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Taut ideal triangulations of 3{manifolds

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Abstract

A taut ideal triangulation of a 3{manifold is a topological ideal triangulation with extra combinatorial structure: a choice of transverse orientation on each ideal 2{simplex, satisfying two simple conditions. The aim of this paper is to demonstrate that taut ideal triangulations are very common, and that their behaviour is very similar to that of a taut foliation. For example, by studying normal surfaces in taut ideal triangulations, we give a new proof of Gabai's result that the singular genus of a knot in the 3{sphere is equal to its genus.

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1 Introduction

In his famous lecture notes [10], Thurston introduced a surprising topological construction of the figure-eight knot complement, by gluing two ideal tetrahedra along their faces. Using this, he gave the knot complement a complete hyperbolic structure. Ideal triangulations are not only a useful tool in hyperbolic geometry (for example [7]), but also provide an elegant way of visualising 3–manifolds with boundary. In this paper, we introduce 'taut' ideal triangulations, which are ideal triangulations with a little extra structure. Instead of relating to hyperbolic geometry, they are more closely associated with taut foliations. In his seminal paper [2], Gabai constructed taut foliations on many Haken 3–manifolds, via his theory of sutured manifolds. Also utilising sutured manifolds, we will prove that many torally bounded 3–manifolds admit a taut ideal triangulation. An analysis of normal surfaces (and their generalisations) in taut ideal triangulations will yield a new proof of Gabai's result that the singular genus of a knot in S^3 is equal to its genus. This avoids many of the foliation technicalities of Gabai's original argument. We hope that taut ideal triangulations will be useful in other areas of 3–manifold theory in the future. Some speculations on possible applications are included in the final section of the paper.

Definition An *ideal 3–simplex* is a 3–simplex with its four vertices removed. An *ideal triangulation* of a 3–manifold M is an expression of $M - \partial M$ as a collection of ideal 3–simplices with their faces glued in pairs. A *taut ideal triangulation* is an ideal triangulation with a transverse orientation assigned to each ideal 2–simplex, such that

- for each ideal 3–simplex, precisely two of its faces are oriented into the 3–simplex, and precisely two are oriented outwards, and
- the faces around each edge are oriented as shown in Figure 1: all but precisely two pairs of adjacent faces encircling the edge have compatible orientations around that edge.

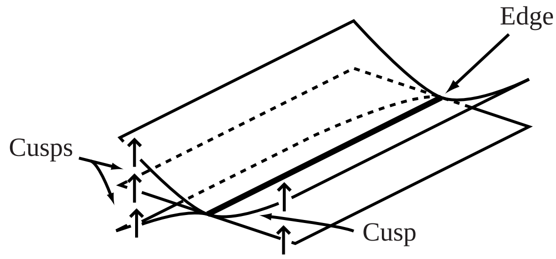


Figure 1

In order to describe situations such as Figure 1 more concisely, we introduce some terminology. Suppose that some transversely oriented surfaces embedded in a 3{manifold meet at a 1{manifold C in each of their boundaries. Then the intersection between adjacent surfaces S_1 and S_2 is *cusped* at C if S_1 and S_2 are compatibly oriented around C. Thus, the second of the above conditions in the de nition of a taut ideal triangulation can be rephrased as follows: all but precisely two pairs of adjacent faces encircling an edge have cusped intersection.

As an example, note that Thurston's ideal triangulation of the gure-eight knot complement can be assigned a transverse orientation, as shown in Figure 2, which makes it taut.

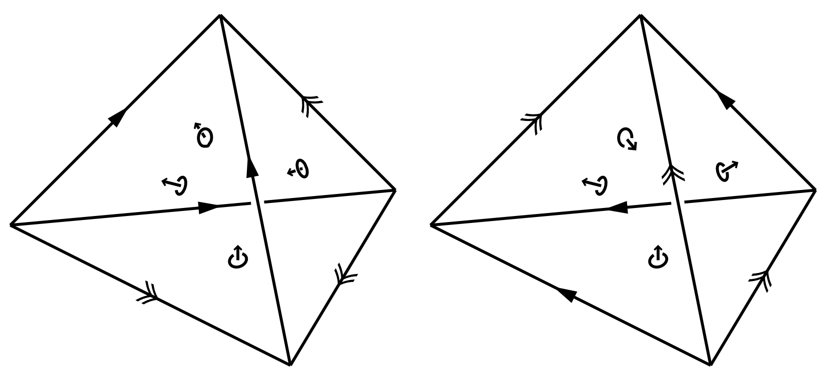


Figure 2

The faces and edges of a taut ideal triangulation form a transversely oriented branched surface in $M - @M$. Its branch locus is not 'generic', since more than three faces may meet at any edge. We will see that surfaces carried by this branched surface have strong genus-minimising properties. Given that taut ideal triangulations are a special sort of branched surface, it is not surprising that they should be related to taut foliations. In fact, it is the absence of 'triple points' in this branched surface that gives taut ideal triangulations many of their special properties.

The aim of this paper is to show that taut ideal triangulations are very common, and that their presence in a 3{manifold has useful consequences. The following is our existence result.

Theorem 1 *Let M be a compact orientable irreducible an-annular 3{manifold with $@M$ a non-empty collection of incompressible tori. Then M has a taut ideal triangulation.*

Many of the conditions in this theorem are necessary: we will see (Proposition 10) that if a compact orientable 3–manifold admits a taut ideal triangulation, then it is irreducible and its boundary is a non-empty collection of incompressible tori. However, the condition that M be an-annular can be weakened.

These taut ideal triangulations are constructed from properly embedded surfaces that are taut, in the sense that they are incompressible and have the smallest possible Thurston complexity in their homology class in $H_2(M;\partial M)$. Recall that the *Thurston complexity* $\chi_-(S)$ of a compact connected surface S is $\max\{0,-\chi(S)\}$. The Thurston complexity of a compact disconnected surface is defined to be the sum of the complexities of its components. We have the following refinement of Theorem 1, which relates taut ideal triangulations to surfaces that minimise Thurston complexity in their homology class.

Theorem 2 *Let M be a compact orientable irreducible 3–manifold with ∂M a non-empty collection of tori. Let S be a properly embedded compact oriented surface in M, such that*

- *every component of S has negative Euler characteristic and has non-empty boundary,*
- *S has minimal Thurston complexity among all embedded surfaces in its class in $H_2(M;\partial M)$,*
- *for any component T of ∂M, the curves $T\cap\partial S$ are all essential in T and coherently oriented, and*
- *there is no properly embedded essential annulus in M disjoint from S.*

Then S is carried by the underlying branched surface of some taut ideal triangulation of M. In fact, $S-\partial S$ is a union of ideal 2–simplices in that ideal triangulation.

For example, the genus one Seifert surface for the figure-eight knot is carried by the underlying branched surface of the ideal triangulation in Figure 2. (It lies in a regular neighbourhood of the front two faces in the ideal 3–simplex on the right of the figure.)

We also present a converse to Theorem 2, which asserts that surfaces carried by taut ideal triangulations minimise Thurston complexity, even when non-embedded surfaces are also considered.

Theorem 3 *Let S be a compact properly embedded surface carried by the underlying branched surface of some taut ideal triangulation of a 3–manifold M. Then S has smallest Thurston complexity among all (possibly non-embedded) surfaces in its class in $H_2(M;\partial M)$.*

By combining Theorems 2 and 3, we retrieve Gabai's result on the singular genus of knots. Recall that the *singular genus* of a knot K in S^3 is the smallest possible genus of a compact orientable surface F mapped into S^3 via a map $f\colon F \to S^3$ with $f^{-1}(K) = \partial F$ and $f|_{\partial F}$ an embedding onto K. Apply Theorem 2, with M being the exterior of K, and S being a minimal genus Seifert surface. Then apply Theorem 3 to obtain the following.

