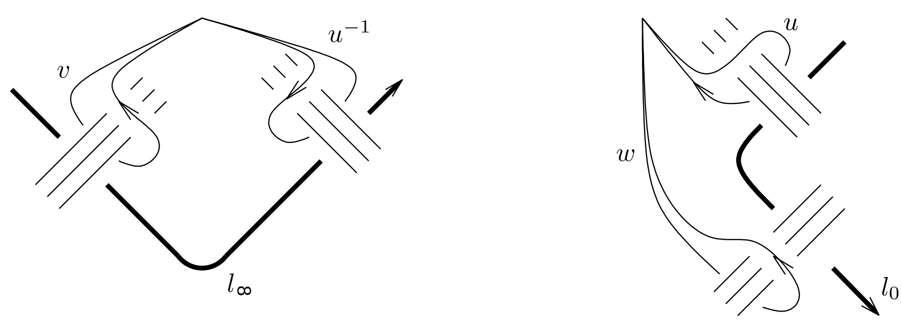

Figure 6: Computing l_∞ and l_0

It remains to use again the definition of μ-invariants to obtain

$$\mu_{1\dots k\dots r}(L, l_+) - \mu_{1\dots k\dots r}(L, l_-) = \mu_{1\dots k-1}(L, l_\infty) \cdot \mu_{k+1\dots r}(L, l_0).$$

In the case when l_+ passes over L_k at the crossing d, the proof is completely similar. This time $l_- = vu^{-1}x_k^{-1}uw$ and $l_+ = vw$, thus basically we just exchange l_+ and l_- in the previous proof. Also, the degree of x_k switches from $+1$ to -1, which leads to the negative sign in the application of (1) and cancels out with the sign appearing from the exchange of L_+ and L_-. □

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