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NON-UNIQUENESS OF CERTAIN HAHN–BANACH EXTENSIONS

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Dedicated to the memory of
Professor Yuri A. Abramovich

Let f be a continuous linear functional defined on a subspace M of a normed space X. If X is real or complex, there are results that characterize uniqueness of continuous extensions F of f to X for every subspace M and those that apply just to M. If X is defined over a non-Archimedean valued field K and the norm also satisfies the strong triangle inequality, the Hahn–Banach theorem holds for all subspaces M of X if and only if K is spherically complete and it is well-known that Hahn–Banach extensions are never unique in this context. We give a different proof of non-uniqueness here that is interesting for its own sake and may point a direction in which further investigation would be fruitful.

1. Introduction

Suppose that K denotes a non-Archimedean, nontrivially valued field, i. e., a field with an absolute value $|\cdot|$ that satisfies the *strong triangle inequality:* for all $a, b \in K$, $|a+b| \leqslant \max(|a|, |b|)$. Let X be a normed space over K in which the norm also satisfies the strong triangle inequality and X' denotes its continuous dual. We refer to X as a *non-Archimedean normed space.* For a subspace M of X, $M^{\perp} = \{f \in X' : f(x) = 0,\ x \in M\}$, the *orthogonal* of M; the orthogonal of $M \subset X'$ is given by $M^{\perp} = \{x \in X : f(x) = 0,\ f \in M\}$.

Definition 1. If each nested sequence $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$ of balls in K has nonempty intersection, K is called *spherically complete.*

Note the absence of any requirement that the diameters shrink to 0 in this stronger version of completeness. If f is a continuous linear functional defined on M, an extension of f to $F \in X'$ of the same norm is called a *Hahn–Banach extension.* In the context of non-Archimedean normed spaces the Hahn–Banach theorem can fail — there exist spaces X and continuous linear functionals defined on a subspace M of X that have no Hahn–Banach extension. If K is spherically complete, however, then any continuous linear functional f defined on any subspace M of X has a Hahn–Banach extension (see [4; p. 78] or [5; p. 102]).

When are Hahn–Banach extensions unique? There are two principal, classical results, one (the Taylor-Foguel theorem) for any subspace M of X and another (Phelps's theorem) that deals with one subspace at a time:

Theorem 1 (Taylor–Foguel). *If E is a normed space over $\mathbb{R}$ or $\mathbb{C}$ then the following conditions are equivalent:*

(a) *For any subspace M of E and any $f \in M'$, f has a unique Hahn–Banach extension;*

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(b) *E' is strictly convex (equivalently «strictly normed») in the sense that for any two unit vectors f and g and any $t \in (0,1)$, $\|tf + (1-t)g\| < 1$.*

Theorem 2 (Phelps)**.** *If M is a subspace of the normed space E over $\mathbb{R}$ or $\mathbb{C}$ then the following conditions are equivalent:*

(a) *For any $f \in M'$, f has a unique Hahn–Banach extension;*

(b) *$M^{\perp}$ has a unique best approximation in E' in the sense that given any $f \in E'$ there exists a unique $m \in M^{\perp}$ such that $\|f - m\| = \inf\{\|f - g\| : g \in M^{\perp}\} = d(f, M^{\perp})$.*

What can be said about uniqueness in the non-Archimedean case. After proving a certain lemma ([5; Lemma 4.4, p. 100] van Rooij observes (p. 103) that Hahn–Banach extensions are unique if and only if the subspace M is dense in X or $f = 0$. We prove the non-uniqueness by different means in the next section.

2. Unique Hahn–Banach Extensions

We obtain a version (Th. 5) of Phelps's theorem concerning uniqueness of Hahn–Banach extensions on a subspace M of X and uniqueness of best approximations from $M^{\perp}$. We then show that the conditions for uniqueness are never satisfied in non-Archimedean spaces.

A subspace M of X is *proximinal* if for all $x \in X$ there exists a «best approximation» $m \in M$ to x, i. e., $m \in M$ with $\|x - m\| = d(x, M)$. We denote the set of all best approximations of x from M by

$$P_M(x) = \{m \in M : \|x - m\| = d(x, M)\}.$$

If $P_M(x)$ is a singleton for every $x \in X$ then M is called *Chebychev*. It is easy to verify that $P_M(x)$ is closed.

As in the real or complex case, for spherically compete K, conventional orthogonal facts are valid as well as orthogonals are proximinal.