Corollary 4 [2] *The singular genus of a knot in S^3 is equal to its genus.*

Interestingly, Theorem 3 is proved using normal surfaces in taut ideal triangulations. Given that normal surfaces have a useful rôle to play in other areas of 3–manifold topology (for example [9]), this suggests that taut ideal triangulations will have other interesting applications.

Taut ideal triangulations are closely related to angled ideal triangulations, defined and studied by Casson, and developed in [4]. An *angled ideal triangulation* is an ideal triangulation with a number in the range $(0,\pi)$ assigned to each edge of each ideal 3–simplex, known as the interior angle at that edge. These angles are required to satisfy two simple conditions: the angles around an edge sum to 2π; and the angles at each ideal vertex of each ideal 3–simplex sum to π. Taut ideal triangulations induce a similar structure, except that there are only two options for the interior angles: the cusped intersections between faces have zero interior angle and the non-cusped intersections have interior angle π. (See Figure 3.)

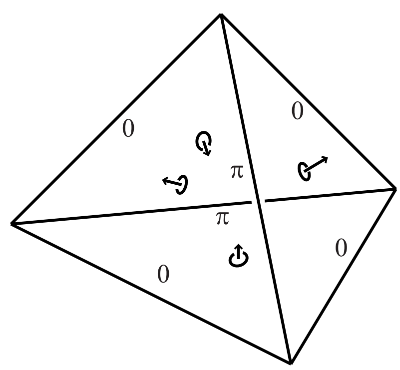


Figure 3

The underlying branched surface of a taut ideal triangulation carries an essential lamination. This is formed by laminating a neighbourhood of each face using a Cantor set transversal, and then patching these laminations together at the edges. Since the branched surface has no vertices, there is no obstruction to performing this procedure. This lamination extends to a foliation of the 3–manifold M, since the complimentary regions of the branched surface are products. It is not hard to find a closed curve transverse to the foliation intersecting every leaf, and so the foliation is taut. Note also that if S is any surface carried by the branched surface, then we may decompose a neighbourhood N of the branched surface along S, then laminate $N - S$, and then extend this to a taut foliation of M in which S is a leaf. Hence, by a theorem of Gabai [2], S must have smallest Thurston complexity among all (possibly non-embedded) surfaces in its class in $H_2(M; \partial M)$. We therefore obtain Theorem 3. However, one of the aims of this paper is, of course, to provide a proof of Theorem 3 using an argument that avoids foliations.

The underlying branched surface of a taut ideal triangulation is, in the terminology of [5], an example of a taut homology Reebless incompressible branched surface. In [5], Oertel analysed this type of branched surface and used them to establish certain facts about the Thurston norm. However, they do not seem to interact so well with singular surfaces as do our taut ideal triangulations.

A purely combinatorial proof of Corollary 4 (and, more generally, of the equivalence of the Thurston norm and the singular norm) has been given by Person in [6], building on Scharlemann's foliation-free approach to sutured manifolds [8]. The argument in [6] is rather different from the proof given here. In particular, it does not deal with ideal triangulations or normal surfaces.

2 Constructing taut ideal triangulations

In this section, we prove Theorems 1 and 2, by constructing the required taut ideal triangulations. Suppose therefore that M is a compact orientable irreducible an-annular 3–manifold, with ∂M a non-empty union of incompressible tori. The simplest case is where M fibres over the circle with fibre S, and then the result is rather easy. Note that S has negative Euler characteristic and non-empty boundary, by our assumptions on M, and hence S has an ideal triangulation. Using this, we will construct a taut ideal triangulation of M. We therefore now recall some well-known facts about ideal triangulations of surfaces.

It will be helpful to consider the following generalisation of an ideal triangulation. An *ideal region* for a compact surface S is a compact submanifold of ∂S having non-empty intersection with each component of ∂S. An *ideal triangulation* of S with ideal region is an expression of $S -$ as a union of ideal 2–simplices with some of their edges glued in pairs. Hence, $\partial S -$ must be a (possibly empty) collection of open arcs, each of which is an edge of an ideal 2{simplex. We define the *triangular number* $t(S;\Delta)$ of a surface S with ideal region Δ to be

$$t(S;\Delta) = -2\chi(S) + |\partial S - \Delta|.$$

Lemma 5 *Let S be a compact orientable surface with non-empty boundary and ideal region Δ. If $t(S;\Delta) > 0$, then S admits an ideal triangulation with ideal region Δ. Any such ideal triangulation contains precisely $t(S;\Delta)$ ideal 2{simplices.*

Proof The surface $S - \Delta$ is obtained from a compact orientable surface $\hat{S}$ by removing a finite number of points P from its boundary and from its interior. If $\hat{S}$ is closed and is not a 2{sphere, it has a one-vertex triangulation. If $\hat{S}$ has non-empty boundary and is not a disc, it has a triangulation with a single vertex on each boundary component and no vertices in its interior. Subdivide these triangulations, if necessary, so that its vertices are precisely P. Then remove these vertices to obtain an ideal triangulation of $S - \Delta$. The argument when $\hat{S}$ is a sphere or a disc is similar. We find a triangulation of $\hat{S}$, and then remove its vertices to obtain an ideal triangulation of $S - \Delta$. The assumption that $t(S;\Delta) > 0$ guarantees that this is possible.

Now consider any such ideal triangulation of S. It is formed by gluing the edges of ideal triangles in pairs. Each ideal triangle has triangular number one. At each gluing of edges, the total Euler characteristic goes down by one, but the number of boundary edges goes down by two. Hence, the total triangular number is unchanged. Thus, this ideal triangulation has $t(S;\Delta)$ ideal 2{simplices. □

It is also very well known that any two ideal triangulations of S with the same ideal region differ by a sequence of the following *elementary moves*: pick two distinct ideal 2{simplices that share an edge; remove this edge, forming a 'square' whose interior is embedded in the interior of S; then subdivide this square along its other diagonal to form two new ideal triangles. This fact is so important for our presentation that we include a proof.

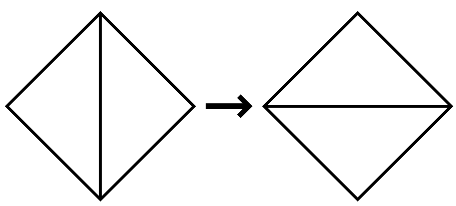


Figure 4

Lemma 6 *Any two ideal triangulations of a compact orientable surface S with the same ideal region   are related by a nite sequence of elementary moves and an ambient isotopy.*

Proof Let T_1 and T_2 be the ideal triangulations of S. Note that T_1 and T_2 both have $t(S;\ )$ ideal triangles. Let E_2 be the edges of T_2. We will work with the dual graph G_1 of T_1, which is a graph embedded in S, the interior vertices having valence three, and vertices on each component of $@S -$ having valence one.

Perform a small ambient isotopy so that the intersection between G_1 and E_2 is transverse and disjoint from the trivalent vertices of G_1. Note that each component of $(S -\ ) - G_1$ is either I $\mathbb{R}$ or S^1 $\mathbb{R}$. We consider three possibilities.

Case 1 Some edge of E_2 is disjoint from G_1.

In this case, the arc lies entirely in a component I $\mathbb{R}$ or S^1 $\mathbb{R}$ of $(S -\ ) - G_1$, and both its ends lie in the same end of I $\mathbb{R}$ or S^1 $\mathbb{R}$. An extrememost such arc separates o a disc of $S - E_2$ with a single end. However, every component of $S - E_2$ is a triangle, and we therefore have a contradiction. Thus, this case does not arise.

Case 2 Every arc of $E_2 - G_1$ runs between G_1 and an end of $S -$.

