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The engulfing property for 3–manifolds

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Abstract We show that there are Haken 3–manifolds whose fundamental groups do not satisfy the engulfing property. In particular one can construct a π_1–injective immersion of a surface into a graph manifold which does not factor through any proper finite cover of the 3–manifold.

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1 Introduction

Definition A subgroup H of a group G is said to be **separable** if it is an intersection of finite index subgroups of G. It is said to be **engulfed** if it is contained in a proper subgroup of finite index in G.

Subgroup separability was first explored as a tool in low dimensional topology by Scott in [7]. He showed that if $f\colon \Sigma \longrightarrow M$ is a π_1–injective immersion of a surface in a 3–manifold and $f_*(\pi_1(\Sigma))$ is a separable subgroup of $\pi_1(M)$ then the immersion factors (up to homotopy) through an embedding in a finite cover of M. This technique has applications to the still open “virtual Haken conjecture” and the “positive virtual first Betti number conjecture”.

The virtual Haken conjecture *If M is a compact, irreducible 3–manifold with infinite fundamental group then M is virtually Haken, that is it has a finite cover which contains an embedded, 2–sided, incompressible surface.*

The positive virtual first Betti number conjecture *If M is a compact, irreducible 3–manifold with infinite fundamental group then it has a finite cover with positive first Betti number.*

Unfortunately it is difficult in general to show that a given subgroup is separable, and it is known that not every subgroup of a 3–manifold group need be separable; the first example was given by Burns, Karrass and Solitar, [1]. On the other hand Shalen has shown that if an aspherical 3–manifold admits a π_1–injective immersion of a surface which factors through infinitely many finite covers then the 3–manifold is virtually Haken [2]. In group theoretic terms Shalen's condition says that the surface subgroup is contained in infinitely many finite index subgroups of the fundamental group of the 3–manifold, and this is clearly a weaker requirement than separability.

The engulfing property is apparently weaker still. It was introduced by Long in [3] to study hyperbolic 3–manifolds, and he was able to show that in some circumstances it implies separability. He remarks that "One of the difficulties with the LERF (separability) property is that there often appears to be nowhere to start, that is, it is conceivable that a finitely generated proper subgroup could be contained in no proper subgroups of finite index at all." In this note we show that this can happen for finitely generated subgroups of the fundamental group of a Haken (though not hyperbolic) 3–manifold. We give two examples, both already known not to be subgroup separable. One is derived from the recent work of Rubinstein and Wang, [6], and we consider it in Theorem 1. The other was the first known example of a 3–manifold group which failed to be subgroup separable and was introduced in [1] and further studied in [4] and [5]. Our proof that it fails to satisfy the engulfing property is more elementary than the original proof that it fails subgroup separability, and we hope that it sheds some light on this fact. Both of the examples are graph manifolds so they leave open the question of whether or not hyperbolic 3–manifold groups are subgroup separable or satisfy the engulfing property. In this connection we note that if every surface subgroup of any closed hyperbolic 3–manifold does satisfy the engulfing property then any such subgroup must be contained in infinitely many finite index subgroups, and Shalen's theorem would give a solution to the "virtual Haken conjecture" for closed hyperbolic 3–manifolds containing surface subgroups.

2 The example of Rubinstein and Wang

We will use the following lemma:

Lemma 1 *Let H be a separable subgroup of a group G. Then the index $[G : H]$ is finite if and only if there is a finite subset $F \subset G$ such that $G = HFH$.*

Proof If $[G : H]$ is finite then $G = FH$ for some finite subset $F \subset G$, so $G = HFH$ as required.

