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SOME VECTOR VALUED MULTIPLIER DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

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In this article we introduce some new difference sequence spaces with a real 2-normed linear space as base space and which are defined using a sequence of Orlicz functions, a bounded sequence of positive real numbers and a sequence of non-zero reals as multiplier sequence. We show that these spaces are complete paranormed spaces when the base space is a 2-Banach space and investigate these spaces for solidity, symmetricity, convergence free, monotonicity and sequence algebra. Further we obtain some relation between these spaces as well as prove some inclusion results.

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1. Introduction

Throughout the paper w, ℓ_∞ , c and c_0 denote the spaces of all bounded, convergent, and null sequences $x = (x_k)$ with complex terms, respectively. The zero sequence is denoted by $\theta = (0, 0, 0, \dots)$.

The notion of difference sequence spaces was introduced by Kizmaz [11] who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [4] by introducing the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. Recently Dutta [2] introduced and studied the following difference sequence spaces:

Let r, s be non-negative integers, then for Z a given sequence space we have

$$Z(\Delta^s_{(r)}) = \Big\{x = (x_k) \in w : (\Delta^s_{(r)} x_k) \in Z\Big\},$$

where $\Delta^s_{(r)} x = (\Delta^s_{(r)} x_k) = (\Delta^{s-1}_{(r)} x_k - \Delta^{s-1}_{(r)} x_{k-r})$ and $\Delta^0_{(r)} x_k = x_k$ for all $k \in N$ and which is equivalent to the binomial representation $\Delta^s_{(r)} x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{k-rv}$.

For $s = 1$, we get the difference operator $\Delta_{(r)}$ introduced and studied by Dutta [3] for sequences of fuzzy numbers. Again $r = s = 1$, we get spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for a sequence space E the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [8] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences

(k^{-1}) and (k) respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence.

The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's. Since then, Gunawan and Mashadi [10], Dutta [1] and many others have studied this concept and obtained various results.

Let X be a real linear space of dimension greater than one and let $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following conditions:

(1) $\|x,y\| = 0$ if and only if x and y are linearly dependent vectors,

(2) $\|x,y\| = \|y,x\|$,

(3) $\|\alpha x,y\| \leqslant |\alpha| \cdot \|x,y\|$, for every $\alpha \in R$

(4) $\|x,y+z\| \leqslant \|x,y\| + \|x,z\|$

then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed linear space.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [14] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leqslant 1 \right\}.$$

Remark 1. An Orlicz function satisfies the inequality $M(\lambda x) < \lambda M(x)$, for all λ with $0 < \lambda < 1$. The following inequality will be used throughout the article.

Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leqslant \sup p_k = G$, $D = \max\left(1, 2^{G-1}\right)$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leqslant D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}$$

and for all $\lambda \in \mathbb{C}$, $|\lambda|^{p_k} \leqslant \max\left(1, |\lambda|^G\right)$.

The studies on paranormed sequence spaces were initiated by Nakano [17] and Simons [20] at the initial stage. Later on it was further studied by Maddox [15], Nanda [18], Lascardies [12], Lascardies and Maddox [13] and many others. Parasar and Choudhary [19], Mursaleen, Khan and Qamaruddin [16] and many others studied paranormed sequence spaces using Orlicz functions.

2. Definition and Preliminaries

A sequence space E is said to be: *solid* (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leqslant 1$ for all $k \in \mathbb{N}$; *monotone* if it contains the canonical preimages of all its step spaces; *symmetric* if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$, where π is a permutation on $\mathbb{N}$; *convergence free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$; *sequence algebra* if $(x_k, y_k) \in E$ whenever $(x_k) \in E$ and $(y_k) \in E$.

A sequence (x_k) in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to converge to some $L \in X$ in the 2-norm if $\lim_{k\to\infty} \|x_k - L, u\| = 0$, for every $u \in X$, and is said to be *Cauchy* sequence with respect to the 2-norm if $\lim_{k,l\to\infty} \|x_k - x_l, u\| = 0$, for every $u \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be 2-*Banach* space.

Now we give the following two familiar examples of 2-norm which will be used in the next section to construct examples.

EXAMPLE 1. Consider the spaces ℓ_∞, c and c_0 of real sequences. Let us define:

$$\|x, y\| = \sup_{i\in N} \sup_{j\in N} |x_i y_j - x_j y_i|,$$

where $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$. Then $\|\cdot,\cdot\|$ is a 2-norm on ℓ_∞, c and c_0.

EXAMPLE 2. Let us take $X = \mathbb{R}^2$ and Consider the function $\|\cdot,\cdot\|$ on X definded as:

$$\|x_1, x_2\|_E = \mathrm{abs}\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right), \quad x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2,\ i = 1, 2.$$

Then $\|\cdot,\cdot\|$ is a 2-norm on X.

Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\Lambda = (\lambda_k)$ be a sequence of non-zero reals. Let m, n be non-negative integers, then for a real linear 2-normed space $(X, \|\cdot,\cdot\|)$ and for a sequence $M = (M_k)$ of Orlicz functions we define the following sequence spaces:

$$c_0(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p) = \Big\{ x = (x_k) \in w(X) :$$
$$\lim_{k\to\infty} \left(M_k \left(\left\| \frac{\Delta^n_{(m)} \lambda_k x_k}{\rho}, z \right\| \right) \right)^{p_k} = 0,\ z \in X,\ \text{for some } \rho > 0 \Big\},$$

$$c(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p) = \Big\{ x = (x_k) \in w(X) :$$
$$\lim_{k\to\infty} \left(M_k \left(\left\| \frac{\Delta^n_{(m)} \lambda_k x_k - L}{\rho}, z \right\| \right) \right)^{p_k} = 0,\ z \in X, L \in X,\ \text{for some } \rho > 0 \Big\},$$

$$\ell_\infty(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p) = \Big\{ x = (x_k) \in w(X) :$$
$$\sup_{k\geqslant 1} \left(M_k \left(\left\| \frac{\Delta^n_{(m)} \lambda_k x_k}{\rho}, z \right\| \right) \right)^{p_k} < \infty,\ z \in X,\ \text{for some } \rho > 0 \Big\},$$

where $(\Delta^n_{(m)} \lambda_k x_k) = (\Delta^{n-1}_{(m)} \lambda_k x_k - \Delta^{n-1}_{(m)} \lambda_{k-m} x_{k-m})$ and $\Delta^0_{(m)} \lambda_k x_k = \lambda_k x_k$ for all $k \in \mathbb{N}$ and which is equivalent to the binomial representation

$$\Delta^n_{(m)} \lambda_k x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} \lambda_{k-mv} x_{k-mv}.$$

In the above expansion it is important to note that we take $x_{k-mv} = 0$ and $\lambda_{k-mv} = 0$, for non-positive values of $k - mv$.

It is obvious that

$$c_0(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p) \subset c(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p) \subset \ell_\infty(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p).$$

The inclusions are strict as follows from the following examples.

Example 3. Let $m = 2$, $n = 2$, $M_k(x) = x^2$ for all k is odd and $M_k(x) = x^6$ for all k is even, for all $x \in [0,\infty)$ and $p_k = 1$ for all $k \geqslant 1$. Consider the 2-normed space as defined in Example 2 and let the sequences $\Lambda = (k^4)$ and $x = \left(\frac{1}{k^2}, \frac{1}{k^2}\right)$. Then $x \in c(M, \|\cdot,\cdot\|, \Delta^2_{(2)}, \Lambda, p)$, but $x \notin c_0(M, \|\cdot,\cdot\|, \Delta^2_{(2)}, \Lambda, p)$.

Example 4. Let $m = 2$, $n = 2$, $M_k(x) = |x|$, for all $k \geqslant 1$ and $x \in [0,\infty)$ and $p_k = 2$ for all k odd and $p_k = 3$ for all k even. Consider the 2-normed space as defined in Example 1 and let the sequences $\Lambda = (1,1,1,\dots)$ and $x = \{1,3,2,4,5,7,6,8,9,11,10,12,\dots\}$. Then $x \in \ell_\infty(M, \|\cdot,\cdot\|, \Delta^2_{(2)}, \Lambda, p)$, but $x \notin c(M, \|\cdot,\cdot\|, \Delta^2_{(2)}, \Lambda, p)$.

Lemma 1. *If a sequence space E is solid, then E is monotone.*

3. Main Results

In this section we prove the main results of this article.

Proposition 1. *The classes of sequences $c_0(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p)$, $c(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p)$ and $\ell_\infty(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p)$ are linear spaces.*

Theorem 2. *For $Z = \ell_\infty$, c and c_0, the spaces $Z(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p)$ are paranormed sapces, paranormed by*

$$g(x) = \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho}, z\right\|\right) \leqslant 1,\ z \in X\right\},$$

where $H = \max(1, \sup_{k\geqslant 1} p_k)$.

$\lhd$ Clearly $g(x) = g(-x)$; $x = \theta$ implies $g(\theta) = 0$. Let (x_k) and (y_k) be any two sequences of the space $c_0(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p)$. Then there exist ρ_1, $\rho_2 > 0$ such that for every z in X,

$$\sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho_1}, z\right\|\right) \leqslant 1, \quad \sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k y_k}{\rho_2}, z\right\|\right) \leqslant 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by the convexity of Orlicz functions, we have for every z in X

$$\sup_k M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k + \Delta^n_{(m)}\lambda_k y_k}{\rho}, z\right\|\right) \leqslant \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_k M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho_1}, z\right\|\right) + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_k M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k y_k}{\rho_2}, z\right\|\right).$$

Hence we have,

$$\begin{aligned} g(x+y) &= \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k + \Delta^n_{(m)}\lambda_k y_k}{\rho}, z\right\|\right) \leqslant 1,\ z \in X\right\} \\ &\leqslant \inf\left\{\rho_1^{\frac{p_k}{H}} : \sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho_1}, z\right\|\right) \leqslant 1,\ z \in X\right\} \\ &+ \inf\left\{\rho_2^{\frac{p_k}{H}} : \sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k y_k}{\rho_2}, z\right\|\right) \leqslant 1,\ z \in X\right\} \Longrightarrow g(x+y) \leqslant g(x) + g(y). \end{aligned}$$

