*ISSN 1464-8997 (on line) 1464-8989 (printed)* 

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 1, pages 5–18

# 1. Higher dimensional local fields

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We give here basic definitions related to $n$-dimensional local fields. For detailed exposition, see [P] in the equal characteristic case, [K1, §8] for the two-dimensional case and [MZ1], [MZ2] for the general case. Several properties of the topology on the multiplicative group are discussed in [F].

## 1.1. Main definitions

Suppose that we are given a surface $S$ over a finite field of characteristic $p$, a curve $C \subset S$, and a point $x \in C$ such that both $S$ and $C$ are regular at $x$. Then one can attach to these data the quotient field of the completion $\widehat{(\widehat{\mathcal{O}}_{S,x})_C}$ of the localization at $C$ of the completion $\widehat{\mathcal{O}}_{S,x}$ of the local ring $\mathcal{O}_{S,x}$ of $S$ at $x$. This is a two-dimensional local field over a finite field, i.e., a complete discrete valuation field with local residue field. More generally, an $n$-dimensional local field $F$ is a complete discrete valuation field with $(n-1)$-dimensional residue field. (Finite fields are considered as 0-dimensional local fields.)

**Definition.** A complete discrete valuation field $K$ is said to have the structure of an $n$-dimensional local field if there is a chain of fields $K = K_n, K_{n-1}, \dots, K_1, K_0$ where $K_{i+1}$ is a complete discrete valuation field with residue field $K_i$ and $K_0$ is a finite field. The field $k_K = K_{n-1}$ (resp. $K_0$) is said to be the *first* (resp. the *last*) residue field of $K$.

**Remark.** Most of the properties of $n$-dimensional local fields do not change if one requires that the last residue $K_0$ is *perfect* rather than finite. To specify the exact meaning of the word, $K$ can be referred to as an $n$-dimensional local field *over* a finite (resp. perfect) field. One can consider an $n$-dimensional local field over an arbitrary field $K_0$ as well. However, in this volume mostly the higher local fields over finite fields are considered.

**Examples.** 1. $\mathbb{F}_q((X_1))\dots((X_n))$. 2. $k((X_1))\dots((X_{n-1}))$, $k$ a finite extension of $\mathbb{Q}_p$.

3. For a complete discrete valuation field $F$ let

$$K = F\{\{T\}\} = \left\{\sum_{-\infty}^{+\infty} a_i T^i : a_i \in F,\ \inf v_F(a_i) > -\infty,\ \lim_{i\to-\infty} v_F(a_i) = +\infty\right\}.$$

Define $v_K(\sum a_i T^i) = \min v_F(a_i)$. Then $K$ is a complete discrete valuation field with residue field $k_F((t))$.

Hence for a local field $k$ the fields

$$k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n)),\quad 0 \leqslant m \leqslant n-1$$

are $n$-dimensional local fields (they are called *standard fields*).

**Remark.** $K((X))\{\{Y\}\}$ is isomorphic to $K((Y))((X))$.

**Definition.** An $n$-tuple of elements $t_1,\dots,t_n \in K$ is called a *system of local parameters* of $K$, if $t_n$ is a prime element of $K_n$, $t_{n-1}$ is a unit in $\mathcal{O}_K$ but its residue in $K_{n-1}$ is a prime element of $K_{n-1}$, and so on.

For example, for $K = k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n))$, a convenient system of local parameter is $T_1,\dots,T_m,\pi,T_{m+2},\dots,T_n$, where $\pi$ is a prime element of $k$.

Consider the maximal $m$ such that $\operatorname{char}(K_m) = p$; we have $0 \leqslant m \leqslant n$. Thus, there are $n+1$ types of $n$-dimensional local fields: fields of characteristic $p$ and fields with $\operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$, $0 \leqslant m \leqslant n-1$. Thus, the mixed characteristic case is the case $m = n-1$.

Suppose that $\operatorname{char}(k_K) = p$, i.e., the above $m$ equals either $n-1$ or $n$. Then the set of Teichmüller representatives $\mathcal{R}$ in $\mathcal{O}_K$ is a field isomorphic to $K_0$.

**Classification Theorem.** *Let $K$ be an $n$-dimensional local field. Then*

(1) $K$ *is isomorphic to* $\mathbb{F}_q((X_1))\dots((X_n))$ *if* $\operatorname{char}(K) = p$*;*

(2) $K$ *is isomorphic to* $k((X_1))\dots((X_{n-1}))$*,* $k$ *is a local field, if* $\operatorname{char}(K_1) = 0$*;*

(3) $K$ *is a finite extension of a standard field* $k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n))$ *and there is a finite extension of* $K$ *which is a standard field if* $\operatorname{char}(K_{m+1}) = 0$*,* $\operatorname{char}(K_m) = p$*.*

*Proof.* In the equal characteristic case the statements follow from the well known classification theorem for complete discrete valuation fields of equal characteristic. In the mixed characteristic case let $k_0$ be the fraction field of $W(\mathbb{F}_q)$ and let $T_1,\dots,T_{n-1},\pi$ be a system of local parameters of $K$. Put

$$K' = k_0\{\{T_1\}\}\dots\{\{T_{n-1}\}\}.$$

Then $K'$ is an absolutely unramified complete discrete valuation field, and the (first) residue fields of $K'$ and $K$ coincide. Therefore, $K$ can be viewed as a finite extension of $K'$ by [FV, II.5.6].

Alternatively, let $t_1, \dots, t_{n-1}$ be any liftings of a system of local parameters of $k_K$. Using the canonical lifting $h_{t_1,\dots,t_{n-1}}$ defined below, one can construct an embedding $K' \hookrightarrow K$ which identifies $T_i$ with $t_i$.