Now suppose that $G = HFH$ for some finite subset $F \subset G$. For each element $g \in F - (H \cap F)$, we can find a finite index subgroup $H_g \subset G$ with $H < H_g$ but $g \notin H_g$. Now let $K = \bigcap_g H_g$. Since F is finite, K has finite index in G, and since $H < K$, K contains every double coset HgH which it intersects non-trivially. It follows that K only intersects a double coset HgH non-trivially if $g \in H$, and so $K = H$. $\square$

Given a subgroup $H < G$ let $\overline{H}$ denote the intersection of the finite index subgroups of G which contain H. ($\overline{H}$ is the closure of H in the profinite topology on G). It is obvious that H is separable if and only if $H = \overline{H}$, and it is engulfed if and only if $G \neq \overline{H}$. If G is a finite union of double cosets of a subgroup H then it is also a finite union of double cosets of $\overline{H}$ and this is clearly a separable subgroup of G so by Lemma 1 it must have finite index. Now if H has infinite index in G and $\overline{H}$ has finite index in G they cannot be equal, and H is not separable. Hence we may interpret a finite double coset decomposition $G = HFH$ as an obstruction to separability for an infinite index subgroup $H < G$.

In [6] Rubinstein and Wang constructed a graph manifold M and a π_1-injective immersion $f: \Sigma \looparrowright M$ of a surface Σ which does not factor through an embedding into any finite cover of M. It follows from [7] that the surface group $H = f_*(\pi_1(\Sigma))$ is not separable in the 3-manifold group $G = \pi_1(M)$. In fact as we shall see G has a finite double coset decomposition $G = HFH$:

Lemma 2 *Let $f: \Sigma \looparrowright M$ be a π_1-injective immersion of a surface Σ in a 3-manifold M, and let M_H be the cover of M defined by the inclusion $f_*(\pi_1(\Sigma)) \hookrightarrow \pi_1(M)$. Let $\tilde{f}: \mathbb{R}^2 \looparrowright \tilde{M}$ be some lift of f to the universal covers, and $\tilde{\Sigma}$ denote the image of $\tilde{f}$. Then the number of H orbits for the action on $G\tilde{\Sigma} = \{g\tilde{\Sigma} \mid g \in G\}$ is precisely the number of distinct double cosets HgH.*

Proof By construction $\tilde{\Sigma}$ is H-invariant, so for each double coset HgH we have $HgH\tilde{\Sigma} = Hg\tilde{\Sigma}$. It follows that if $F = \{g_i \mid i \in I\}$ is a complete family of representatives for the distinct double cosets Hg_iH in G then the G-orbit $G\tilde{\Sigma}$ breaks into $|F|$ H-orbits as required. $\square$

Now in the example in [6] we are told in Corollary 2.5 that the image of each orbit $Hg(\tilde{\Sigma})$ intersects the image of $H\tilde{\Sigma}$ which by construction of H is compact. Hence there are only finitely many such images, and therefore only finitely many H-orbits for the action of H on the set $G\tilde{\Sigma}$. Hence $G = HFH$ for some finite subset $F \subset G$.

Corollary *The profinite closure of H must have finite index in G, ie there are only finitely many finite index subgroups of G containing H, or, in topological terms, there are only finitely many finite covers of the 3–manifold M to which the surface Σ lifts by degree 1.*

Now as in the proof of Lemma 1, let K denote the intersection of the finite index subgroups of G containing H, and let M_K denote the finite cover of M corresponding to the finite index subgroup $K < G$. Then the immersion of Σ in M lifts to an immersion $\phi: \Sigma \looparrowright M_K$ which does not lift to any finite cover of M_K. Hence:

Theorem 1 *There is a compact 3–manifold M_K and a π_1–injective immersion $\phi: \Sigma \looparrowright M_K$ which does not factor through any proper finite cover of M_K.*

3 The example of Burns, Karrass and Solitar

In [1], Burns Karrass and Solitar gave an example of a 3–manifold group with a finitely generated subgroup which is not separable. Their example is a free by $\mathbb{Z}$ group with presentation $\langle \alpha, \beta, y \mid \alpha^y = \alpha\beta, \beta^y = \beta \rangle$. It is easy to show that their example is isomorphic to the group G with presentation $\langle a, b, t \mid [a, b], a^t = b \rangle$, and it is in this form that we shall work with G. Note that here and below we use the notation $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{1}xy$.