Then each edge of E_2 intersects G_1 at a single point. Hence, each triangle of T_2 has three arcs of G_1 entering it. It therefore has at least one trivalent vertex of G_1 in its interior. However, there are as many trivalent vertices of G_1 as there are triangles of T_2, and so each triangle of T_2 contains a single trivalent vertex of G_1. Hence, G_1 is the dual of T_2, and so (up to ambient isotopy) T_1 and T_2 are the same ideal triangulation.

Case 3 Some arc of $E_2 - G_1$ has both endpoints in G_1.

Pick such an arc   extrememost in one component of $S - G_1$. This separates o a disc D which lies in some triangle of T_2. Let   be $@D -$, which is a path in G_1. If   runs through at most one vertex of G_1, then there is an ambient isotopy of G_1 which reduces $jG_1 \setminus E_2j$. Suppose therefore that   contains at least two vertices of G_1. Apply an elementary move to adjacent vertices of $G_1 \setminus$ to reduce the number of vertices of . Repeat this until   contains only one vertex, and then perform an ambient isotopy to reduce $jG_1 \setminus E_2j$. In this way, we remove all arcs of $E_2 - G_1$ with both endpoints in G_1. Hence, after a nite number of elementary moves, we end with T_1 ambient isotopic to T_2. □

Let us now return to the manifold M which fibres over the circle, with fibre S. Let $h\colon S \to S$ be the monodromy homeomorphism. Pick some ideal triangulation T of S with ideal region ∂S. Then $h(T)$ may be transformed into T by sequence of elementary moves and an isotopy. Of course, in this sequence of moves, we may guarantee that no edge of T is left untouched. (For example, if an edge is adjacent to two distinct 2–simplices, perform an elementary move and its reverse). The taut ideal triangulation of M is constructed as follows. Start with S and its triangulation T. Each time that an elementary move is performed, glue an ideal tetrahedron (as in Figure 5) onto one side of S, attaching it to the two ideal triangles involved in the elementary move. This side of S then inherits the new ideal triangulation. Repeat this process for each elementary move. Then we have constructed $(S - \partial S) \times I$, since every edge of T was modified by the elementary moves. Now glue the two components of $(S - \partial S) \times \partial I$ via h. The ideal triangulations match up to form a taut ideal triangulation of M. The choice of elementary moves realizing h was highly non-unique, and hence M has many taut ideal triangulations. It is interesting to note that the taut ideal triangulation of the figure-eight knot complement given in Figure 2 can be constructed in this way, except that only two elementary moves are used, and hence one edge of the ideal triangulation of the fibre remains unmodified.

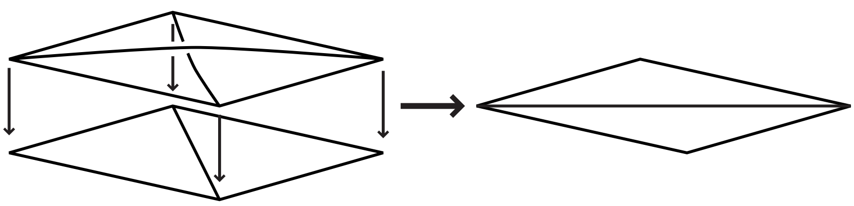


Figure 5

When M does not fibre over S^1, it is significantly more difficult to construct taut ideal triangulations. The main technical tool is sutured manifold theory, which Gabai originally used to construct taut foliations on many irreducible 3–manifolds. We will use the version developed by Scharlemann in [8], and, in this section of the paper, we will assume that the reader is reasonably familiar with [8]. Recall that a *sutured manifold* (M, γ) is a compact oriented 3–manifold M, with ∂M decomposed into subsurfaces R_-, R_+, $A(\gamma)$ and $T(\gamma)$, which intersect in simple closed curves, such that

- each component of $A(\gamma)$ is an annulus adjacent to both R_- and R_+,
- each component of $T(\gamma)$ is a torus, and

- $R_- \cap R_+ = \emptyset$.

We let γ be the core curves of $A(\gamma)$. The surface R_- (respectively, R_+) is assigned an orientation into (respectively, out of) M. The annuli $A(\gamma)$ and tori $T(\gamma)$ are not assigned a specific orientation.

When S is a transversely oriented surface properly embedded in a sutured manifold (M, γ), the transverse orientations on S and $R_\pm$ induce a cusp on one side of each component of $\partial S \cap R_\pm$. The manifold $M - \mathrm{int}(N(S))$ inherits a sutured manifold structure, providing S satisfies various simple properties. These properties have a variety of names in the literature: d–surface or conditioned in [8], groomed or well-groomed in [3]. In this paper, we introduce a variant of these. We allow S to intersect $A(\gamma)$ in simple closed curves, *transverse* arcs (which run between distinct components of $\partial A(\gamma)$) and *glancing* arcs (which run between the same component of $\partial A(\gamma)$).

Definition A transversely oriented surface S properly embedded in (M, γ) is *styled* if

- for each component T of $T(\gamma)$, the curves $T \cap \partial S$ are all essential in T and coherently oriented,
- near each simple closed curve of $S \cap A(\gamma)$, S has the same orientation as R_- and R_+ near that component of $A(\gamma)$,
- the transverse arcs of intersection between S and any component of $A(\gamma)$ are all coherently oriented, and
- near any glancing arc α of $S \cap A(\gamma)$, the cusped side of ∂S runs into the disc component of $A(\gamma) - \alpha$.

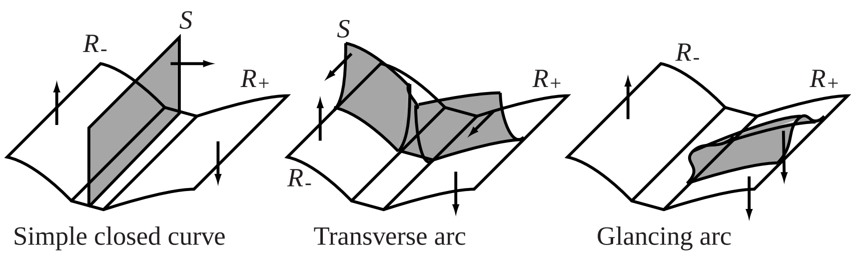


Figure 6

(See Figure 6.) When S is styled, then $M_S = M - \mathrm{int}(N(S))$ inherits a sutured manifold structure (M_S, γ_S) in a natural way: $R_\pm(\gamma_S)$ is composed of $R_\pm(\gamma) - \mathrm{int}(N(S))$ and the copies of S. The tori of $T(\gamma)$ disjoint from

S yield $T(\gamma_S)$. The annuli $A(\gamma_S)$ lie between $R_-(\gamma_S)$ and $R_+(\gamma_S)$, and also arise from $T(\gamma) - \mathrm{int}(N(S))$. The orientation conditions imposed on glancing arcs and simple closed curves of $S \cap A(\gamma)$ guarantee that each component of $A(\gamma) - \mathrm{int}(N(S))$ lies in $A(\gamma_S)$.

We introduce the following definition.

Definition An *ideal region* Δ of a sutured manifold (M, γ) is a collection of the following subsets of $A(\gamma) \cup T(\gamma)$:

- all of $T(\gamma)$,
- possibly some components of $A(\gamma)$, and
- 'squares', each of which is a subset of a component A of $A(\gamma)$, being the region between two properly embedded transverse arcs in A.

We insist, in addition, that no component of $A(\gamma)$ is disjoint from Δ.

The idea behind the above definition is that we start with a sutured manifold (M, γ) having $\partial M = T(\gamma)$, and so the whole of ∂M is the ideal region. Then we perform a sequence of sutured manifold decompositions, resulting in sutured manifolds embedded in M. Their ideal regions will be their intersection with ∂M.

