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THE ORDER CONTINUOUS DUAL OF THE REGULAR INTEGRAL OPERATORS ON L^p

Dedicated to Safak Alpay on the occasion of his sixtieth birthday

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In this paper we give two descriptions of the order continuous dual of the Banach lattics of regular integral operators on L^p. The first description is in terms of a Calderon space, while the second one in terms of the ideal generated by the finite rank operators.

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Introduction

Let $\mathscr{I}_r(L^p)$ denote the collection of all regular integral operators on L^p $(1 < p < \infty)$ with the regular norm $\|\cdot\|_r$. Then $\mathscr{I}_r(L^p)$ is a Banach function space on $X \times X$ with the Fatou property. In this paper we shall give two distinct descriptions of the order continuous dual, or associate space, of $\mathscr{I}_r(L^p)$. Our first description follows from a result on $\mathscr{I}_r(L^p)$. In [3] we showed that $\mathscr{I}_r(L^p)$ is equal to the Calderon space $(L_{\infty,1})^{\frac{1}{p'}}(L^t_{\infty,1})^{\frac{1}{p}}$. As a consequence we derive our first description via Lozanovskii's duality theorem. Then we will present a different description of this space. We will prove that the order ideal generated by the finite rank operators on L^p provided with an extension of the positive projective tensor norm is a Banach function space with the Fatou property, from which it follows that it is isometric to the order continuous dual of $\mathscr{I}_r(L^p)$.

The order continuous dual as Calderon space

Throughout this paper (X, μ) will denote a σ-finite measure space and p will be a fixed real number with $1 < p < \infty$. Recall the definition of a regular integral or kernel operator on L^p-spaces. Let $T(x, y)$ be a $\mu \times \mu$-measurable function on $X \times X$. Then $T(x, y)$ is the kernel of an integral operator T from L^p into L^p if

$$\int_X |T(x,y)f(y)|\,d\mu(y) < \infty \quad \text{a. e.}$$

for all $f \in L^p$ and

$$Tf(x) = \int_X T(x,y)f(y)\,d\mu(y) \in L^p$$

for all $f \in L^p$. If in addition $|T(x,y)|$ is the kernel of an integral operator (denoted by $|T|$) from L^p into L^p, then T is called a regular (or order bounded) integral operator. By $\mathscr{I}_r(L^p)$ we shall denote the collection of all such regular integral operators on L^p. If we equip $\mathscr{I}_r(L^p)$ with the regular norm $\|\cdot\|_r$, i. e., the operator norm of the modulus operator $|T|$, then it is well-known that $\mathscr{I}_r(L^p)$ becomes a Banach function space on $X \times X$ with the Fatou property. Many order-theoretic properties of $\mathscr{I}_r(L^p)$ are known, i. e., $\mathscr{I}_r(L^p)$ equals the band generated by $L^{p'} \otimes L^p$ in $\mathscr{L}_r(L^p)$ (see e. g. [4]). By $\mathscr{K}_r(L^p)$ we shall denote the closure of $L^{p'} \otimes L^p$ in $\mathscr{L}_r(L^p)$ with respect to the regular norm. It is a consequence of e. g. the Dodds-Fremlin theorem (see [4]) that $\mathscr{K}_r(L^p)$ is an ideal in $\mathscr{I}_r(L^p)$ and is equal to the ideal of elements of order continuous norm in $\mathscr{I}_r(L^p)$. Finding a description of the order continous dual of $\mathscr{I}_r(L^p)$ is the same as finding a description of the norm dual of $\mathscr{K}_r(L^p)$. For a measurable function F on $X \times X$ we define for $1 \leqslant p < \infty$ the norm $\|F\|_{\infty,p}$ as follows

$$\|F\|_{\infty,p} = \left\| \left(\int |F(x,y)|^p \, dy \right)^{\frac{1}{p}} \right\|_{\infty}.$$

We denote

$$L_{\infty,p} = \big\{ F \in L_0(X \times X) : \|F\|_{\infty,p} < \infty \big\}.$$

Given F on $X \times X$ we define the transpose of F by $F^t(x,y) = F(y,x)$. Then $L^t_{\infty,p}$ will denote the collection of all F such that $F^t \in L_{\infty,p}$ and the norm on $L^t_{\infty,p}$ will be defined by $\|F^t\|_{\infty,p}$. In[3] we proved the following theorem.