To prove the last assertion of the theorem, one can use *Epp's theorem on elimination of wild ramification* (see 17.1) which asserts that there is a finite extension $l/k_0$ such that $e\left(lK/lK'\right) = 1$. Then $lK'$ is standard and $lK$ is standard, so $K$ is a subfield of $lK$. See [Z] or [KZ] for details and a stronger statement. $\square$

**Definition.** The *lexicographic order* of $\mathbb{Z}^n$: $\mathbf{i} = (i_1, \dots, i_n) \leqslant \mathbf{j} = (j_1, \dots, j_n)$ if and only if

$$i_l \leqslant j_l,\ i_{l+1} = j_{l+1}, \dots, i_n = j_n \text{ for some } l \leqslant n.$$

Introduce $\mathbf{v} = (v_1, \dots, v_n)\colon K^* \to \mathbb{Z}^n$ as $v_n = v_{K_n}$, $v_{n-1}(\alpha) = v_{K_{n-1}}(\alpha_{n-1})$ where $\alpha_{n-1}$ is the residue of $\alpha t_n^{-v_n(\alpha)}$ in $K_{n-1}$, and so on. The map $\mathbf{v}$ is a valuation; this is a so called discrete valuation of rank $n$. Observe that for $n > 1$ the valuation $\mathbf{v}$ does depend on the choice of $t_2, \dots, t_n$. However, all the valuations obtained this way are in the same class of equivalent valuations.

Now we define several objects which do not depend on the choice of a system of local parameters.

**Definition.**

$O_K = \{\alpha \in K : \mathbf{v}(\alpha) \geqslant 0\}$, $M_K = \{\alpha \in K : \mathbf{v}(\alpha) > 0\}$, so $O_K/M_K \simeq K_0$. The group of *principal units* of $K$ with respect to the valuation $\mathbf{v}$ is $V_K = 1 + M_K$.

**Definition.**

$$P(i_l, \dots, i_n) = P_K(i_l, \dots, i_n) = \{\alpha \in K : (v_l(\alpha), \dots, v_n(\alpha)) \geqslant (i_l, \dots, i_n)\}.$$

In particular, $O_K = P\big(\underbrace{0, \dots, 0}_{n}\big)$, $M_K = P\big(1, \underbrace{0, \dots, 0}_{n-1}\big)$, whereas $\mathcal{O}_K = P(0)$, $\mathcal{M}_K = P(1)$. Note that if $n > 1$, then

$$\cap_i M_K^i = P\big(1, \underbrace{0, \dots, 0}_{n-2}\big),$$

since $t_2 = t_1^{i-1}(t_2/t_1^{i-1})$.

**Lemma.** *The set of all non-zero ideals of $O_K$ consists of all*

$$\{P(i_l, \dots, i_n) : (i_l, \dots, i_n) \geqslant (0, \dots, 0),\quad 1 \leqslant l \leqslant n\}.$$

*The ring $O_K$ is not Noetherian for $n > 1$.*

*Proof.* Let $J$ be a non-zero ideal of $O_K$. Put $i_n = \min\{v_n(\alpha) : \alpha \in J\}$. If $J = P(i_n)$, then we are done. Otherwise, it is clear that

$$i_{n-1} := \inf\{v_{n-1}(\alpha) : \alpha \in J, v_n(\alpha) = i_n\} > -\infty.$$

If $i_n = 0$, then obviously $i_{n-1} \geqslant 0$. Continuing this way, we construct $(i_l, \dots, i_n) \geqslant (0, \dots, 0)$, where either $l = 1$ or

$$i_{l-1} = \inf\{v_{l-1}(\alpha) : \alpha \in J, v_n(\alpha) = i_n, \dots, v_l(\alpha) = i_l\} = -\infty.$$

In both cases it is clear that $J = P(i_l, \dots, i_n)$.

The second statement is immediate from $P(0,1) \subset P(-1,1) \subset P(-2,1) \dots$. □

For more on ideals in $O_K$ see subsection 3.0 of Part II.

## 1.2. Extensions

Let $L/K$ be a finite extension. If $K$ is an $n$-dimensional local field, then so is $L$.

**Definition.** Let $t_1, \dots, t_n$ be a system of local parameters of $K$ and let $t_1', \dots, t_n'$ be a system of local parameters of $L$. Let $\mathbf{v}, \mathbf{v}'$ be the corresponding valuations. Put

$$E(L|K) := \left(v_j'(t_i)\right)_{i,j} = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ \dots & e_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & e_n \end{pmatrix},$$

where $e_i = e_i(L|K) = e(L_i|K_i)$, $i = 1, \dots, n$. Then $e_i$ do not depend on the choice of parameters, and $|L : K| = f(L|K)\prod_{i=1}^n e_i(L|K)$, where $f(L|K) = |L_0 : K_0|$ .

The expression "unramified extension" can be used for extensions $L/K$ with $e_n(L|K) = 1$ and $L_{n-1}/K_{n-1}$ separable. It can be also used in a narrower sense, namely, for extensions $L/K$ with $\prod_{i=1}^n e_i(L|K) = 1$. To avoid ambiguity, sometimes one speaks of a "semiramified extension" in the former case and a "purely unramified extension" in the latter case.