The taut ideal triangulations in Theorems 1 and 2 will be constructed using a sutured manifold hierarchy. At each stage of this hierarchy, we will construct a taut triangulation, in the following sense.

Definition A *taut triangulation* of a sutured manifold (M, γ) with ideal region Δ is an expression of $M - \Delta$ as a collection of ideal 3–simplices with some of their faces identified in pairs, and with a transverse orientation assigned to each ideal 2–simplex, such that

- for each ideal 3–simplex, precisely two of its faces are oriented into the 3–simplex, and precisely two are oriented outwards,
- each component of $\gamma - \Delta$ is an edge of the triangulation,
- each 2–simplex in ∂M lies entirely in R_- or R_+ (apart from a collar neighbourhood of some of its edges, which may lie in $A(\gamma)$),
- the transverse orientation of each 2–simplex in ∂M agrees with that of $R_\pm$,
- for each edge not in γ, all but precisely two pairs of adjacent faces around that edge have cusped intersection, and
- for each edge in γ, all faces around that edge have cusped intersection.

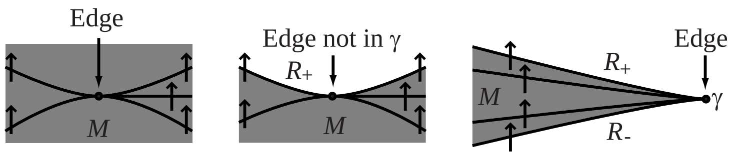


Figure 7

When $(M;\gamma)$ has a taut triangulation, R inherits an ideal triangulation with ideal region $\backslash R$. Note also that when $@M$ is a collection of tori, and $= T(\gamma)$ is all of these tori, then a taut triangulation of M is a taut ideal triangulation.

The case where M is a 3{ball and γ is a single curve is an instructive example. If is precisely four squares, then a single ideal 3{simplex forms a taut triangulation of M, as in Figure 3. If is more than four squares, then we may pick an ideal triangulation of $R_- -$. Then, as in the bred case, a taut triangulation of M arises by attaching a collection of ideal 3{simplices to realize a suitable sequence of elementary moves. However, if is fewer than four squares or the whole of $A(\gamma)$, then $t(R_-;)$ and $t(R_+;)$ are each at most one. Thus, any ideal triangulation of $R_- -$ and $R_+ -$ consists of at most one ideal triangle. It is then not hard to see that $(M;\gamma)$ admits no taut triangulation having as an ideal region. The following theorem demonstrates that a similar pattern arises for other taut sutured manifolds. We say that an annulus embedded in M is {*essential* if

its boundary is in ,

it is incompressible, and

it is not parallel to an annulus in .

Theorem 7 *Let $(M;\gamma)$ be a connected taut sutured manifold. Suppose that $@M$ is non-empty and that no component of $@M$ is disjoint from $A(\gamma)$ [$T(\gamma)$. Let be any choice of ideal region, such that $t(R_-;)$ 2 and $t(R_+;)$ 2, and where M contains no {essential annulus. Then $(M;\gamma)$ admits a taut triangulation with as ideal region.*

Proof of Theorems 1 and 2 from Theorem 7 We rst show that Theorem 1 follows from Theorem 2. Let M be a compact orientable irreducible an-annular 3{manifold with $@M$ a non-empty collection of incompressible tori. Then, it is well known that some element of $H_2(M;@M)$ has non-trivial image in $H_1(@M)$ under the boundary map of the homology exact sequence of the pair $(M;@M)$. This element of $H_2(M;@M)$ is represented by a properly embedded compact orientable surface S with non-empty boundary. Take S to have minimal Thurston complexity among all embedded surfaces in its homology class. We may cap off curves of ∂S bounding discs in ∂M, and also, by attaching annuli, we may assume that, for each component T of ∂M, $\partial S \cap T$ is a collection of coherently oriented simple closed curves, each essential in T. This does not increase its Thurston complexity or alter its class in $H_2(M; \partial M)$. By construction, at least one component of S has non-empty boundary. Restrict attention to this component, which we will now call S. Then, S is neither a disc nor an annulus, since M has incompressible boundary and is an-annular. Hence, S satisfies all of the conditions of Theorem 2, and so Theorem 1 follows from Theorem 2.

We now show that Theorem 2 follows from Theorem 7. Let M and S be as in Theorem 2, and let $T(\gamma) = \partial M$. Then $(M; \gamma)$ is taut since M is irreducible and not a solid torus. Perform the taut sutured manifold decomposition

$$(M; \gamma) \xrightarrow{S} (M_S; \gamma_S);$$

and let $\Delta_S = A(\gamma_S) \cup T(\gamma_S)$. Note that there is no Δ_S–essential annulus in M_S, since this would be an essential annulus in M disjoint from S. Also,

$$t(R_-(\gamma_S); \Delta_S) = t(R_+(\gamma_S); \Delta_S) = -2\chi(S) \geq 2.$$

Thus, using Theorem 7, find a taut triangulation for $(M_S; \gamma_S)$ with ideal region Δ_S. Glue the two copies of S in $R_-(\gamma_S)$ and $R_+(\gamma_S)$ together, ensuring that their ideal triangulations agree using a sequence of elementary moves, as in the fibred case. The result is a taut ideal triangulation of M in which S is a union of ideal 2–simplices and is carried by the underlying branched surface. □

The remainder of this section is devoted to proving Theorem 7. Let $(M; \gamma)$ and Δ be as in Theorem 7. We will prove the theorem by induction backwards along a sutured manifold hierarchy for $(M; \gamma)$. Each decomposing surface (other than product discs) will end up as a collection of ideal 2–simplices in the resulting taut triangulation. Hence it is vital that each surface meets Δ. We do this by 'sliding' the boundary of the surface towards Δ along arcs. The following lemma guarantees that this is possible.

Lemma 8 *Let $(M; \gamma)$ be a connected taut sutured manifold where ∂M is non-empty and no component of ∂M is disjoint from $A(\gamma) \cup T(\gamma)$. Then there is a taut sutured manifold hierarchy*

$$(M; \gamma) = (M_1; \gamma_1) \xrightarrow{S_1} (M_2; \gamma_2) \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} (M_n; \gamma_n);$$

such that, for each i, S_i is a connected styled non-separating surface with non-empty boundary, and, for any point p on $R_\pm(\gamma_i)$, there is an embedded arc α in ∂M_i such that

- *p is one endpoint of $\;$,*
- *$\cap A(\gamma_i)$ is the other endpoint of $\;$, and*
- *if $\;$ is oriented from p to $\partial \; - p$, then at each point of intersection between $\;$ and ∂S_i, $\;$ runs from the cusped side of ∂S_i to the uncusped side.*

Proof We must return to the proof of the existence of sutured manifold hierarchies (Theorem 4.19 of [8]). There are two main ingredients: to show that given any non-zero class $z \in H_2(M_i; \partial M_i)$, we can perform a taut decomposition along a styled surface S_i with $z = [S_i; \partial S_i]$; then to show that, with the correct choice of decomposing surfaces, a sequence of taut decompositions can be made to terminate in a collection of 3–balls. The second part is dealt with in §4 of [8]. There, a complexity for a sutured manifold is defined. If a taut sutured manifold contains a non-trivial product disc, then decomposing along this disc does not increase the complexity. (A product disc is *non-trivial* if it does not separate off a 3–ball.) If a taut sutured manifold contains no non-trivial product disc, then any taut decomposition along a connected non-separating incompressible surface decreases sutured manifold complexity (Theorem 4.17 of [8]). If a sutured manifold is decomposed along a product disc, then future decomposing surfaces can be ambient isotoped so that they avoid the two copies of this disc. Hence, by Lemma 4.2 of [8], it is possible to postpone all the decompositions along product discs until the final step. Hence, a sequence of taut decompositions along connected non-separating incompressible surfaces must eventually terminate with a product sutured manifold. Decompose this along non-separating product discs to obtain a 3–ball. Hence, providing at each stage we can find a surface satisfying the requirements of Lemma 8, this sequence of sutured manifolds can be guaranteed to terminate.

