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Nondi eomorphic Symplectic 4{Manifolds with the same Seiberg{Witten Invariants

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Abstract The goal of this paper is to demonstrate that, at least for non-simply connected 4{manifolds, the Seiberg{Witten invariant alone does not determine di eomorphism type within the same homeomorphism type.

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Dedicated to Robion C Kirby on the occasion of his 60^{th} *birthday*

1 Introduction

The goal of this paper is to demonstrate that, at least for nonsimply connected 4{manifolds, the Seiberg{Witten invariant alone does not determine di eomorphism type within the same homeomorphism type. The rst examples which demonstrate this phenomenon were constructed by Shuguang Wang [13]. These are examples of two homeomorphic 4{manifolds with $_1 = Z_2$ and trivial Seiberg{Witten invariants. One of these manifolds is irreducible and the other splits as a connected sum. It is our goal here to exhibit examples among symplectic 4{manifolds, where the Seiberg{Witten invariants are known to be nontrivial. We shall construct symplectic 4{manifolds with $_1 = Z_p$ which have the same nontrivial Seiberg{Witten invariant but whose universal covers have di erent Seiberg{Witten invariants. Thus, at the very least, in order to determine di eomorphism type, one needs to consider the Seiberg{Witten invariants of nite covers.

Recall that the Seiberg{Witten invariant of a smooth closed oriented 4{manifold X with $b_2^+(X) > 1$ is an integer-valued function which is de ned on the set of $spin^c$ structures over X (cf [14]). In case $H_1(X;Z)$ has no 2{torsion there is a natural identification of the $spin^c$ structures of X with the characteristic elements of $H_2(X;\mathbf{Z})$ (ie, those elements k whose Poincare duals $\hat{k}$ reduce mod 2 to $w_2(X)$). In this case we view the Seiberg–Witten invariant as

$$\mathrm{SW}_X : \{k \in H_2(X;\mathbf{Z}) \mid \hat{k} \equiv w_2(TX) \pmod 2\} \to \mathbf{Z}.$$

The sign of SW_X depends on an orientation of $H^0(X;\mathbf{R}) \otimes \det H^2_+(X;\mathbf{R}) \otimes \det H^1(X;\mathbf{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then β is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with $\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$ where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of X.

Now let $\{\pm\beta_1, \dots, \pm\beta_n\}$ be the set of nonzero basic classes for X. Consider variables $t_\beta = \exp(\beta)$ for each $\beta \in H^2(X;\mathbf{Z})$ which satisfy the relations $t_{\alpha+\beta} = t_\alpha t_\beta$. We may then view the Seiberg–Witten invariant of X as the Laurent polynomial

$$SW_X = \mathrm{SW}_X(0) + \sum_{j=1}^{n} \mathrm{SW}_X(\beta_j) \cdot (t_{\beta_j} + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_{\beta_j}^{-1}).$$

2 The Knot and Link Surgery Construction

We shall need the knot surgery construction of [3]: Suppose that we are given a smooth simply connected oriented 4–manifold X with $b^+ > 1$ containing an essential smoothly embedded torus T of self-intersection 0. Suppose further that $\pi_1(X \setminus T) = 1$ and that T is contained in a cusp neighborhood. Let $K \subset S^3$ be a smooth knot and M_K the 3–manifold obtained from 0–framed surgery on K. The meridional loop m to K defines a 1–dimensional homology class $[m]$ both in $S^3 \setminus K$ and in M_K. Denote by T_m the torus $S^1 \times m \subset S^1 \times M_K$. Then X_K is defined to be the fiber sum

$$X_K = X \#_{T=T_m} S^1 \times M_K = (X \setminus N(T)) \cup (S^1 \times (S^3 \setminus N(K))),$$

where $N(T) = D^2 \times T^2$ is a tubular neighborhood of T in X and $N(K)$ is a neighborhood of K in S^3. If λ denotes the longitude of K (λ bounds a surface in $S^3 \setminus K$) then the gluing of this fiber sum identifies $\{pt\} \times \lambda$ with a normal circle to T in X. The main theorem of [3] is:

Theorem [3] *With the assumptions above, X_K is homeomorphic to X, and*

$$SW_{X_K} = SW_X \cdot \Delta_K(t)$$

where Δ_K is the symmetrized Alexander polynomial of K and $t = \exp(2[T])$.