Theorem 3. *Let K be spherically complete and let $\sigma(X', X)$ denotes the weak-* topology on X'. Then*

(a) [3; p. 211] *For $M \subset X'$, $M^{\perp\perp} = \mathrm{cl}_{\sigma(X',X)} M$. Thus, if M is $\sigma(X', X)$-closed, $M = M^{\perp\perp}$.*

(b) [3; p. 215] *For $M \subset X$, $(X/M)'$ is algebraically isomorphic to $M^{\perp}$ and M' is algebraically isomorphic to $X'/M^{\perp}$.*

Theorem 4. *Let K be spherically complete, M be a subspace of X and $f \in X'$. If F is any extension of $f\big|_M$ of the same norm, then $F - f$ is a best approximation to f from $M^{\perp}$ and $d(f, M^{\perp}) = \|f|_M\|$, i. e., $M^{\perp}$ is proximal.*

$\lhd$ Let $f \in X'$. Then for every $m' \in M^{\perp}$,

$$\begin{aligned} \|f\big|_M\| &= \sup\{|f(x)| : x \in U \cap M\} = \sup\{|f(x) - m'(x)| : x \in U \cap M\} \\ &\leqslant \sup\{|f(x) - m'(x)| : x \in U\} = \|f - m'\|. \end{aligned}$$

Since $m' \in M^{\perp}$ is arbitrary, it follows that $\|f|_M\| \leqslant d(f, M^{\perp})$. To obtain the reverse inequality, consider an extension $F \in X'$ of $f|_M$ with $\|F\| = \|f|_M\|$. Since $f - F \in M^{\perp}$,

$$\|f|_M\| = \|F\| = \|f - (f - F))\| \geqslant d(f, M^{\perp}).$$

In other words, $f - F$ is a best approximation to f from $M^{\perp}$ and $\|f|_M\| = d(f, M^{\perp})$. $\rhd$

Using a technique of Herrero's [1], we now obtain a version of Phelps's theorem that a subspace M of X has unique Hahn–Banach extensions if and only if $M^{\perp}$ is Chebysev.

Theorem 5. *For $M \subset X$ over a spherically complete field K, the following assertions are equivalent:*

(a) *each $f \in M'$ has a unique Hahn–Banach extension;*

(b) *$M^{\perp}$ is Chebychev.*

$\triangleleft$ (a) $\Longrightarrow$ (b): Let $f \in X'$. By Theorem 4, $M^{\perp}$ is proximinal, so it only remains to prove uniqueness of best approximations. If $g, h \in P_{M^{\perp}}(f)$, then $f - g$ and $f - h$ are extensions of $f|_M$; since $g, h \in P_{M^{\perp}}(f)$,

$$\|f - g\| = \|f - h\| = d\left(f, M^{\perp}\right).$$

Since extensions of $f|_M$ of the same norm are unique, $f - g = f - h$ which implies $g = h$.

(b) $\Longrightarrow$ (a): Suppose $f \in M'$ has extensions $g, h \in X'$ of the same norm as f. Then h is an extension of $g|_M$ to h of the same norm. Therefore, by Theorem 4, $g - h$ is a best approximation to g from $M^{\perp}$. Since $\|h\| = \|g\| = \|f\|$ and

$$\|g\| = \|g - 0\| = \|h\| = \|g - (g - h)\| = d(g, M^{\perp})$$

it follows that $0 \in P_M(g)$ as well. By the uniqueness of best approximation, $g - h = 0$. $\triangleright$

Since a weak-* closed subspace M of X' is the orthogonal of $M^{\perp}$, it follows that:

Corollary 1. *A weak-* closed subspace M of X' is Chebychev if and only if each bounded linear map $f : M^{\perp} \to K$ has a unique extension $F \in X'$ of the same norm.*

The following result establishes that non-Archimedean spaces are never Chebychev.

Theorem 6 (cf. [2]). *Suppose $M \subset X$ is a closed subspace and $x \notin M$. If $m \in P_M(x)$ and $m' \in M$ is such that $\|m' - m\| < \|x - m\|$, then $m' \in P_M(x)$.*

$\triangleleft$ Since $x \notin M$ and $m' \in M$, it follows that $\|x - m'\| > 0$. By the strong triangle inequality, $\|x - m'\| = \|x - m\|$. $\triangleright$

It follows from Corolary 1 and Theorem 6 that Hahn–Banach extensions are never unique.

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