Suppose therefore that we have constructed the sutured manifold sequence as far as $(M_i; \gamma_i)$. We claim that no component F of ∂M_i is disjoint from $A(\gamma_i) \cup T(\gamma_i)$. Suppose that, on the contrary, F is disjoint from $A(\gamma_i) \cup T(\gamma_i)$. We are assuming that no component of ∂M is disjoint from $A(\gamma) \cup T(\gamma)$, and hence F must intersect some S_j, $j < i$. Let j be the largest such integer. Note that ∂S_j has non-empty boundary. If ∂S_j intersects $A(\gamma_j) \cup T(\gamma_j)$, then $F \cap A(\gamma_i)$ is non-empty. If ∂S_j is disjoint from $A(\gamma_j) \cup T(\gamma_j)$, then there is an arc $\;$ as in the lemma, which runs from ∂S_j to $A(\gamma_j)$. Whether or not the interior of $\;$ intersects ∂S_j, we obtain a component of $A(\gamma_i)$ in F.

Suppose that ∂M_i is not a collection of 2–spheres. For otherwise we have constructed the required hierarchy. Let C be a finite collection of disjoint oriented simple closed curves in ∂M_i satisfying the following:

- $[C] \neq 0 \in H_1(\partial M_i)$,
- $[C] = 0 \in H_1(M_i)$,
- C intersects $A(\gamma_i)$ only in transverse arcs, and
- $|C|$ is minimal among all oriented curves in ∂M_i with the above three properties.

The existence of C is a consequence of the well-known fact that $H_1(\partial M_i) \to H_1(M_i)$ has non-zero kernel. The orientation on C and some orientation on ∂M_i induce a transverse orientation on C. Our aim is to construct a taut decomposing surface S_i such that, away from a regular neighbourhood of $A(\gamma_i) \cup T(\gamma_i)$, ∂S_i agrees with C. Hence, we now check that for any point p on $R_-(\gamma_i)$, we may find a path as in the lemma.

Construct a graph in ∂M_i, with a single vertex in each component of $\partial M_i - C$ and with an edge for each component of C intersecting that component transversely and missing all other components of C. Orient the edges according to the transverse orientation of C. Note that each vertex of the graph has valence more than one. Otherwise, there would be a separating component of C, which would contradict the minimality assumption on $|C|$. Note also that no vertex of the graph can have more than one edge entering it, or more than edge leaving it. For, in this case, these edges correspond to distinct components of C, which we may join by an arc in $\partial M_i - C$, and using this arc, we may reduce the number of components of C. This contradicts the minimality assumption again. Hence, each vertex of the graph has precisely one edge entering and one edge leaving. Therefore, the graph is a disjoint union of circles. Let F be the component of ∂M_i containing p. The component of $F - C$ containing p corresponds to a vertex v_1 in the graph. Some component of $F - C$ must intersect γ_i. This corresponds to a vertex v_2 of the graph. There are paths in the graph from v_1 to v_2 that are compatible and incompatible with the orientation on the graph. Truncate these paths at their first intersection points with $A(\gamma_i)$. One of these paths is the path as required.

Transversely orient the curves γ_i so that they point towards $R_+(\gamma_i)$. Let C' be the double-curve sum of C with a sufficient number of parallel copies of γ_i, so that, after a small ambient isotopy, C' intersects any component of $A(\gamma_i)$ in a collection of coherently oriented transverse arcs. Apply Theorem 2.5 of [8] to these curves C'. This results in a surface S_i properly embedded in M_i, such that $(M_i, \gamma_i) \xrightarrow{S_i} (M_{i+1}, \gamma_{i+1})$ is taut, and such that $\partial S_i - A(\gamma_i)$ and $C' - A(\gamma_i)$ are the same 1–manifolds with the same transverse orientations. After possibly capping off oppositely oriented simple closed curves of ∂S_i in $A(\gamma_i)$ and $T(\gamma_i)$, S_i becomes styled. For any point p on $R_-(\gamma_i)$, we may find an arc running from p to $A(\gamma_i)$ intersecting ∂S_i correctly. The particular choice of transverse orientation on γ_i guarantees that this is still true at the intersection points between and ∂S_i coming from the parts of γ_i in C'. By restricting to some component of S_i, we may ensure that S_i is connected and non-separating, and has non-empty boundary. □

Proof of Theorem 7 Let (M, γ) and be as in Theorem 7. We consider a sutured manifold hierarchy as in Lemma 8, and prove the theorem by induction backwards along the hierarchy. The hierarchy ends with a 3–ball. We have already shown in this case that Theorem 7 holds. We now prove the inductive step. Consider a taut sutured manifold decomposition

$$(M, \gamma) \xrightarrow{S} (M_S, \gamma_S)$$

where S is a surface satisfying the requirements of Lemma 8. We assume inductively that (M_S, γ_S) satisfies the conclusion of the theorem.

Note that S has non-negative triangular number. For, otherwise, it is a disc intersecting in at most one arc (which must be glancing) or in a simple closed curve. Since S is non-separating, $R(\gamma)$ is compressible, which is a contradiction.

We now perform a sequence of ambient isotopies to S so that, afterwards

- S remains styled,
- no component of ∂S is disjoint from ,
- each arc of intersection between S and $A(\gamma)$ lies in (but closed curves of $S \cap A(\gamma)$ need not lie wholly in), and
- (M_S, γ_S) remains unchanged, up to homeomorphism.

Note that when the above conditions hold, the manifold (M_S, γ_S) inherits the ideal region $_S = \cap (A(\gamma_S) \cup T(\gamma_S))$. Note in particular that no component of $A(\gamma_S)$ is disjoint from $_S$. Also, M_S contains no $_S$–essential annulus, since this would be a –essential annulus in M.

We may clearly perform an ambient isotopy of S, supported in a neighbourhood of $A(\gamma)$ to ensure that each arc of intersection between S and $A(\gamma)$ lies in . Suppose that some component of ∂S is disjoint from $A(\gamma) \cup T(\gamma)$. Pick a point p on this component. Let be the arc running from p to $A(\gamma)$, as in Lemma 8. We may assume that avoids all components of ∂S intersecting $A(\gamma)$, and that the endpoint of is in . Then let p' be the point of $\cap S$ closest to $A(\gamma)$. Ambient isotope the component of ∂S containing p' along so that afterwards it does intersect $A(\gamma)$ in a glancing arc. Note that the assumption in Lemma

8 that at each intersection point with ∂S, runs from the cusped side of ∂S to the uncusped side guarantees that S is styled and that $(M_S; \gamma_S)$ remains unchanged. Repeat as necessary, until S satis es the above four conditions.

Note that M_S is connected, since S is connected and non-separating. We have that

$$t(R_+(\gamma_S);\ _S) = t(R_+(\gamma);\ ) + t(S;\ ) \quad t(R_+(\gamma);\ ) \quad 2;$$

and a similar inequality holds for $t(R_-(\gamma_S);\ _S)$.