In case the knot K is fibered, the 3–manifold M_K is a surface bundle over the circle; hence $S^1 \times M_K$ is a surface bundle over T^2. It follows from [12] that $S^1 \times M_K$ admits a symplectic structure and T_m is a symplectic submanifold. Hence, if $T \subset X$ is a torus satisfying the conditions above, and if in addition X is a symplectic 4–manifold and T is a symplectic submanifold, then the fiber sum $X_K = X \#_{T=T_m} S^1 \times M_K$ carries a symplectic structure [4]. Since K is a fibered knot, its Alexander polynomial is the characteristic polynomial of its monodromy φ; in particular, $M_K = S^1 \times_\varphi \Sigma$ for some surface Σ and $\Delta_K(t) = \det(\varphi_* - tI)$, where φ_* is the induced map on H_1.

There is a generalization of the above theorem in this case due to Ionel and Parker [7] and to Lorek [8].

Theorem [7, 8] *Let X be a symplectic 4–manifold with $b^+ > 1$, and let T be a symplectic self-intersection 0 torus in X which is contained in a cusp neighborhood. Also, let Σ be a symplectic 2–manifold with a symplectomorphism $\varphi\colon \Sigma \to \Sigma$ which has a fixed point $\varphi(x_0) = x_0$. Let $m_0 = S^1 \times_\varphi \{x_0\}$ and $T_0 = S^1 \times m_0 \subset S^1 \times (S^1 \times_\varphi \Sigma)$. Then $X_\varphi = X \#_{T=T_0} S^1 \times (S^1 \times_\varphi \Sigma)$ is a symplectic manifold whose Seiberg–Witten invariant is*

$$SW_{X_\varphi} = SW_X \cdot \Delta_\varphi(t)$$

where $t = \exp(2[T])$ and $\Delta_\varphi(t)$ is the obvious symmetrization of $\det(\varphi_* - tI)$.

Note that in case K is a fibered knot and $M_K = S^1 \times_\varphi \Sigma$, Moser's theorem [9] guarantees that the monodromy map φ can be chosen to be a symplectomorphism with a fixed point.

There is a related link surgery construction which starts with an oriented n–component link $L = \{K_1, \dots, K_n\}$ in S^3 and n pairs (X_i, T_i) of smoothly embedded self-intersection 0 tori in simply connected 4–manifolds as above. Let

$$\lambda_L\colon \pi_1(S^3 \setminus L) \to \mathbf{Z}$$

denote the homomorphism characterized by the property that it send the meridian m_i of each component K_i to 1. Let $N(L)$ be a tubular neighborhood of L. Then if ℓ_i denotes the longitude of the component K_i, the curves $\gamma_i = \ell_i + \lambda_L(\ell_i) m_i$ on $\partial N(L)$ given by the $\lambda_L(\ell_i)$ framing of K_i form the boundary of a Seifert surface for the link. In $S^1 \times (S^3 \setminus N(L))$ let $T_{m_i} = S^1 \times m_i$ and define the 4–manifold $X(X_1, \dots, X_n; L)$ by

$$X(X_1, \dots, X_n; L) = (S^1 \times (S^3 \setminus N(L))) \cup \bigcup_{i=1}^{n} (X_i \setminus (T_i \times D^2))$$

where $S^1 \times \partial N(K_i)$ is identified with $\partial N(T_i)$ so that for each i:

$$[T_{m_i}] = [T_i]; \quad \text{and} \quad [\gamma_i] = [\mathrm{pt} \times \partial D^2].$$

Theorem [3] *If each T_i is homologically essential and contained in a cusp neighborhood in X_i and if each $\pi_1(X \setminus T_i) = 1$, then $X(X_1, \dots, X_n; L)$ is simply connected and its Seiberg–Witten invariant is*

$$SW_{X(X_1,\dots,X_n;L)} = \Delta_L(t_1, \dots, t_n) \cdot \prod_{j=1}^{n} SW_{E(1)\#_{F=T_j} X_j}$$

where $t_j = \exp(2[T_j])$ and $\Delta_L(t_1, \dots, t_n)$ is the symmetric multivariable Alexander polynomial.