In this section we show that G has a proper subgroup $K \subset G$ such that K is not engulfed. In particular, this yields an easier proof that G has non-separable subgroups.

Lemma 3 *Let $J = \langle abb, t \rangle$. Let H be a finite index subgroup of G containing J. Then $G = H\langle a \rangle$.*

Proof We express the argument in terms of covering spaces. Let X denote the standard based 2–complex for the presentation of G. Let T denote the torus subcomplex $\langle a, b \mid [a, b] \rangle$ of X. The complex X is formed from T by the addition of a cylinder C whose top and bottom boundary components are attached to the loops a and b respectively, and C is subdivided by a single edge labeled t which is oriented from the a loop to the b loop.

Let $\hat{X}$ denote the finite based cover of X corresponding to the subgroup H. Let $\hat{T}$ denote the cover of T at the basepoint of $\hat{X}$. Let $\hat{a}$ and $\hat{b}$ denote the covers of the loops a and b at the basepoint.

Since t lifts to a closed path in $\hat{X}$, we see that C has a finite cover $\hat{C}$ which lifts at the basepoint to a cylinder attached at its ends to $\hat{a}$ and $\hat{b}$. Now $\hat{C}$ gives a one-to-one correspondence between 0–cells on $\hat{a}$ and 0–cells on $\hat{b}$. In particular, each t edge of $\hat{C}$ is directed from some 0–cell in $\hat{a}$ to some 0–cell in $\hat{b}$ and therefore Degree($\hat{a}$) = Degree($\hat{b}$).

Because $abb \in J \subset H$ and hence $abb \in \pi_1(\hat{T})$, we see that b generates the covering group of the regular cover $\hat{T} \longrightarrow T$, and therefore Degree($\hat{b}$) = Degree($\hat{T}$). Thus we have Degree($\hat{T}$) = Degree($\hat{b}$) = Degree($\hat{a}$), and because Degree($\hat{T}$) is finite, we see that every 0–cell of $\hat{T}$ lies in both $\hat{a}$ and $\hat{b}$.

As above, each 0–cell of $\hat{a}$ has an outgoing t edge in $\hat{C}$ and each 0–cell of $\hat{b}$ has an incoming t edge in $\hat{C}$, and so we see that each 0–cell of $\hat{T} \cup \hat{C}$ has an incoming and outgoing t edge. Since 0–cells of $\hat{T} \cup \hat{C}$ obviously have incoming and outgoing a and b edges in $\hat{T}$, we see that $\hat{X} = \hat{T} \cup \hat{C}$ and in particular, every 0–cell of $\hat{X}$ is contained in $\hat{T}$ and therefore in $\hat{a}$. Thus $\langle a \rangle$ contains a set of right coset representatives for H in G, and consequently $G = H\langle a \rangle$. □

Lemma 4 *Let $K = \langle J \cup a^g \rangle$ for some $g \in G$. Then K is not engulfed.*

Proof Let H be a subgroup of finite index containing K. Since $J \subset H$ we may apply Lemma 3 to conclude that $G = H\langle a \rangle$ and so it is sufficient to show that $a \in H$. Observe that $g^{-1} = ha^n$ for some $h \in H$ and $n \in \mathbb{Z}$. But $a^g = (ha^n)aa^{-n}h^{-1} = hah^{-1}$, and obviously $hah^{-1} \in H$ implies that $a \in H$. □

Theorem 2 *Let K be the subgroup $\langle abb, t, btat^{-1}b^{-1} \rangle$. Then the engulfing property fails for K, that is, $K \neq G$ and the only subgroup of finite index containing K is G.*

Proof Lemma 4 with $g = t^{-1}b^{-1}$ shows that K is not engulfed. To see that $K \neq G$ we observe that the normal form theorem for an HNN extension shows that there is no non-trivial cancellation between the generators of K so it is a rank 3 free group, but G is not free. □

Remark It is not difficult to see that there are many finitely generated subgroups J for which some version of Lemma 3 is true. In addition, one has some freedom to vary the choice of g in theorem 2. Consequently subgroups of G which are not engulfed are numerous.

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