Our aim now is to alter S so that afterwards each component of $R\ (\gamma) - \mathrm{int}(N(S))$ has non-negative triangular number. A component D of $R\ (\gamma) - \mathrm{int}(N(S))$ with negative triangular number must be a disc intersecting in at most one arc or simple closed curve. It cannot be a component of $R\ (\gamma)$, since that would imply that M was a 3–ball with $t(R_-(\gamma);\ ) = t(R_+(\gamma);\ ) \quad -1$. Hence D must intersect ∂S in a single arc. The cusped side of ∂S cannot lie in D, for otherwise $R\ (\gamma_S)$ would be a disc and so $(M_S; \gamma_S)$ would be a 3–ball with $t(R_-(\gamma_S);\ _S) = t(R_+(\gamma_S);\ _S) = -1$, which is a contradiction. Ambient isotope this arc $\partial D \cap S$ across D into $A(\gamma)$ to reduce the number of components of $\partial S -\ $. It is straightforward to check that S remains styled. Hence, eventually, each component of $R\ (\gamma) - \mathrm{int}(N(S))$ has non-negative triangular number.

The components of $R\ (\gamma) - \mathrm{int}(N(S))$ with zero triangular number are of two possible types:

(i) a parallelity region between an arc C of $\partial S \cap R\ (\gamma)$ and an arc of $\gamma -\ $, or

(ii) a region that lies between two parallel arcs C_1 and C_2 of $\partial S \cap R\ (\gamma)$ and that misses $\gamma -\ $.

In case (i), note that the cusped side of C cannot lie in the parallelity region. For this would create a disc component of $R\ (\gamma_S)$ with zero triangular number. Then $(M_S; \gamma_S)$ would be a 3–ball with $t(R_-(\gamma_S);\ _S) = t(R_+(\gamma_S);\ _S) = 0$, which is a contradiction. We examine the arcs of $\partial S \cap A(\gamma)$ adjacent to C. They cannot both be transverse arcs, since S is styled. If they are the same glancing arc, then we ambient isotope C into $A(\gamma)$. Otherwise, we perform an ambient isotopy of S which removes a glancing arc of $S \cap A(\gamma)$. Note that S remains styled.

In case (ii), this parallelity region cannot contain the cusped sides of both C_1 and C_2. For, again, this would create a disc component of $R\ (\gamma_S)$ with zero triangular number. We perform a small homotopy which amalgamates these parallel arcs into one. Of course, this renders S no longer embedded. Note that the only possible obstruction to performing all these homotopies is when the type (ii) parallelity regions lie in an annular component of $R\ (\gamma)$ with its entire boundary in . However, if such an annulus existed, we could push its interior a little into the interior of M, forming a {essential annulus, which is a contradiction.

These alterations to S will not alter the homeomorphism type of $(M_S; \gamma_S)$. They may alter $_S$, but both $t(R_-(\gamma_S); \ _S)$ and $t(R_+(\gamma_S); \ _S)$ remain at least two. Inductively, therefore, $(M_S; \gamma_S)$ has a taut triangulation with ideal region $_S$. We now use this to construct a taut triangulation of $(M; \gamma)$.

Consider first the case where S has positive triangular number. Then, $S -$ has an ideal triangulation. We extend this to an ideal triangulation T of $R_-(\gamma_S)$ and $R_+(\gamma_S)$, which is possible since each component of $R\ (\gamma) - \mathrm{int}(N(S))$ has positive triangular number. However, $R_-(\gamma_S)$ and $R_+(\gamma_S)$ already come equipped with an ideal triangulation, inherited from the taut triangulation of $(M_S; \gamma_S)$. Using elementary moves, we may alter this to T. Then glue the two copies of S in $R_-(\gamma_S)$ and $R_+(\gamma_S)$. We claim that the result is a taut triangulation of $(M; \gamma)$. By construction, we have guaranteed that each ideal 3{simplex has two inward-pointing faces and two outward-pointing faces. Also, the transverse orientations on the ideal 2{simplices in $R\ (\gamma)$ are as they should be. We now check that the orientations of the faces around edges of M are correct.

Consider an edge e of M. If e is in the interior of M, then there are two possibilities: either it comes from a single edge in the interior of M_S, in which case the faces around it are already correctly oriented; or e arises by identifying two edges in $@M_S - \gamma_S$, one in $R_-(\gamma_S)$ and one in $R_+(\gamma_S)$, and so in this case also, all but precisely two pairs of adjacent faces around e have cusped intersection. If e is an edge in $@M - \gamma$, then it came from an edge in $R\ (\gamma_S)$ and possibly several edges of $\gamma_S - \ _S$. Again, the faces around e are correctly oriented. Finally, if e is in $\gamma -$, then it is formed from one or more edges of $\gamma_S - \ _S$, and the faces around e in this case all have cusped intersection. This verifies that this is a taut triangulation of $(M; \gamma)$.

The case where S has zero triangular number is similar. Since M has no { essential annuli, S is not an annulus with its boundary in . Hence, S must be a disc intersecting twice. Since γ separates $@M - T(\gamma)$ into $R_-(\gamma)$ and $R_+(\gamma)$, either both arcs of $@S \setminus$ are glancing or they are both transverse. However, in the former case, S would be boundary-parallel, which is a contradiction. Hence, both arcs are transverse and S is a product disc. There are two copies of this product disc in $R_+(M_S)$ and $R_-(M_S)$, adjacent to two arcs of $\gamma_S - \ _S$. Glue these two edges together, forming an edge e. The result is not quite a copy of (M, γ), since it is not a 3–manifold in a neighbourhood of e. Let T_1 and T_2 be the two ideal triangles in $R_-(M)$ adjacent to e. If necessary, we may perform a elementary move to $R_-(M)$ to ensure that T_1 and T_2 are distinct. Then, attach an ideal 3–simplex to T_1 and T_2, realizing an elementary move. The result is now a taut triangulation of (M, γ). □

We finish this section with a simple result. When assigning a transverse orientation to an ideal triangulation, one need not check all the conditions of the definition of tautness; instead one can apply the following proposition.

Proposition 9 *Suppose that a transverse orientation is assigned to the faces of an ideal triangulation of a 3–manifold M, with ∂M a collection of tori. Then this specifies a taut ideal triangulation, providing that:*

- *for each ideal 3–simplex, precisely two of its faces are oriented into the 3–simplex, and precisely two are oriented outwards, and*
- *around each edge, at least one pair of adjacent faces encircling the edge do not have cusped intersection.*

Proof The ideal triangulation of M induces a triangulation of ∂M. The transverse orientation on the 2–simplices of M induces a transverse orientation on the 1–simplices of ∂M. This specifies interior angles of either 0 or π at each corner of each triangle of ∂M. Let V, E and F be the number of vertices, edges and faces of ∂M. Let N be the number of π interior angles. The first of the conditions in the proposition gives that each triangle contains precisely two zero interior angles and one π interior angle, and hence $N = F$. Note that for orientation reasons, each vertex of ∂M has an even number of π interior angles. Hence, the second of the above conditions gives that each vertex of ∂M has at least two interior angles of π, and so $N \geq 2V$. However, $0 = V - E + F = V - F/2$, and so $F = 2V$. There must therefore be precisely two interior angles of π at each vertex of ∂M, and hence this is a taut ideal triangulation. □

3 Singular surfaces and taut ideal triangulations

In this section, we will prove that the singular genus of a knot in S^3 is equal to its genus. We emphasise that no further sutured manifold theory will be required. We now recall the outline of Gabai's original proof of this result. More generally, he showed that, for any compact orientable irreducible 3–manifold M with ∂M a (possibly empty) collection of tori, the minimal Thurston complexity of embedded surfaces representing a class in $H_2(M, \partial M)$ is equal to the minimal Thurston complexity of singular surfaces representing that class. He first constructed, for each non-zero class in $H_2(M;\partial M)$, a taut transversely oriented foliation with a compact leaf S (or leaves) representing that class. This foliation defines a 2–plane field, which has an associated Euler class in $e \in H^2(M;\partial M)$.