## 1.3. Topology on $K$

Consider an example of $n$-dimensional local field

$$K = k\,\{\{T_1\}\}\dots\{\{T_m\}\}\,((T_{m+2}))\dots((T_n)).$$

Expanding elements of $k$ into power series in $\pi$ with coefficients in $\mathcal{R}_k$, one can write elements of $K$ as formal power series in $n$ parameters. To make them convergent power series we should introduce a topology in $K$ which takes into account topologies of the residue fields. We do not make $K$ a topological field this way, since multiplication is only sequentially continuous in this topology. However, for class field theory sequential continuity seems to be more important than continuity.

**1.3.1.**

**Definition.**

(a) If $F$ has a topology, consider the following topology on $K = F((X))$. For a sequence of neighbourhoods of zero $(U_i)_{i\in\mathbb{Z}}$ in $F$, $U_i = F$ for $i \gg 0$, denote $U_{\{U_i\}} = \left\{\sum a_i X^i : a_i \in U_i\right\}$. Then all $U_{\{U_i\}}$ constitute a base of open neighbourhoods of 0 in $F((X))$. In particular, a sequence $u^{(n)} = \sum a_i^{(n)} X^i$ tends to 0 if and only if there is an integer $m$ such that $u^{(n)} \in X^m F[[X]]$ for all $n$ and the sequences $a_i^{(n)}$ tend to 0 for every $i$.

Starting with the discrete topology on the last residue field, this construction is used to obtain a well-defined topology on an $n$-dimensional local field of characteristic $p$.

(b) Let $K_n$ be of mixed characteristic. Choose a system of local parameters $t_1, \dots, t_n = \pi$ of $K$. The choice of $t_1, \dots, t_{n-1}$ determines a canonical lifting

$$h = h_{t_1,\dots,t_{n-1}} \colon K_{n-1} \to \mathcal{O}_K$$

(see below). Let $(U_i)_{i\in\mathbb{Z}}$ be a system of neighbourhoods of zero in $K_{n-1}$, $U_i = K_{n-1}$ for $i \gg 0$. Take the system of all $U_{\{U_i\}} = \left\{\sum h(a_i)\pi^i,\ a_i \in U_i\right\}$ as a base of open neighbourhoods of 0 in $K$. This topology is well defined.

(c) In the case $\operatorname{char}(K) = \operatorname{char}(K_{n-1}) = 0$ we apply constructions (a) and (b) to obtain a topology on $K$ which depends on the choice of the coefficient subfield of $K_{n-1}$ in $\mathcal{O}_K$.

The definition of the canonical lifting $h_{t_1,\dots,t_{n-1}}$ is rather complicated. In fact, it is worthwhile to define it for any $(n-1)$-tuple $(t_1, \dots, t_{n-1})$ such that $v_i(t_i) > 0$ and $v_j(t_i) = 0$ for $i < j \leqslant n$. We shall give an outline of this construction, and the details can be found in [MZ1, §1].

Let $F = K_0((\overline{t_1}))\dots((\overline{t_{n-1}})) \subset K_{n-1}$. By a lifting we mean a map $h\colon F \to \mathcal{O}_K$ such that the residue of $h(a)$ coincides with $a$ for any $a \in F$.

Step 1. An auxiliary lifting $H_{t_1,\dots,t_{n-1}}$ is uniquely determined by the condition

$$H_{t_1,\dots,t_{n-1}}\left(\sum_{i_1=0}^{p-1}\cdots\sum_{i_{n-1}=0}^{p-1} \overline{t_1}^{\,i_1}\dots\overline{t_{n-1}}^{\,i_{n-1}} a^p_{i_1,\dots,i_{n-1}}\right) = \sum_{i_1=0}^{p-1}\cdots\sum_{i_{n-1}=0}^{p-1} t_1^{i_1}\dots t_{n-1}^{i_{n-1}} (H_{t_1,\dots,t_{n-1}}(a_{i_1,\dots,i_{n-1}}))^p.$$

Step 2. Let $k_0$ be the fraction field of $W(K_0)$. Then $K' = k_0\{\{T_1\}\}\dots\{\{T_{n-1}\}\}$ is an $n$-dimensional local field with the residue field $F$. Comparing the lifting $H = H_{T_1,\dots,T_{n-1}}$ with the lifting $h$ defined by

$$h\Big(\sum_{\mathbf{r}\in\mathbb{Z}^{n-1}} \theta_{\mathbf{r}}\overline{T_1}^{r_1}\dots\overline{T_{n-1}}^{r_{n-1}}\Big) = \sum_{\mathbf{r}\in\mathbb{Z}^{n-1}} [\theta_{\mathbf{r}}]T_1^{r_1}\dots T_{n-1}^{r_{n-1}},$$

we introduce the maps $\lambda_i \colon F \to F$ by the formula

$$h(a) = H(a) + pH(\lambda_1(a)) + p^2H(\lambda_2(a)) + \dots$$

Step 3. Introduce $h_{t_1,\dots,t_{n-1}} \colon F \to \mathcal{O}_K$ by the formula

$$h_{t_1,\dots,t_{n-1}}(a) = H_{t_1,\dots,t_{n-1}}(a) + pH_{t_1,\dots,t_{n-1}}(\lambda_1(a)) + p^2H_{t_1,\dots,t_{n-1}}(\lambda_2(a)) + \dots .$$

**Remarks.** 1. Observe that for a standard field $K = k\{\{T_1\}\}\dots\{\{T_{n-1}\}\}$, we have

$$h_{T_1,\dots,T_{n-1}} \colon \sum \theta_{\mathbf{i}}\overline{T_1}^{i_1}\dots\overline{T_{n-1}}^{i_{n-1}} \mapsto \sum [\theta_{\mathbf{i}}]T_1^{i_1}\dots T_{n-1}^{i_{n-1}},$$

where $\overline{T_j}$ is the residue of $T_j$ in $k_K$, $j = 1,\dots,n-1$.

