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SOME STABILITY RESULTS FOR PICARD ITERATIVE PROCESS IN UNIFORM SPACE

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We prove some stability results for Picard iteration in uniform space by introducing the concept of an M_e-distance as well as using some contractive conditions. Our results generalize, extend and improve some earlier results.

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1. Introduction

The following concepts shall be required in the sequel:

Definition 1.1 [5, 27]. A *uniform space* (X, Φ) is a nonempty set X equipped with a nonempty family Φ of subsets of $X \times X$ satisfying the following properties:

(i) if U is in Φ, then U contains the diagonal $\{(x, x)\,|x \in X\}$;

(ii) if U is in Φ and V is a subset of $X \times X$ which contains U, then V is in Φ;

(iii) if U and V are in Φ, then $U \cap V$ is in Φ;

(iv) if U is in Φ, then there exists V in Φ, such that, whenever (x, y) and (y, z) are in V, then (x, z) is in U;

(v) if U is in Φ, then $\{(y, x)\,|(x, y) \in U\}$ is also in Φ.

Φ is called the *uniform structure* of X and its elements are called *entourages or neighbourhoods or surroundings.*

The space (X, Φ) is called *quasiuniform* if property (v) is omitted.

The notions of an A-distance and an E-distance were introduced by Aamri and El Moutawakil [1] to prove some common fixed point theorems for some new contractive or expansive maps in uniform space. In [1], the following contractive definition was employed: Let $f, g : X \to X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leqslant \psi(p(g(x), g(y))) \quad (x, y \in X), \tag{1}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying

(i) for each $t \in (0, +\infty)$, $0 < \psi(t)$;

(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0$, $t \in (0, +\infty)$.

The function ψ satisfies also the condition $\psi(t) < t$, for each $t > 0$.

Several stability results established in metric spaces and normed linear spaces are available in the literature, but to the best our knowledge, up till now, no stability result has been proved in uniform space. For excellent study of stability of the fixed point iterative procedures, we refer our readers to Ostrowski [20], Harder and Hicks [8], Rhoades [21, 23], Osilike [18], Osilike and Udomene [19], Jachymski [17], Berinde [3, 4] and the papers of the author [9, 10, 11, 12, 13, 14, 15]. Harder and Hicks [8], Rhoades [21, 23], Osilike [18], Osilike and Udomene [19] as well as Berinde [3] proved various stability results for certain contractive definitions. The first stability result on T-stable mappings was due to Ostrowski [20].

In this paper, our purpose is to obtain some stability results for Picard iteration in uniform space by introducing some new concept of stability in uniform space, notion of an M_e-distance as well as using two contractive conditions which are more general than (1) above.

Our results are improvements, generalizations and extensions of some of the results of Harder and Hicks [8], Rhoades [21, 22], Osilike [18], Osilike and Udomene [19], Berinde [3, 4] as well as the recent results of the author [9, 10].

2. Preliminaries

Let (X, Φ) be a uniform space.

Remark 2.1. When topological concepts are mentioned in the context of a uniform space (X, Φ), they always refer to the topological space $(X, \tau(\Phi))$.

Definition 2.1 [1]. If $V \in \Phi$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be *V-close.* A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a *Cauchy sequence* for Φ if for any $V \in \Phi$, there exists $N \geqslant 1$ such that x_n and x_m are V-close for $n, m \geqslant N$.

Definition 2.2 [1]. A function $p : X \times X \to \mathbb{R}^+$ is said to be an *A-distance* if for any $V \in \Phi$, there exists $\delta > 0$ such that $(x, y) \in V$ wherever $p(z, x) \leqslant \delta$ and $p(z, y) \leqslant \delta$ for some $z \in X$.

Definition 2.3 [1]. A uniform space (X, Φ) is said to be *Hausdorff* if the intersection of all $V \in \Phi$ reduces to the diagonal $\{(x, x) | x \in X\}$, i. e. if $(x, y) \in V$ for all $V \in \Phi$ implies $x = y$. A surrounding $V \in \Phi$ is said to be *symmetric* if $V = V^{-1} = \{(y, x) | (x, y) \in V\}$.

Definition 2.4 [1]. Let (X, Φ) be a uniform space and p be an A-distance on X.

(i) X is said to be *S-complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$.

(ii) X is said to be *p-Cauchy complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$.

Definition 2.5 [4]. A function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is called a *comparison function* if:

(i) ψ is monotone increasing;

(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0, \ \forall\ t \geqslant 0$.

Remark 2.2. Every comparison function satisfies the condition $\psi(0) = 0$.

Also, both conditions (i) and (ii) imply that $\psi(t) < t$ for all $t > 0$.

Remark 2.3. A sequence in X is p-Cauchy if it satisfies the usual metric condition.