Gabai then showed using a doubling argument that it sufficed to consider the case where $\partial M = \emptyset$. Then he considered how another closed oriented surface F mapped into M interacts with the taut foliation, where no component of F is a 2–sphere and F is *homotopically incompressible*, meaning that the only simple closed curves in F which are homotopically trivial in M are those which bound discs in F. He showed that F may be homotoped so that the non-transverse intersections between F and the foliation are of two types: saddle tangencies and circle tangencies. The evaluation of the cohomology class e on the oriented surface F is equal to the number of saddle singularities, counted with sign, the sign depending on whether the transverse orientations of F and the foliation agree or disagree at a particular saddle. If $[F;\partial F] = [S;\partial S] \in H_2(M;\partial M)$, then of course $|e([F;\partial F])| = |e([S;\partial S])|$, which is precisely $-\chi(S)$. Hence, $-\chi(S)$ is the number of saddles of F counted with sign. However, $-\chi(F)$ is the number of saddles of F counted without sign. Hence, $\chi_-(F) = -\chi(F) \geq -\chi(S) = \chi_-(S)$.

Our combinatorial substitute for the taut foliation in the above argument is the taut ideal triangulation of Theorem 2. We therefore need to consider how another (possibly non-embedded) surface F interacts with this ideal triangulation. The obvious way to analyse this is to use a version of normal surface theory. Similar considerations arose when dealing with angled ideal triangulations in [4]. We recall the main points made there.

Since we are considering surfaces with boundary, it is best to truncate each of the tetrahedra of the ideal triangulation. The boundary of each truncated tetrahedron Δ is decomposed into four triangles (which are the intersection with ∂M) and four hexagons (the truncated 2–simplices). Note that $\partial\Delta$ inherits a 1–complex, which is the union of the truncated 1–simplices and the boundary of the triangles $\Delta \cap \partial M$.

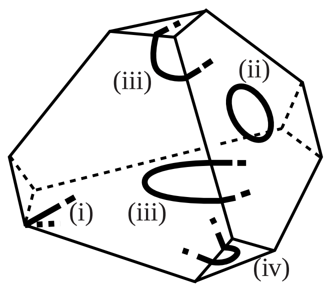
Let F be a compact orientable surface mapped into M, with ∂F sent to ∂M. Suppose that F is homotopically incompressible and also *homotopically ∂–incompressible*, meaning that no embedded essential arc in F can be homotoped in M (keeping its endpoints fixed) to an arc in ∂M. Suppose also that F contains no 2–sphere components and no discs parallel to discs in ∂M. We will see later (in Proposition 10) that M is irreducible and has incompressible boundary. Hence, by the discussion in [4], there is a homotopy of F taking it into *admissible form*, which means that it satisfies the following conditions for each truncated 3–simplex Δ:

(i) $F \cap$ is a collection of discs in intersecting ∂ in closed curves transverse to the interior of the 1{cells in ∂ ;

(ii) no curve of $F \cap \partial$ is disjoint from the 1{cells in ∂ ;

(iii) no arc of intersection between F and a hexagon H has endpoints lying in the same 1{cell of ∂ or in adjacent 1{cells of ∂ ; and

(iv) no arc of intersection between F and a triangle of $\partial M \cap$ has endpoints lying in the same 1{cell of ∂ .



Non-admissible arcs

Figure 8

When F is an admissible surface in a taut ideal triangulation, its intersection with the 2{skeleton of $M - \partial M$ forms a transversely oriented branched 1{ manifold in F. Its complimentary regions are discs. The boundary of such a disc D inherits a number of *cusps* which arise in two possible situations:

either when ∂D runs over a cusped intersection between adjacent 2{ simplices of M, or

when ∂D runs over ∂M.

If $c(D)$ is the number of cusps of D, we de ne the *combinatorial area* of D to be

$$\text{Area}(D) = \quad (c(D) - 2).$$

This concurs with the de nition of combinatorial area in [4], which was given in terms of interior angles. (Recall that the transverse orientation on the ideal triangulation speci es an interior angle of either zero or at each edge of each ideal 3{simplex.) It is clear that Area(D) is non-negative for any admissible disc D. We de ne Area(F) to be the sum of the combinatorial areas of its discs. With this de nition, the argument in Proposition 4.3 of [4] gives that

$$\text{Area}(F) = -2 \quad (F).$$

Thus, we have the following immediate corollary.

Proposition 10 *A taut ideal triangulation contains no admissible 2–spheres or discs. Hence, the underlying 3–manifold is irreducible and its boundary is a collection of incompressible tori.*

Placing F into admissible form is the analogue of Gabai's method of homotoping F so that its non-transverse intersections with a foliation are saddles and centres. The above formula for Area(F) plays the rôle of the formula for the Euler characteristic of F in terms of the number of saddles.

We now need an analogue of the Euler class of a foliation. Instead of finding a class in $H^2(M;\partial M)$, we construct a class in $H_1(M)$. Let G be the 4–valent graph which is the 1–skeleton of the spine dual to the ideal triangulation. The transverse orientation on the ideal 2–simplices determines an orientation on each edge of G. Since two edges point into each vertex of G and two edges point out, this forms a 1–cycle $[G] \in H_1(M)$. When S is a surface as in Theorem 3 that is carried by the underlying branched surface of the taut ideal triangulation, then G intersects each ideal 2–simplex of $S - \partial S$ precisely once, and these intersection points all have the same sign. Hence, we have the following formula for the intersection at the level of homology:

$$|[G] \cdot [S,\partial S]| = t(S,\partial S) = -2\chi(S).$$

The above formula and the following proposition will complete the proof of Theorem 3.

Proposition 11 *Let F be a compact orientable surface with no sphere or disc components. Then for any map $(F,\partial F) \to (M,\partial M)$, we have*

$$|[G] \cdot [F,\partial F]| \leq -2\chi(F).$$

We may reduce to the case where F is connected, and where $[F,\partial F] \neq 0 \in H_2(M;\partial M)$. If we homotopically compress and homotopically ∂–compress F, this does not change its class in $H_2(M;\partial M)$ and it increases its Euler characteristic. Also, it does not create any spheres or discs, since M is irreducible and has incompressible boundary, and $[F,\partial F] \neq 0 \in H_2(M;\partial M)$. Hence, we may assume that F is homotopically incompressible and homotopically ∂–incompressible. We may therefore homotope it into admissible form. We may also homotope F so that each point of intersection with each 1–cell of $\partial\Delta$ is at the midpoint of the 1–cell, and each arc of intersection between F and a 2–cell of $\partial\Delta$ runs linearly between these points. Hence, each truncated hexagonal 2–simplex intersects F only in one of nine possible curves (shown in Figure 10), and F intersects each triangle in ∂M in at most three possible curves. Note that, for each hexagon H, three of the possible curves of $F \cap H$ run straight through $G \cap H$. This causes a few minor technical problems. If D is a disc of $F \cap \Delta$ with ∂D avoiding G, then there is a well-defined signed intersection number $G \cdot D$ between G and D which is invariant under homotopies of D in Δ that keep ∂D fixed. If ∂D hits G in a number of points, we may perturb ∂D at each of these points so as to miss G, in one of two possible ways. After this perturbation, there is then a well-defined intersection number with G. We define $G \cdot D$ to be the average of these signed intersection numbers over all such perturbations and all points of $D \cap G$. It follows fairly rapidly from this definition that $[G] \cdot [F, \partial F]$ is simply the sum of $G \cdot D$ over all discs D of $F \cap \Delta$ and all truncated 3–simplices Δ of M. Hence, Proposition 11 follows from the following result.

Proposition 12 *For any admissible disc D in Δ,* Area$(D) \geq \pi |G \cdot D|$.