2. The idea of the above construction is to find a field $k_0\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$ isomorphic to $K'$ inside $K$ without a priori given topologies on $K$ and $K'$. More precisely, let $t_1,\dots,t_{n-1}$ be as above. For $a = \sum_{-\infty}^{\infty} p^i h(a_i) \in K'$, let

$$f_{t_1,\dots,t_{n-1}}(a) = \sum_{-\infty}^{\infty} p^i h_{t_1,\dots,t_{n-1}}(a_i)$$

Then $f_{t_1,\dots,t_{n-1}} \colon K' \to K$ is an embedding of $n$-dimensional complete fields such that

$$f_{t_1,\dots,t_{n-1}}(T_j) = t_j, \qquad j = 1,\dots,n-1$$

(see [MZ1, Prop. 1.1]).

3. In the case of a standard mixed characteristic field the following alternative construction of the same topology is very useful.

Let $K = E\{\{X\}\}$, where $E$ is an $(n-1)$-dimensional local field; assume that the topology of $E$ is already defined. Let $\{V_i\}_{i\in\mathbb{Z}}$ be a sequence of neighbourhoods of zero in $E$ such that

(i) there is $c \in \mathbb{Z}$ such that $P_E(c) \subset V_i$ for all $i \in \mathbb{Z}$;

(ii) for every $l \in \mathbb{Z}$ we have $P_E(l) \subset V_i$ for all sufficiently large $i$.

Put

$$\mathcal{V}_{\{V_i\}} = \Big\{\sum b_iX^i : b_i \in V_i\Big\}.$$

Then all the sets $\mathcal{V}_{\{V_i\}}$ form a base of neighbourhoods of 0 in $K$. (This is an easy but useful exercise in the 2-dimensional case; in general, see Lemma 1.6 in [MZ1]).

4. The formal construction of $h_{t_1,\dots,t_{n-1}}$ works also in case $\mathrm{char}(K) = p$, and one need not consider this case separately. However, if one is interested in equal

characteristic case only, all the treatment can be considerably simplified. (In fact, in this case $h_{t_1,\dots,t_{n-1}}$ is just the obvious embedding of $F \subset k_K$ into $\mathcal{O}_K = k_K[[t_n]]$.)

**1.3.2. Properties.**

(1) $K$ is a topological group which is complete and separated.

(2) If $n > 1$, then every base of neighbourhoods of 0 is uncountable. In particular, there are maps which are sequentially continuous but not continuous.

(3) If $n > 1$, multiplication in $K$ is not continuous. In fact, $UU = K$ for every open subgroup $U$, since $U \supset P(c)$ for some $c$ and $U \not\subset P(s)$ for any $s$. However, multiplication is sequentially continuous:

$$\alpha_i \to \alpha, \quad 0 \neq \beta_i \to \beta \neq 0 \Longrightarrow \alpha_i\beta_i^{-1} \to \alpha\beta^{-1}.$$

(4) The map $K \to K$, $\quad \alpha \mapsto c\alpha$ for $c \neq 0$ is a homeomorphism.

(5) For a finite extension $L/K$ the topology of $L$ = the topology of finite dimensional vector spaces over $K$ (i.e., the product topology on $K^{|L:K|}$). Using this property one can redefine the topology first for "standard" fields

$$k\,\{\{T_1\}\}\,\dots\{\{T_m\}\}\,((T_{m+2}))\,\dots((T_n))$$

using the canonical lifting $h$, and then for arbitrary fields as the topology of finite dimensional vector spaces.

(6) For a finite extension $L/K$ the topology of $K$ = the topology induced from $L$. Therefore, one can use the Classification Theorem and define the topology on $K$ as induced by that on $L$, where $L$ is taken to be a standard $n$-dimensional local field.

**Remark.** In practical work with higher local fields, both (5) and (6) enables one to use the original definition of topology only in the simple case of a standard field.

**1.3.3. About proofs.** The outline of the proof of assertions in 1.3.1–1.3.2 is as follows. (Here we concentrate on the most complicated case $\operatorname{char}(K) = 0$, $\operatorname{char}(K_{n-1}) = p$; the case of $\operatorname{char}(K) = p$ is similar and easier, for details see [P]).

*Step 1* (see [MZ1, §1]). Fix first $n-1$ local parameters (or, more generally, any elements $t_1,\dots,t_{n-1} \in K$ such that $v_i(t_i) > 0$ and $v_j(t_i) = 0$ for $j > i$).

Temporarily fix $\pi_i \in K$ ($i \in \mathbb{Z}$), $v_n(\pi_i) = i$, and $e_j \in P_K(0)$, $j = 1,\dots,d$, so that $\{\overline{e_j}\}_{j=1}^d$ is a basis of the $F$-linear space $K_{n-1}$. (Here $F$ is as in 1.3.1, and $\bar{\alpha}$ denotes the residue of $\alpha$ in $K_{n-1}$.) Let $\{U_i\}_{i\in\mathbb{Z}}$ be a sequence of neighbourhoods of zero in $F$, $U_i = F$ for all sufficiently large $i$. Put

$$\mathcal{U}_{\{U_i\}} = \Big\{\sum_{i\geqslant i_0} \pi_i \cdot \sum_{j=1}^{d} e_j h_{t_1,\dots,t_{n-1}}(a_{ij}) : a_{ij} \in U_i, i_0 \in \mathbb{Z}\Big\}.$$

The collection of all such sets $\mathcal{U}_{\{U_i\}}$ is denoted by $B_U$.