We shall employ the following *contractive conditions*: For a selfmapping $T\colon X \to X$:

(i) there exist $a \in [0, 1)$ and a monotone increasing function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$, and for all $x, y \in X$,

$$p(Tx, Ty) \leqslant \varphi(p(x, Tx)) + ap(x, y); \tag{2}$$

(ii) there exist a continuous comparison function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ and a monotone increasing function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$, and for all $x, y \in X$, we have

$$p(Tx, Ty) \leqslant \varphi(p(x, Tx)) + \psi(p(x, y)). \tag{3}$$

REMARK 2.4. The contractive condition (2) is more general than (1) in the sense that if in (2), $\varphi(u) = 0$ for all $u \in \mathbb{R}^+$, then we obtain (1) above.

Lemma 2.1 [3, 4]**.** *If δ is a real number such that $0 \leqslant \delta < 1$, and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leqslant \delta u_n + \epsilon_n, \quad n = 0, 1, 2, \dots,$$

we have $\lim_{n\to\infty} u_n = 0$.

In addition to the notions above, we shall state the following definitions of an M_e-distance and stability of iterative process in a uniform space:

Lemma 2.2 [10, 15]**.** *If $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a subadditive comparison function and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leqslant \sum_{k=0}^{m} \delta_k \psi^k(u_n) + \epsilon_n, \quad n = 0, 1, \dots,$$

where $\delta_0, \delta_1, \dots, \delta_m \in [0, 1]$ with $0 \leqslant \sum_{k=0}^{m} \delta_k \leqslant 1$, we have $\lim_{n\to\infty} u_n = 0$.

DEFINITION 2.6. A function $p\colon X \times X \to \mathbb{R}^+$ is said to be an M_e*-distance* if

(p_1) p is an A-distance;

(p_2) $p(x, y) \leqslant p(x, z) + p(z, y)$ for all $x, y \in X$;

(p_3) $p(x, y) \leqslant p(y, x)$ for all $x, y \in X$.

REMARK 2.5. The function $p\colon X \times X \to \mathbb{R}^+$ is called an E*-distance* if p satisfies only the properties (p_1) and (p_2). See Aamri and El Moutawakil [1] and Olatinwo [16] for more on the concept of an E-distance.

DEFINITION 2.7. Let (X, Φ) be a Hausdorff uniform space and $p\colon X \times X \to \mathbb{R}^+$ an M_e-distance. Suppose that $F_T = \{u \in X | Tu = u\} \neq \varnothing$ is the set of the fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be a p-Cauchy sequence generated by an iterative process involving T defined by

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \dots, \tag{4}$$

where $x_0 \in X$ is an initial approximation and f is some function. Suppose that X is S-complete with respect to $\{x_n\}_{n=0}^{\infty}$, or X is p-Cauchy complete with respect to $\{x_n\}_{n=0}^{\infty}$. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary p-Cauchy sequence and set $\epsilon_n = p(y_{n+1}, f(T, y_n))$, $n = 0, 1, 2, \dots$. Then, the iterative process (4) is said to be T*-stable* or *stable with respect to* T if $\lim_{n\to\infty} \epsilon_n = 0$ implies that X is S-complete with respect to $\{y_n\}_{n=0}^{\infty}$, or X is p-Cauchy complete with respect to $\{y_n\}_{n=0}^{\infty}$.

REMARK 2.6. (i) Definition 2.7 reduces to that of Harder and Hicks [8] if $p = d$ and X becomes a complete metric space and the notion of S-completeness or p-Cauchy completeness reduces to the concept of convergence in complete metric space.

(ii) If in (4),

$$x_{n+1} = f(T, x_n) = Tx_n, \quad x_0 \in X,\ n = 0, 1, 2, \dots, \tag{5}$$

then we obtain the Picard iterative process.

3. Main Results

Theorem 3.1. *Let (X, Φ) be a Hausdorff uniform space and $p\colon X \times X \to \mathbb{R}^+$ an M_e-distance such that $p(x', x') = 0$ for some $x' \in \mathbb{R}^+$. Let $T\colon X \to X$ be a selfmap of X satisfying the contractive condition (2) and $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\varphi(0) = 0$. Let $x_0 \in X$ and $\{x_n\}_{n=0}^{\infty}$ the Picard iterative process associated to T defined by (5). Suppose that T has a fixed point u and let X be S-complete with respect to $\{x_n\}_{n=0}^{\infty}$. Then, the Picard iterative process is T-stable.*

$\lhd$ Let $\{y_n\}_{n=0}^{\infty} \subset X$, $\epsilon_n = p(y_{n+1}, Ty_n)$ and assume that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Then, we shall employ the properties of an M_e-distance, condition (2) as well as Lemma 2.1 to establish that X is S-complete with respect to $\{y_n\}_{n=0}^{\infty}$. That is, we shall show that $\lim_{n\to\infty} p(y_n, u) = 0$, for some $u \in F_T \subset X$:

$$\begin{aligned} p(y_{n+1}, u) &\leqslant p(y_{n+1}, Ty_n) + p(Ty_n, Tu) = p(Tu, Ty_n) + \epsilon_n \\ &\leqslant \varphi(p(u, Tu)) + ap(u, y_n) + \epsilon_n = ap(y_n, u) + \epsilon_n. \end{aligned} \tag{6}$$

Since $0 \leqslant a < 1$, Lemma 2.1 and (6) yield $\lim\limits_{n\to\infty} p(y_{n+1}, u) = 0$. Since $\lim_{n\to\infty} p(y_{n+1}, u) = 0$ for some $u \in X$, we conclude that X is S-complete with respect to $\{y_n\}_{n=0}^{\infty}$.