3 2–bridge knots

Recall that 2–bridge knots, K, are classified by the double covers of S^3 branched over K, which are lens spaces. Let $K(p/q)$ denote the 2–bridge knot whose double branched cover is the lens space $L(p,q)$. Here, p is odd and q is relatively prime to p. Notice that $L(p,q) = L(p,q-p)$; so we may assume at will that either q is even or odd. We are first interested in finding a pair of distinct fibered 2–bridge knots $K(p/q_i)$, $i = 1,2$ with the same Alexander polynomial. Since 2–bridge knots are alternating, they are fibered if and only if their Alexander polynomials are monic [2]. There is a simple combinatorial scheme for calculating the Alexander polynomial of a 2–bridge knot $K(p/q)$; it is described as follows in [10]. Assume that q is even and let $\mathrm{b}(p/q) = (b_1, \dots, b_n)$ where p/q is written as a continued fraction:

$$\frac{p}{q} = 2b_1 + \cfrac{1}{-2b_2 + \cfrac{1}{2b_3 + \cfrac{1}{\ddots + \cfrac{1}{2b_n}}}}$$

There is then a Seifert surface for $K(p/q)$ whose corresponding Seifert matrix is:

$$V(p/q) = \begin{pmatrix} b_1 & 0 & 0 & 0 & 0 & \\ 1 & b_2 & 1 & 0 & 0 & \\ 0 & 0 & b_3 & 0 & 0 & \\ 0 & 0 & 1 & b_4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Thus the Alexander polynomial for $K(p/q)$ is

$$\Delta_{K(p/q)}(t) = \det(t\, V(p/q) - V(p/q)^{\mathrm{tr}}).$$

Using this technique we calculate:

Proposition 3.1 *The 2–bridge knots $K(105/64)$ and $K(105/76)$ share the Alexander polynomial*

$$\Delta(t) = t^4 - 5t^3 + 13t^2 - 21t + 25 - 21t^{-1} + 13t^{-2} - 5t^{-3} + t^{-4}.$$

In particular, these knots are fibered.

Proof The knots $K(105/64)$ and $K(105/76)$ correspond to the vectors

$$\mathrm{b}(105/64) = (1,1,-1,-1,-1,-1,1,1)$$
$$\mathrm{b}(105/76) = (1,1,1,-1,-1,1,1,1). \qquad \square$$

4 The examples

Consider any pair of inequivalent fibered 2–bridge knots $K_i = K(p/q_i)$, $i = 1,2$, with the same Alexander polynomial $\Delta(t)$. Let $\tilde{K}_i = \pi_i^{-1}(K_i)$ denote the branch knot in the 2–fold branched covering space $\pi_i\colon L(p,q_i) \to S^3$, and let $\tilde{m}_i = \pi_i^{-1}(m_i)$, with m_i the meridian of K_i. Then $M_{K_i} = S^1 \times_{\varphi_i} \Sigma$ with double cover $\tilde{M}_{K_i} = S^1 \times_{\varphi_i^2} \Sigma$.

Let X be the K3–surface and let F denote a smooth torus of self-intersection 0 which is a fiber of an elliptic fibration on X. Our examples are

$$X_{K_i} = X \#_{F=T_{\tilde{m}_i}} (S^1 \times \tilde{M}_{K_i}).$$

The gluing is chosen so that the boundary of a normal disk to F is matched with the lift $\tilde{\lambda}_i$ of a longitude to K_i. A simple calculation and our above discussion implies that X_{K_1} and X_{K_2} are homeomorphic [5] and have the same Seiberg–Witten invariant:

Theorem 4.1 *The manifolds X_{K_i} are homeomorphic symplectic rational homology K3–surfaces with fundamental groups $\pi_1(X_{K_i}) = \mathbf{Z}_p$. Their Seiberg–Witten invariants are*

$$SW_{X_{K_i}} = \det(\varphi^2_{i,*} - \tau^2 I) = \Delta(\tau)\cdot\Delta(-\tau)$$

where $\tau = \exp([F])$.