*Step 2* ([MZ1, Th. 1.1]). In parallel one proves that

– the set $B_U$ has a cofinal subset which consists of subgroups of $K$; thus, $B_U$ is a base of neighbourhoods of zero of a certain topological group $K_{t_1,\dots,t_{n-1}}$ with the underlying (additive) group $K$;

– $K_{t_1,\dots,t_{n-1}}$ does not depend on the choice of $\{\pi_i\}$ and $\{e_j\}$;

– property (4) in 1.3.2 is valid for $K_{t_1,\dots,t_{n-1}}$.

*Step 3* ([MZ1, §2]). Some properties of $K_{t_1,\dots,t_{n-1}}$ are established, in particular, (1) in 1.3.2, the sequential continuity of multiplication.

*Step 4* ([MZ1, §3]). The independence from the choice of $t_1, \dots, t_{n-1}$ is proved.

We give here a short proof of some statements in Step 3.

Observe that the topology of $K_{t_1,\dots,t_{n-1}}$ is essentially defined as a topology of a finite-dimensional vector space over a standard field $k_0\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$. (It will be precisely so, if we take $\{\pi_i e_j : 0 \leqslant i \leqslant e-1, 1 \leqslant j \leqslant d\}$ as a basis of this vector space, where $e$ is the absolute ramification index of $K$, and $\pi_{i+e} = p\pi_i$ for any $i$.) This enables one to reduce the statements to the case of a standard field $K$.

If $K$ is standard, then either $K = E((X))$ or $K = E\{\{X\}\}$, where $E$ is of smaller dimension. Looking at expansions in $X$, it is easy to construct a limit of any Cauchy sequence in $K$ and to prove the uniqueness of it. (In the case $K = E\{\{X\}\}$ one should use the alternative construction of topology in Remark 3 in 1.3.1.) This proves (1) in 1.3.2.

To prove the sequential continuity of multiplication in the mixed characteristic case, let $\alpha_i \to 0$ and $\beta_i \to 0$, we shall show that $\alpha_i\beta_i \to 0$.

Since $\alpha_i \to 0, \beta_i \to 0$, one can easily see that there is $c \in \mathbb{Z}$ such that $v_n(\alpha_i) \geqslant c$, $v_n(\beta_i) \geqslant c$ for $i \geqslant 1$.

By the above remark, we may assume that $K$ is standard, i.e., $K = E\{\{t\}\}$. Fix an open subgroup $U$ in $K$; we have $P(d) \subset U$ for some integer $d$. One can assume that $U = \mathcal{V}_{\{V_i\}}$, $V_i$ are open subgroups in $E$. Then there is $m_0$ such that $P_E(d-c) \subset V_m$ for $m > m_0$. Let

$$\alpha_i = \sum_{-\infty}^{\infty} a_i^{(r)} t^r, \quad \beta_i = \sum_{-\infty}^{\infty} b_i^{(l)} t^l, \qquad a_i^{(r)}, b_i^{(l)} \in E.$$

Notice that one can find an $r_0$ such that $a_i^{(r)} \in P_E(d-c)$ for $r < r_0$ and all $i$. Indeed, if this were not so, one could choose a sequence $r_1 > r_2 > \dots$ such that $a_{i_j}^{(r_j)} \notin P_E(d-c)$ for some $i_j$. It is easy to construct a neighbourhood of zero $V'_{r_j}$ in $E$ such that $P_E(d-c) \subset V'_{r_j}$, $a_{i_j}^{(r_j)} \notin V_{r_j}$. Now put $V'_r = E$ when $r$ is distinct from any of $r_j$, and $U' = \mathcal{V}_{\{V'_r\}}$. Then $a_{i_j} \notin U'$, $j = 1, 2, \dots$ The set $\{i_j\}$ is obviously infinite, which contradicts the condition $\alpha_i \to 0$.

Similarly, $b_i^{(l)} \in P_E(d-c)$ for $l < l_0$ and all $i$. Therefore,

$$\alpha_i\beta_i \equiv \sum_{r=r_0}^{m_0} a_i^{(r)} t^r \cdot \sum_{l=l_0}^{m_0} b_i^{(l)} t^l \mod U,$$

and the condition $a_i^{(r)} b_i^{(l)} \to 0$ for all $r$ and $l$ immediately implies $\alpha_i \beta_i \to 0$.

**1.3.4. Expansion into power series.** Let $n = 2$. Then in characteristic $p$ we have $\mathbb{F}_q((X))((Y)) = \{\sum \theta_{ij} X^j Y^i\}$, where $\theta_{ij}$ are elements of $\mathbb{F}_q$ such that for some $i_0$ we have $\theta_{ij} = 0$ for $i \leqslant i_0$ and for every $i$ there is $j(i)$ such that $\theta_{ij} = 0$ for $j \leqslant j(i)$.

On the other hand, the definition of the topology implies that for every neighbourhood of zero $U$ there exists $i_0$ and for every $i < i_0$ there exists $j(i)$ such that $\theta X^j Y^i \in U$ whenever either $i \geqslant i_0$ or $i < i_0, j \geqslant j(i)$.

So every formal power series has only finitely many terms $\theta X^j Y^i$ outside $U$. Therefore, it is in fact a *convergent* power series in the just defined topology.