Conversely, let X be S-complete with respect to $\{y_n\}_{n=0}^{\infty}$. Then, we prove that $\lim_{n\to\infty} \epsilon_n = 0$. Again, by the properties of an M_e-distance and condition (2) we have, for $u \in F_T \subset X$,

$$\begin{gathered} \epsilon_n = p(y_{n+1}, Ty_n) \leqslant p(y_{n+1}, u) + p(Tu, Ty_n) \\ \leqslant p(y_{n+1}, u) + \varphi(p(u, Tu)) + ap(u, y_n) = p(y_{n+1}, u) + ap(u, y_n) \to 0 \quad \text{as} \quad n \to \infty, \end{gathered}$$

since X is S-complete with respect to $\{y_n\}_{n=0}^{\infty}$. So, it follows that $\lim_{n\to\infty} \epsilon_n = 0$.

Hence, the Picard iterative process is T-stable. $\rhd$

Remark 3.1. Theorem 3.1 is a generalization and extension of Theorem 1 and Theorem 2 of Berinde [3], Theorem 1 of Osilike [18] (and same result in [19]), Theorem 1 of Rhoades [21, 23], Theorem 2 of Harder and Hicks [8] as well as Theorem 3.1 of the author [9].

Theorem 3.1 is further generalized to the following result:

Theorem 3.2. *Let (X, Φ) be a Hausdorff uniform space and $p\colon X \times X \to \mathbb{R}^+$ an M_e-distance such that $p(x', x') = 0$ for some $x' \in \mathbb{R}^+$. Let $T\colon X \to X$ be a selfmap of X satisfying the contractive condition (3). Suppose also that $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous subadditive comparison function and $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\varphi(0) = 0$. Let $x_0 \in X$ and $\{x_n\}_{n=0}^{\infty}$ the Picard iterative process associated to T defined by (5). Suppose that T has a fixed point u and let X be S-complete with respect to $\{x_n\}_{n=0}^{\infty}$. Then, the Picard iterative process is T-stable.*

$\lhd$ Again, let $\{y_n\}_{n=0}^{\infty} \subset X$, $\epsilon_n = p(y_{n+1}, Ty_n)$ and assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then, by the properties of an M_e-distance, condition (3) and Lemma 2.2, we have for some $u \in F_T \subset X$:

$$p(y_{n+1}, u) \leqslant p(Tu, Ty_n) + \epsilon_n \leqslant \psi(p(y_n, u)) + \epsilon_n. \tag{7}$$

By using Lemma 2.2 in (7), we get $\lim_{n\to\infty} p(y_{n+1}, u) = 0$. Since $\lim_{n\to\infty} p(y_{n+1}, u) = 0$ for some $u \in X$, we deduce that X is S-complete with respect to $\{y_n\}_{n=0}^{\infty}$.

Conversely, let X be S-complete with respect to $\{y_n\}_{n=0}^{\infty}$. Then, for $u \in F_T \subset X$, we have

$$\epsilon_n = p(y_{n+1}, Ty_n) \leqslant p(y_{n+1}, u) + \psi(p(u, y_n)) \to 0 \quad \text{as} \quad n \to \infty,$$

since X is S-complete with respect to $\{y_n\}_{n=0}^{\infty}$. So, it follows that $\lim_{n\to\infty} \epsilon_n = 0$.

Hence, the Picard iterative process is T-stable. $\rhd$

Remark 3.2. Theorem 3.1 is a generalization and extension of Theorem 1 and Theorem 2 of Berinde [3], Theorem 1 of Osilike [18] (and same result in [19]), Theorem 1 of Rhoades [21, 23], Theorem 2 of Harder and Hicks [8] as well as similar results of the author [9, 10].

Remark 3.3. The proof is similar when X is p-Cauchy complete with respect to $\{y_n\}_{n=0}^{\infty}$ as S-completeness implies p-Cauchy completeness.

Remark 3.4. We obtain a direct extension of Theorem 1 of Osilike [18] if we replace the contractive conditions (2) and (3) in both Theorem 3.1 and Theorem 3.2 by the following: there exist constants $L \geqslant 0$ and $a \in [0, 1)$ such that for all $x, y \in X$,

$$p(Tx, Ty) \leqslant Lp(x, Tx) + ap(x, y). \tag{8}$$

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ОБ УСТОЙЧИВОСТИ ИТЕРАЦИОННОГО ПРОЦЕССА ПИКАРА В РАВНОМЕРНЫХ ПРОСТРАНСТВАХ

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Доказаны результаты об итерационном процессе Пикара в равномерных пространствах. С этой целью введены понятие M_e-расстояния и некоторые условия сжатия. Полученные факты обобщают, расширяют и улучшают некоторые ранее опубликованные результаты.

Ключевые слова: итерационный процесс Пикара, равномерные пространства, условия сжатия.