5 Their universal covers

The purpose of this final section is to prove our main theorem.

Theorem 5.1 *$X_{K(105/64)}$ and $X_{K(105/76)}$ are homeomorphic but not diffeomorphic symplectic 4–manifolds with the same Seiberg–Witten invariant.*

Let $K_1 = K(105/64)$ and $K_2 = K(105/76)$. We have already shown that X_{K_1} and X_{K_2} are homeomorphic symplectic 4–manifolds with the same Seiberg–Witten invariant. Suppose that $f\colon X_{K_1} \to X_{K_2}$ is a diffeomorphism. It then satisfies $f_*(SW_{X_{K_1}}) = SW_{X_{K_2}}$. Since these are both Laurent polynomials in the single variable $t = \exp([F])$, and $[F] = [T_{m_i}]$ in X_{K_i}, after appropriately orienting T_{m_2}, we must have

$$f_*[T_{m_1}] = [T_{m_2}].$$

We study the induced diffeomorphism $\tilde f\colon \widetilde X_{K_1} \to \widetilde X_{K_2}$ of universal covers. The universal cover $\widetilde X_{K_i}$ of X_{K_i} is obtained as follows. Let $\pi_i\colon S^3 \to L(p;q_i)$ be the universal covering ($p = 105$, $q_1 = 64$, $q_2 = 76$) which induces the universal covering $\tilde\pi_i\colon \widetilde X_{K_i} \to X_{K_i}$, and let $\widetilde L_i$ be the p–component link $\widetilde L_i = \pi_i^{-1}(\widetilde K_i)$. The composition of the maps $\varphi\circ\pi_i\colon S^3 \to S^3$ is a dihedral covering space branched over K_i, and the link $\widetilde L_i = \widetilde L(p/q_i)$ is classically known as the 'dihedral covering link' of $K(p/q_i)$. This is a symmetric link, and in fact, the deck transformations $\tau_{i,k}$ of the cover $\pi_i\colon S^3 \to L(p;q_i)$ permute the link components. The collection of linking numbers of $\widetilde L_i$ (the dihedral linking numbers of $K(p/q_i)$) classify the 2–bridge knots [2]. The universal cover $\widetilde X_{K_i}$ is obtained via the construction $\widetilde X_{K_i} = X(X_1,\dots,X_p;L_i)$ of section 2, where each $(X_i,T_i) = (K3,F)$. Hence it follows from section 2 that

$$SW_{\widetilde X_{K_i}} = \Delta_{\widetilde L_i}(t_{i,1},\dots,t_{i,p})\prod_{j=1}^{p} SW_{E(1)\#_F K3} = \Delta_{\widetilde L_i}(t_{i,1},\dots,t_{i,p})\prod_{j=1}^{p}(t_{i,j}^{1/2} - t_{i,j}^{-1/2})$$

where $t_{i,j} = \exp([2T_{i,j}])$ and $T_{i,j}$ is the fiber F in the jth copy of K3. Let $L_{i,1},\dots,L_{i,p}$ denote the components of the covering link $\widetilde L_i$ in S^3, and let $m_{i,j}$ denote a meridian to $L_{i,j}$. Then $[T_{i,j}] = [S^1\times m_{i,j}]$ in $H_2(\widetilde X_{K_i};\mathbf{Z})$, and so $\tilde\pi_{i*}[T_{i,j}] = [T_i]$.