**Definition.** $\Omega \subset \mathbb{Z}^n$ is called admissible if for every $1 \leqslant l \leqslant n$ and every $j_{l+1}, \dots, j_n$ there is $i = i(j_{l+1}, \dots, j_n) \in \mathbb{Z}$ such that

$$(i_1, \dots, i_n) \in \Omega,\ i_{l+1} = j_{l+1}, \dots, i_n = j_n \Rightarrow i_l \geqslant i.$$

**Theorem.** *Let $t_1, \dots, t_n$ be a system of local parameters of $K$. Let $s$ be a section of the residue map $O_K \to O_K/M_K$ such that $s(0) = 0$. Let $\Omega$ be an admissible subset of $\mathbb{Z}^n$. Then the series*

$$\sum_{(i_1,\dots,i_n)\in\Omega} b_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n} \textit{ converges} \quad (b_{i_1,\dots,i_n} \in s(O_K/M_K))$$

*and every element of $K$ can be uniquely written this way.*

**Remark.** In this statement it is essential that the last residue field is finite. In a more general setting, one should take a "good enough" section. For example, for $K = k\{\{T_1\}\} \dots \{\{T_m\}\}((T_{m+2}))\dots((T_n))$, where $k$ is a finite extension of the fraction field of $W(K_0)$ and $K_0$ is perfect of prime characteristic, one may take the Teichmüller section $K_0 \to K_{m+1} = k\{\{T_1\}\} \dots \{\{T_m\}\}$ composed with the obvious embedding $K_{m+1} \hookrightarrow K$.

*Proof.* We have

$$\sum_{(i_1,\dots,i_n)\in\Omega} b_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n} = \sum_{b\in s(O_K/M_K)} \Big(b \cdot \sum_{(i_1,\dots,i_n)\in\Omega_b} t_1^{i_1} \dots t_n^{i_n}\Big),$$

where $\Omega_b = \{(i_1, \dots, i_n) \in \Omega : b_{i_1,\dots,i_n} = b\}$. In view of the property (4), it is sufficient to show that the inner sums converge. Equivalently, one has to show that given a neighbourhood of zero $U$ in $K$, for almost all $(i_1, \dots, i_n) \in \Omega$ we have $t_1^{i_1} \dots t_n^{i_n} \in U$. This follows easily by induction on $n$ if we observe that $t_1^{i_1} \dots t_{n-1}^{i_{n-1}} = h_{t_1,\dots,t_{n-1}}(\overline{t_1}^{i_1} \dots \overline{t_{n-1}}^{i_{n-1}})$.

To prove the second statement, apply induction on $n$ once again. Let $r = v_n(\alpha)$, where $\alpha$ is a given element of $K$. Then by the induction hypothesis

$$\overline{t_n^{-r}\alpha} = \sum_{(i_1,\dots,i_{n-1})\in\Omega_r} \overline{b_{i_1,\dots,i_n}}\,\left(\overline{t_1}\right)^{i_1}\dots\left(\overline{t_{n-1}}\right)^{i_{n-1}},$$

where $\Omega_r \subset \mathbb{Z}^{n-1}$ is a certain admissible set. Hence

$$\alpha = \sum_{(i_1,\dots,i_{n-1})\in\Omega_r} b_{i_1,\dots,i_n} t_1^{i_1}\dots t_{n-1}^{i_{n-1}} t_n^r + \alpha',$$

where $v_n(\alpha') > r$. Continuing this way, we obtain the desired expansion into a sum over the admissible set $\Omega = (\Omega_r \times \{r\}) \cup (\Omega_{r+1} \times \{r+1\}) \cup \dots$

The uniqueness follows from the continuity of the residue map $\mathcal{O}_K \to K_{n-1}$. □

## 1.4. Topology on $K^*$

**1.4.1. 2-dimensional case,** $\operatorname{char}(k_K) = p$**.**

Let $A$ be the last residue field $K_0$ if $\operatorname{char}(K) = p$, and let $A = W(K_0)$ if $\operatorname{char}(K) = 0$. Then $A$ is canonically embedded into $\mathcal{O}_K$, and it is in fact the subring generated by the set $\mathcal{R}$.

For a 2-dimensional local field $K$ with a system of local parameters $t_2, t_1$ define a base of neighbourhoods of 1 as the set of all $1 + t_2^i\mathcal{O}_K + t_1^j A[[t_1, t_2]]$, $i \geqslant 1$, $j \geqslant 1$. Then every element $\alpha \in K^*$ can be expanded as a convergent (with respect to the just defined topology) product

$$\alpha = t_2^{a_2} t_1^{a_1}\theta \prod (1 + \theta_{ij} t_2^i t_1^j)$$

with $\theta \in \mathcal{R}^*, \theta_{ij} \in \mathcal{R}, a_1, a_2 \in \mathbb{Z}$. The set $S = \{(j,i) : \theta_{ij} \neq 0\}$ is admissible.

**1.4.2.** In the general case, following Parshin's approach in characteristic $p$ [P], we define the topology $\tau$ on $K^*$ as follows.

**Definition.** If $\operatorname{char}(K_{n-1}) = p$, then define the topology $\tau$ on

$$K^* \simeq V_K \times \langle t_1 \rangle \times \dots \times \langle t_n \rangle \times \mathcal{R}^*$$

as the product of the induced from $K$ topology on the group of principal units $V_K$ and the discrete topology on $\langle t_1 \rangle \times \dots \times \langle t_n \rangle \times \mathcal{R}^*$.

If $\operatorname{char}(K) = \operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$, where $m \leqslant n-2$, then we have a canonical exact sequence

$$1 \to 1 + P_K\bigl(1, \underbrace{0, \dots, 0}_{n-m-2}\bigr) \to O_K^* \to O_{K_{m+1}}^* \to 1.$$

Define the topology $\tau$ on $K^* \simeq O_K^* \times \langle t_1 \rangle \times \cdots \times \langle t_n \rangle$ as the product of the discrete topology on $\langle t_1 \rangle \times \cdots \times \langle t_n \rangle$ and the inverse image of the topology $\tau$ on $O_{K_{m+1}}^*$. Then the intersection of all neighbourhoods of 1 is equal to $1 + P_K\big(1, \underbrace{0, \ldots, 0}_{n-m-2}\big)$ which is a uniquely divisible group.