For this implies that

$$|[G] \cdot [F, \partial F]| = \left|\sum_D G \cdot D\right| \leq \sum_D |G \cdot D|$$
$$\leq \sum_D \mathrm{Area}(D)/\pi = \mathrm{Area}(F)/\pi = -2\chi(F).$$

Proof of Proposition 12 It is possible to compute $G \cdot D$ in terms of the arcs of intersection between ∂D and the hexagons of $\partial \Delta$. Label the four edges of Δ having zero interior angle with $e_1, \dots, e_4$. For $i = 1$ to 4, let α_i be the arc in $\partial \Delta$ running from a point of $G \cap \partial \Delta$ linearly to the midpoint of e_i and then continuing linearly on to another point of $G \cap \partial \Delta$. Let G_i be the arc in G running between the endpoints of α_i. Orient G_i according to the transverse orientation of the 2–simplices of Δ.

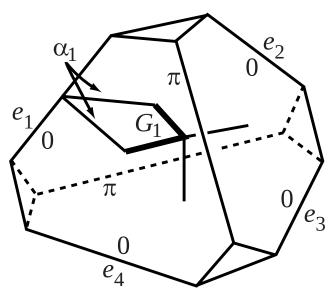


Figure 9

Note that

$$\sum_{i=1} [G_i, \partial G_i] = 2[G, G \cap \partial\ ] \in H_1(\ , G \cap \partial\ );$$

and so

$$\sum_{i=1} G_i \ D = 2(G \ D).$$

But $G_i \ D$ is (providing ∂D misses $G_i \cap \partial$) simply the winding number of ∂D around the annulus $\partial \ - G_i$, which is the signed intersection number between ∂D and $_i$. Hence, we can calculate $G \ D$ by computing, for each arc A of intersection between a hexagon H of ∂ and ∂D, the signed intersection number between A and $_i \cap H$ (weighted by 1/2 if A and $_i \cap H$ intersect at an endpoint of $_i \cap H$), and then summing these contributions over all the i, all the arcs A and all the hexagons H, and then dividing by two (since $\sum_i G_i \ D = 2G \ D$). Figure 10 shows the nine possibilities for A in each hexagon and the contribution that each makes to $G \ D$:

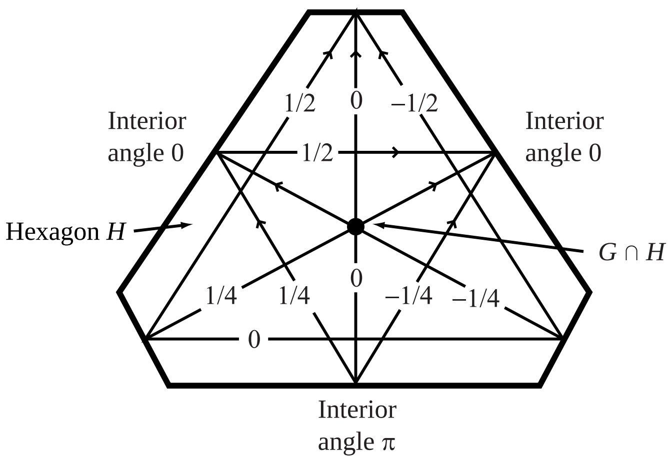


Figure 10

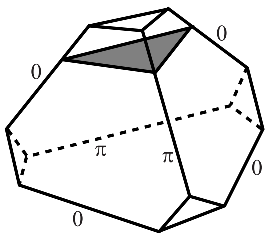
Note that the modulus of the contribution of each arc A to $G \ D$ is at most one quarter the number of cusps at the endpoints of A. Hence,

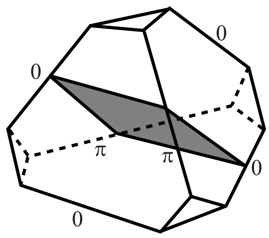
$$|G \ D| \ c(D)/2.$$

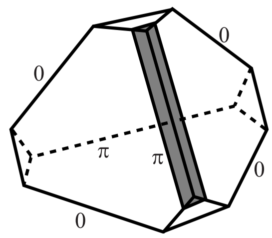
When $c(D) \ 4$, this proves the proposition, since then

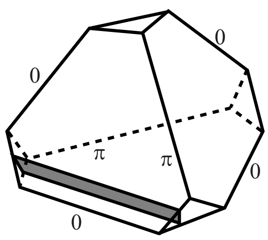
$$\text{Area}(D) = \ (c(D) - 2) \ c(D)/2 \ |G \ D|.$$

There are only a nitely many admissible discs D with $c(D) < 4$. These are shown in Figure 11 (up to obvious symmetries of) and are easily checked to satisfy the proposition. ◻


Area(D) = π |G·D| = 0


Area(D) = π |G·D| = 0


Area(D) = π |G·D| = 0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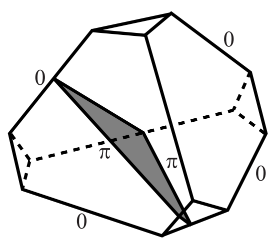
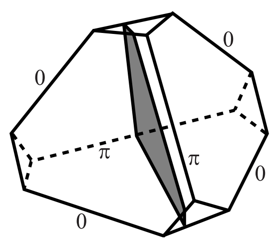
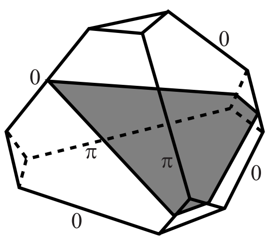
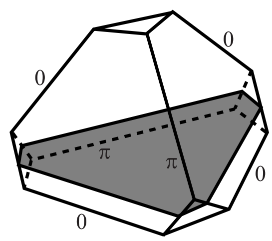
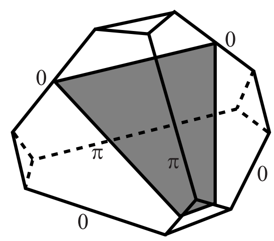

Area(D) = π |G·D| = 0


Area(D) = π |G·D| = 0


Area(D) = π |G·D| = π


Area(D) = π |G·D| = π


Area(D) = π |G·D| = π

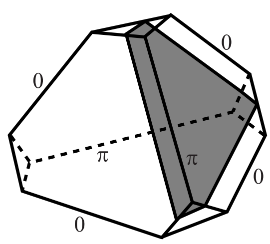

Area(D) = π |G·D| = π

Figure 11

4 Further questions

Although this paper presents simplified proofs of several results that had previously been proved using foliation theory, it is in no way meant as a replacement for that theory. One of the principal limitations of taut ideal triangulations is that they do not occur in closed 3–manifolds, whereas taut foliations may of course arise. The first of the following questions addresses this issue. The remaining questions relate to other possible applications of taut ideal triangulations.

(1) Is there a version of taut ideal triangulations for closed 3–manifolds? One candidate is the structures on triangulations defined by Calegari in [1].

(2) Let T be a taut ideal triangulation of a 3–manifold M, with ∂M a single torus. We say that a slope s on ∂M is *carried* by T if there is a lamination fully carried by the underlying branched surface of T which intersects ∂M only in simple closed curves of slope s. Which slopes are carried by T? Is the set of slopes open? Certainly, if a slope is carried by T, then the manifold obtained by Dehn filling M along that slope has a taut transversely oriented foliation transverse to the surgery curve. This yields topological restraints on the possible slopes carried by any taut ideal triangulation. For example, the meridian of a knot exterior in S^3 is never carried by a taut ideal triangulation.

(3) Let M be a compact orientable irreducible atoroidal 3–manifold with ∂M a non-empty union of incompressible tori. Does M have a taut ideal triangulation whose angles can be perturbed to give an angled ideal triangulation, as in [4]? Certainly, atoroidality is essential here. A positive answer to this question would give a new construction of angled ideal triangulations, which might provide a useful insight into the geometrisation of M.

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