Now we have $\tilde f_*(SW_{\widetilde X_{K_1}}) = SW_{\widetilde X_{K_2}}$ as elements of the integral group ring of $H_2(\widetilde X_{K_2};\mathbf{Z})$. The formula given for $SW_{\widetilde X_{K_i}}$ shows that each basic class may be written in the form $\alpha = \sum_{j=1}^{p} a_j [T_{i,j}]$. Thus if α is a basic class of X_{K_1}, then

$$\hat{f}_*(\alpha) = \hat{f}_*\Big(\sum_{j=1}^{p} a_j [T_{1,j}]\Big) = \sum_{j=1}^{p} b_j [T_{2,j}]$$

for some integers, $b_1, \dots, b_p$. But since $f_*[T_1] = [T_2]$ in $H_2(X_{K_2}; \mathbf{Z})$ we have

$$\Big(\sum_{j=1}^{p} a_j\Big)[T_2] = f_*\Big(\sum_{j=1}^{p} a_j [T_1]\Big) = f_* \#_{1*}(\alpha) = \#_{2*} \hat{f}_*(\alpha) =$$
$$\#_{2*}\Big(\sum_{j=1}^{p} b_j [T_{2,j}]\Big) = \sum_{j=1}^{p} b_j [T_2].$$

Hence $\sum_{j=1}^{p} a_j = \sum_{j=1}^{p} b_j$.

Form the 1–variable Laurent polynomials $P_i(t) = \Delta_{\hat{L}_i}(t, \dots, t) \cdot (t^{1/2} - t^{-1/2})^p$ by equating all the variables $t_{i,j}$ in $SW_{X_{K_i}}$. The coefficient of a fixed term t^k in $P_i(t)$ is

$$\sum \Big\{ \mathrm{SW}_{X_{K_i}}\Big(\sum_{j=1}^{p} a_j [T_{i,j}]\Big) \;\Big|\; \sum_{j=1}^{p} a_j = k \Big\}.$$

Our argument above (and the invariance of the Seiberg–Witten invariant under diffeomorphisms) shows that $\hat{f}_*$ takes $P_1(t)$ to $P_2(t)$; ie, $P_1(t) = P_2(t)$ as Laurent polynomials.

The reduced Alexander polynomials $\Delta_{\hat{L}_i}(t, \dots, t)$ have the form

$$\Delta_{\hat{L}_i}(t, \dots, t) = (t^{1/2} - t^{-1/2})^{p-2} \cdot \nabla_{\hat{L}_i}(t),$$

where the polynomial $\nabla_{\hat{L}_i}(t)$ is called the Hosokawa polynomial [6]. Consider the matrix:

$$\Lambda(p/q) = \begin{pmatrix} & \epsilon_1 & \cdots & \epsilon_{p-1} \\ \epsilon_{p-1} & & \cdots & \epsilon_{p-2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \epsilon_1 & \epsilon_2 & \cdots & \end{pmatrix}$$

(Burde has shown that this is the linking matrix of $\hat{L}(p/q)$.)

It is a theorem of Hosokawa [6] that $\nabla_{\hat{L}(p/q)}(1)$ can be calculated as the determinant of any $(p-1)$ by $(p-1)$ minor $\Lambda'(p/q)$ of $\Lambda(p/q)$. In particular, we have the following Mathematica calculations. (Note that $K(105/64) = K(105/-41)$ and $K(105/76) = K(105/-29)$.)

$$\det(\Delta'(105/-41))/105 = 13^2 \cdot 61^2 \cdot 127^2 \cdot 463^2 \cdot 631^4 \cdot 1358281^4$$
$$\det(\Delta'(105/-29))/105 = 139^4 \cdot 211^4 \cdot 491^2 \cdot 8761^2 \cdot 10005451^4.$$

This means that $\nabla_{\hat{L}_1}(1) \neq \nabla_{\hat{L}_2}(1)$. However, if we let $Q(t) = (t^{1/2} - t^{-1/2})^{2p-2}$, then $P_i(t) = \nabla_{\hat{L}_i}(t) \cdot Q(t)$. For $|u-1|$ small enough, $P_1(u)/Q(u) \neq P_2(u)/Q(u)$. Hence for $u \neq 1$ in this range, $P_1(u) \neq P_2(u)$. This contradicts the existence of the diffeomorphism f and completes the proof of Theorem 5.1.

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