**Remarks.** 1. Observe that $K_{m+1}$ is a mixed characteristic field and therefore its topology is well defined. Thus, the topology $\tau$ is well defined in all cases.

2. A base of neighbourhoods of 1 in $V_K$ is formed by the sets

$$h(U_0) + h(U_1)t_n + \ldots + h(U_{c-1})t_n^{c-1} + P_K(c),$$

where $c \geqslant 1$, $U_0$ is a neighbourhood of 1 in $V_{k_K}$, $U_1, \ldots, U_{c-1}$ are neighbourhoods of zero in $k_K$, $h$ is the canonical lifting associated with some local parameters, $t_n$ is the last local parameter of $K$. In particular, in the two-dimensional case $\tau$ coincides with the topology of 1.4.1.

**Properties.**

(1) Each Cauchy sequence with respect to the topology $\tau$ converges in $K^*$.

(2) Multiplication in $K^*$ is sequentially continuous.

(3) If $n \leqslant 2$, then the multiplicative group $K^*$ is a topological group and it has a countable base of open subgroups. $K^*$ is not a topological group with respect to $\tau$ if $m \geqslant 3$.

*Proof.* (1) and (2) follow immediately from the corresponding properties of the topology defined in subsection 1.3. In the 2-dimensional case (3) is obvious from the description given in 1.4.1. Next, let $m \geqslant 3$, and let $U$ be an arbitrary neighbourhood of 1. We may assume that $n = m$ and $U \subset V_K$. From the definition of the topology on $V_K$ we see that $U \supset 1 + h(U_1)t_n + h(U_2)t_n^2$, where $U_1, U_2$ are neighbourhoods of 0 in $k_K$, $t_n$ a prime element in $K$, and $h$ the canonical lifting corresponding to some choice of local parameters. Therefore,

$$\begin{aligned} UU + P(4) &\supset (1 + h(U_1)t_n)(1 + h(U_2)t_n^2) + P(4) \\ &= \{1 + h(a)t_n + h(b)t_n^2 + h(ab)t_n^3 : a \in U_1, b \in U_2\} + P(4). \end{aligned}$$

(Indeed, $h(a)h(b) - h(ab) \in P(1)$.) Since $U_1 U_2 = k_K$ (see property (3) in 1.3.2), it is clear that $UU$ cannot lie in a neighbourhood of 1 in $V_K$ of the form $1 + h(k_K)t_n + h(k_K)t_n^2 + h(U')t_n^3 + P(4)$, where $U' \neq k_K$ is a neighbourhood of 0 in $k_K$. Thus, $K^*$ is not a topological group. □

**Remarks.** 1. From the point of view of class field theory and the existence theorem one needs a stronger topology on $K^*$ than the topology $\tau$ (in order to have more open subgroups). For example, for $n \geqslant 3$ each open subgroup $A$ in $K^*$ with respect to the topology $\tau$ possesses the property: $1 + t_n^2 \mathcal{O}_K \subset (1 + t_n^3 \mathcal{O}_K)A$.

A topology $\lambda_*$ which is the sequential saturation of $\tau$ is introduced in subsection 6.2; it has the same set of convergence sequences as $\tau$ but more open subgroups. For example [F1], the subgroup in $1 + t_n \mathcal{O}_K$ topologically generated by $1 + \theta t_n^{i_n} \dots t_1^{i_1}$ with $(i_1, \dots, i_n) \neq (0, 0, \dots, 1, 2)$, $i_n \geqslant 1$ (i.e., the sequential closure of the subgroup generated by these elements) is open in $\lambda_*$ and does not satisfy the above-mentioned property.

One can even introduce a topology on $K^*$ which has the same set of convergence sequences as $\tau$ and with respect to which $K^*$ is a topological group, see [F2].

2. For another approach to define open subgroups of $K^*$ see the paper of K. Kato in this volume.

**1.4.3. Expansion into convergent products.** To simplify the following statements we assume here char $k_K = p$. Let $B$ be a fixed set of representatives of non-zero elements of the last residue field in $K$.

**Lemma.** *Let $\{\alpha_i : i \in I\}$ be a subset of $V_K$ such that*

$$(*) \qquad \alpha_i = 1 + \sum_{\mathbf{r} \in \Omega_i} b_{\mathbf{r}}^{(i)} t_1^{r_1} \dots t_n^{r_n},$$

*where $b \in B$, and $\Omega_i \subset \mathbb{Z}_+^n$ are admissible sets satisfying the following two conditions:*

(i) $\Omega = \bigcup_{i \in I} \Omega_i$ *is an admissible set;*

(ii) $\bigcap_{j \in J} \Omega_j = \emptyset$, *where $J$ is any infinite subset of $I$.*

*Then $\prod_{i \in I} \alpha_i$ converges.*

*Proof.* Fix a neighbourhood of 1 in $V_K$; by definition it is of the form $(1 + U) \cap V_K$, where $U$ is a neighbourhood of 0 in $K$. Consider various finite products of $b_{\mathbf{r}}^{(i)} t_1^{r_1} \dots t_n^{r_n}$ which occur in $(*)$. It is sufficient to show that almost all such products belong to $U$.