Theorem 1.1. *Let $1 < p < \infty$. Then $L_{\infty,p'} \cdot L^t_{\infty,p}$ is a product Banach function space isometrically equal to $\mathscr{I}_r(L_p)$ and for any $T \in \mathscr{I}_r(L_p)$ we have a factorization $T(x,y) = T_1(x,y) \cdot T_{2(x,y)}$ with $\|T\|_r = ||T_1\|_{\infty,p'} \|T_2^t\|_{\infty,p}$. In particular $\mathscr{I}_r(L_p) = (L_{\infty,1})^{\frac{1}{p'}} (L^t_{\infty,1})^{\frac{1}{p}}$.* As a consequence we get our first description of the order continuous dual of $\mathscr{I}_r(L^p)$.

Theorem 1.2. *Let $1 < p < \infty$. Then the order continuous dual of $\mathscr{I}_r(L^p)$ is equal to $(L_{1,\infty})^{\frac{1}{p'}} (L^t_{1,\infty})^{\frac{1}{p}} = L_{p',\infty} L^t_{p,\infty}$.*

$\lhd$ Recall first Lozanovskii's theorem. If E and F are Banach function spaces with the Fatou property and $1 < p < \infty$, then $\left(E^{\frac{1}{p}} F^{\frac{1}{p'}} \right)' = (E')^{\frac{1}{p}} (F')^{\frac{1}{p'}}$. Applying this to the situation at hand we get $(\mathscr{I}_r(L^p))' = (L'_{\infty,1})^{\frac{1}{p'}} ((L^t_{\infty,1})')^{\frac{1}{p}}$. Now it is easy to see that $L'_{\infty,1} = L_{1,\infty}$ and similarly $(L^t_{\infty,1})' = L^t_{1,\infty}$. Hence the result follows. $\rhd$

The order continuous dual as the ideal generated by the finite rank operators

We now recall from [2] some facts about the positive projective tensor product of Banach lattices, applied to our situation. The Riesz space tensor product $L^{p'} \overline{\otimes} L^p$ of $L^{p'} \otimes L^p$ can be identified with the Riesz subspace generated by $L^{p'} \otimes L^p$ in $\mathscr{I}_r(L^p)$. On $L^{p'} \overline{\otimes} L^p$ we can define the positive projective tensor norm $\|\cdot\|_{|\pi|}$, which by Theorems 2.1 and 2.2 of [2] is equal to

$$\|f\|_{|\pi|} = \inf\{ \|g\|_{p'} \|h\|_p : \ |f| \leqslant g \otimes h, \ 0 \leqslant g \in L^{p'}, \ 0 \leqslant h \in L^p \}.$$

Note, in general we can not restrict ourselves to majorants consisting of single tensors, but we can in this particular case. Moreover the positive projective tensor product $L^{p'} \widetilde{\otimes}_{|\pi|} L^p$ of $L^{p'}$ with L^p is now the completion of $L^{p'} \overline{\otimes} L^p$ with respect to this norm. In Theorem 2.1 of [2] it was also observed that $L^{p'}\widetilde{\otimes}_{|\pi|}L^p$ is contained in the ideal generated by $L^{p'} \otimes L^p$ in $\mathscr{I}_r(L^p)$. It was proved by Fremlin [1]that $L^2\widetilde{\otimes}_{|\pi|}L^2$ is not Dedekind complete in general and similar arguments show that the same is true for $L^{p'}\widetilde{\otimes}_{|\pi|}L^p$. We consider therefore the extension of the positive projective norm to the ideal generated by the finite rank operators. More precisely, denote by $\mathscr{F}(L^p)$ the ideal in $\mathscr{I}_r(L^p)$ generated by the finite rank operators and define for $f \in \mathscr{F}(L^p)$

$$\|f\|_{|\pi|} = \inf\{\|g\|_{p'}\|h\|_p : \ |f| \leqslant g \otimes h, \ 0 \leqslant g \in L^{p'}, \ 0 \leqslant h \in L^p\}.$$

It is clear that this defines a norm on $\mathscr{F}(L^p)$ if we realize that the unit ball $B_{\mathscr{F}} = \{f \in \mathscr{F}(L^p) : \|f\|_{|\pi|} \leqslant 1\}$ is the solid hull of the unit ball of $L^{p'}\overline{\otimes}L^p$ (and in fact also of the unit ball of $L^{p'}\widetilde{\otimes}_{|\pi|}L^p$). The main result of this section is now the following theorem.