Any product under consideration has the form

$$(**) \qquad \gamma = b_1^{k_1} \dots b_s^{k_s} t_1^{l_1} \dots t_n^{l_n}$$

with $l_n > 0$, where $B = \{b_1, \dots, b_s\}$. We prove by induction on $j$ the following claim: for $0 \leqslant j \leqslant n$ and fixed $l_{j+1}, \dots, l_n$ the element $\gamma$ almost always lies in $U$ (in case $j = n$ we obtain the original claim). Let

$$\hat{\Omega} = \{\mathbf{r}_1 + \dots + \mathbf{r}_t : t \geqslant 1, \mathbf{r}_1, \dots, \mathbf{r}_t \in \Omega\}.$$

It is easy to see that $\hat{\Omega}$ is an admissible set and any element of $\hat{\Omega}$ can be written as a sum of elements of $\Omega$ in finitely many ways only. This fact and condition (ii) imply that any particular $n$-tuple $(l_1, \dots, l_n)$ can occur at the right hand side of $(**)$ only finitely many times. This proves the base of induction ($j = 0$).

For $j > 0$, we see that $l_j$ is bounded from below since $(l_1, \dots, l_n) \in \hat{\Omega}$ and $l_{j+1}, \dots, l_n$ are fixed. On the other hand, $\gamma \in U$ for sufficiently large $l_j$ and arbitrary $k_1, \dots, k_s, l_1, \dots, l_{j-1}$ in view of [MZ1, Prop. 1.4] applied to the neighbourhood of zero $t_{j+1}^{-l_{j+1}} \dots t_n^{-l_n} U$ in $K$. Therefore, we have to consider only a finite range of values $c \leqslant l_j \leqslant c'$. For any $l_j$ in this range the induction hypothesis is applicable. □

**Theorem.** *For any* $\mathbf{r} \in \mathbb{Z}_+^n$ *and any* $b \in B$ *fix an element*

$$a_{\mathbf{r},b} = \sum_{\mathbf{s}\in\boldsymbol{\Omega}_{\mathbf{r},b}} b_{\mathbf{s}}^{\mathbf{r},b} t_1^{s_1} \dots t_n^{s_n},$$

*such that* $b_{\mathbf{r}}^{\mathbf{r},b} = b$, *and* $b_{\mathbf{s}}^{\mathbf{r},b} = 0$ *for* $\mathbf{s} < \mathbf{r}$. *Suppose that the admissible sets*

$$\{\boldsymbol{\Omega}_{\mathbf{r},b} : \mathbf{r} \in \boldsymbol{\Omega}_*, b \in B\}$$

*satisfy conditions* (i) *and* (ii) *of the Lemma for any given admissible set* $\boldsymbol{\Omega}_*$.

*1. Every element* $a \in K$ *can be uniquely expanded into a convergent series*

$$a = \sum_{\mathbf{r}\in\boldsymbol{\Omega}_a} a_{\mathbf{r},b_{\mathbf{r}}},$$

*where* $b_{\mathbf{r}} \in B$, $\boldsymbol{\Omega}_a \subset \mathbb{Z}_n$ *is an admissible set.*

*2. Every element* $\alpha \in K^*$ *can be uniquely expanded into a convergent product:*

$$\alpha = t_n^{a_n} \dots t_1^{a_1} b_0 \prod_{\mathbf{r}\in\boldsymbol{\Omega}_\alpha} \left(1 + a_{\mathbf{r},b_{\mathbf{r}}}\right),$$

*where* $b_0 \in B$, $b_{\mathbf{r}} \in B$, $\boldsymbol{\Omega}_\alpha \subset \mathbb{Z}_n^+$ *is an admissible set.*

*Proof.* The additive part of the theorem is [MZ2, Theorem 1]. The proof of it is parallel to that of Theorem 1.3.4.

To prove the multiplicative part, we apply induction on $n$. This reduces the statement to the case $\alpha \in 1+P(1)$. Here one can construct an expansion and prove its uniqueness applying the additive part of the theorem to the residue of $t_n^{-v_n(\alpha-1)}(\alpha-1)$ in $k_K$. The convergence of all series which appear in this process follows from the above Lemma. For details, see [MZ2, Theorem 2]. □

**Remarks.** 1. Conditions (i) and (ii) in the Lemma are essential. Indeed, the infinite products $\prod_{i=1}^{\infty}(1 + t_1^i + t_1^{-i}t_2)$ and $\prod_{i=1}^{\infty}(1 + t_1^i + t_2)$ do not converge. This means that the statements of Theorems 2.1 and 2.2 in [MZ1] have to be corrected and conditions (i) and (ii) for elements $\varepsilon_{\mathbf{r},\theta}$ $(\mathbf{r} \in \boldsymbol{\Omega}_*)$ should be added.

2. If the last residue field is not finite, the statements are still true if the system of representatives $B$ is not too pathological. For example, the system of Teichmüller representatives is always suitable. The above proof works with the only ammendment: instead of Prop. 1.4 of [MZ1] we apply the definition of topology directly.

**Corollary.** *If* $\operatorname{char}(K_{n-1}) = p$, *then every element* $\alpha \in K^*$ *can be expanded into a convergent product:*

$$(***) \qquad \alpha = t_n^{a_n} \dots t_1^{a_1} \theta \prod (1 + \theta_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n}), \quad \theta \in \mathcal{R}^*, \quad \theta_{i_1,\dots,i_n} \in \mathcal{R},$$

*with* $\{(i_1, \dots, i_n) : \theta_{i_1,\dots,i_n} \neq 0\}$ *being an admissible set. Any series* $(***)$ *converges.*

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