Theorem 2.1. *The normed Köthe space $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$ has the Fatou property. In particular $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$ is complete.*

$\lhd$ It suffices to prove that the unit ball $B_{\mathscr{F}} = \{f \in \mathscr{F}(L^p) : \|f\|_{|\pi|} \leqslant 1\}$ is closed in measure in $L^0(X \times X, \mu \times \mu)$. Let $0 \leqslant f_n(x, y) \in B_{\mathscr{F}}$ and assume $f_n(x, y) \to f(x, y)$ a. e. on $X \times X$. Without loss of generality we can assume that $\|f_n\|_{|\pi|} < 1$ for all n. Then there exist g_n and h_n with $\|g_n\|_{p'} \leqslant 1$, $\|h_n\|_p \leqslant 1$ such that $f_n \leqslant g_n \otimes h_n$. By using Komlos' Theorem (see Theorem 3.1 of [3]) we can find subsequences g_{n_k} and h_{n_k} such that $g_{n_k}(y)$ Cesaro converges a. e. on X to $g(y)$ and $h_{n_k}(x)$ Cesaro converges a. e to $h(x)$ on X. This implies that $\|g\|_{p'} \leqslant 1$ and $\|h\|_p \leqslant 1$. Now $g_{n_k} \otimes \chi_X$ Cesaro converges a. e. to $g \otimes \chi_X$ and $\chi_X \otimes h_{n_k}$ Cesaro converges a. e. to $\chi_X \otimes h$ on $X \times X$. From $0 \leqslant f_{n_k} \leqslant g_{n_k} \otimes \chi_X \cdot \chi_X \otimes h_{n_k}$ it follows now from Theorem 2.3 of [3] that $f(x, y) \leqslant g \otimes \chi_X \cdot \chi_X \otimes h = g \otimes h$, which shows that $f \in B_{\mathscr{F}}$. This shows that $B_{\mathscr{F}}$ is closed in measure and thus the normed Köthe space $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$ has the Fatou property. $\rhd$

As a consequence of the above theorem we get our second description of the order continuous dual of $\mathscr{I}_r(L^p)$.

Theorem 2.2. *The order continuous dual of the space $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$ is $\mathscr{I}_r(L^p)$. Therefore the norm dual of $\mathscr{K}_r(L^p)$ and the order continuous dual of $\mathscr{I}_r(L^p)$ is equal to $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$.*

$\lhd$ Let $k(x, y)$ define an order continuous functional T of norm less or equal to one on $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$. Then $\langle |k|, |g| \otimes |h|\rangle < \infty$ for all $g \otimes h \in L^{p'} \otimes L^p$ implies that k defines an integral operator $T_k \in \mathscr{I}_r(L^p)$. Moreover

$$\|T_k\|_r = \sup(\langle |k|, |g| \otimes |h|\rangle : \|g \otimes h\|_{|\pi|} \leqslant 1) = \sup((\langle |k|, |f|\rangle : \|f\|_{|\pi|} \leqslant 1) = \|k\|_{\mathscr{F}'}.$$

Hence the map $k \mapsto T_k$ is an isometric lattice isomorphism from $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$ into $\mathscr{I}_r(L^p)$. Now it is straightforward to verify that this mapping is onto, as the kernel of a positive kernel operator in $\mathscr{I}_r(L^p)$ of norm less or equal to one defines a positive linear functional on $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$ of norm less or equal to one and the proof of the first assertion follows. The remaining statements are now an immediate consequence from the Fatou property of the norm on $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$. $\rhd$

Remark. (1) One can ask why the above results are restricted to L^p-spaces and whether the results can't be extended to more general Banach lattices. For the first part of the last theorem that seems possible, but the second part is not clear as it depends on the Fatou property of $(\mathscr{F}(L^p), \|\cdot\|_{|\pi|})$. The proof of this fact in Theorem 2.1 depended essentially on the fact that the positive projective norm on $L^{p'} \otimes_{|\pi|} L^p$ can be obtained using only single tensors instead of finite sums of tensors. This property is known only for $E' \otimes E$ when E is isomorphic to an L^p-space (see [2]).

(2) We observe that the second part of the above theorem can be considered as an order continuous analogue of the classical duality of compact operators, trace class operators and all bounded operators.

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