The continuity of the scalar multiplication follows from the following equality:

$$g(\alpha x)=\inf\left\{\rho^{\frac{p_k}{H}}:\sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\alpha\lambda_k x_k}{\rho},z\right\|\right)\leqslant 1,\ z\in X\right\}$$
$$=\inf\left\{(t|\alpha|)^{\frac{p_k}{H}}:\sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{t},z\right\|\right)\leqslant 1,\ z\in X\right\},$$

where $t=\frac{\rho}{|\alpha|}$ Hence the spaces $c_0(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p)$ is a paranormed space, paranormed by g. The rest of the cases will follow similarly. $\rhd$

Theorem 3. *If $(X,\|\cdot,\cdot\|)$ is a 2-Banach space, then the spaces $Z(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p)$, for $Z=\ell_\infty$, c and c_0 are complete paranormed spaces, paranormed by*

$$g(x)=\inf\left\{\rho^{\frac{p_k}{H}}:\sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\leqslant 1,\ z\in X\right\}$$

where $H=\max(1,\sup_{k\geqslant 1}p_k)$.

$\lhd$ We prove the result for the space $\ell_\infty(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p)$ and for other spaces it will follow on applying similar arguments.

Let (x^i) be any Cauchy sequence in $\ell_\infty(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p)$. Let $x_0>0$ be fixed and $t>0$ be such that for $0<\varepsilon<1$, $\frac{\varepsilon}{x_0 t}>0$ and $x_0 t>0$. Then there exists a positive integer n_0 such that $g(x^i-x^j)<\frac{\varepsilon}{x_0 t}$, for all $i,j\geqslant n_0$. Using the definition of paranorm, we get

$$\inf\left\{\rho^{\frac{p_k}{H}}:\sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k(x^i_k-x^j_k)}{\rho},z\right\|\right)\leqslant 1,\ z\in X\right\}<\frac{\varepsilon}{x_0 t}\quad(\forall\, i,j\geqslant n_0).$$

Then we get for every z in X

$$\sup_{k\geqslant 1} M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k(x^i_k-x^j_k)}{g(x^i-x^j)},z\right\|\right)\leqslant 1\quad(\forall\, i,j\geqslant n_0).$$

It follows that for every $z\in X$ and $k\geqslant 1$

$$M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k(x^i_k-x^j_k)}{g(x^i-x^j)},z\right\|\right)\leqslant 1\quad(\forall\, i,j\geqslant n_0).$$

Now for $t>0$ with $M_k\left(\frac{tx_0}{2}\right)\geqslant 1$, for each $k\geqslant 1$

$$M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k(x^i_k-x^j_k)}{g(x^i-x^j)},z\right\|\right)\leqslant M_k\left(\frac{tx_0}{2}\right),\quad z\in X.$$

This implies that

$$\left\|\Delta^n_{(m)}\lambda_k x^i_k-\Delta^n_{(m)}\lambda_k x^j_k,z\right\|\leqslant\left(\frac{tx_0}{2}\right)\left(\frac{\varepsilon}{tx_0}\right)=\frac{\varepsilon}{2},\quad z\in X.$$

Hence $\left(\Delta^n_{(m)}\lambda_k x^i_k\right)$ is a Cauchy sequence in 2-Banach space X for all $k\in N$. $\Rightarrow$ $\left(\Delta^n_{(m)}\lambda_k x^i_k\right)$ is convergent in X for all $k\in N$. For simplicity, let $\lim_{i\to\infty}\Delta^n_{(m)}\lambda_k x^i_k=y_k$ for each $k\in N$. Let $k=1$, we have

$$\lim_{i\to\infty}\Delta^n_{(m)}\lambda_1 x^i_1=\lim_{i\to\infty}\sum_{v=0}^{n}(-1)^v\binom{n}{v}\lambda_{1-mv}x^i_{1-mv}=\lim_{i\to\infty}\lambda_1 x^i_1=y_1. \tag{1}$$

Similarly we have

$$\lim_{i\to\infty}\Delta^n_{(m)}\lambda_k x^i_k=y_k,\quad k=1,2,\dots,nm. \tag{2}$$

Thus from (1) and (2) we have $\lim_{i\to\infty}x^i_{1+nm}$ exists. Let $\lim_{i\to\infty}x^i_{1+nm}=x_{1+nm}$. Proceeding in this way inductively, we have $\lim_{i\to\infty}x^i_k=x_k$ exists for each $k\in\mathbb{N}$. Now we have for all $i,j\geqslant n_0$.

$$\inf\left\{\rho^{\frac{p_k}{H}}:\sup_{k\geqslant1}M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k(x^i_k-x^j_k)}{\rho},z\right\|\right)\leqslant1,\ z\in X\right\}\leqslant\varepsilon$$

$$\Longrightarrow\lim_{j\to\infty}\inf\left\{\rho^{\frac{p_k}{H}}:\sup_{k\geqslant1}M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x^i_k-\Delta^n_{(m)}\lambda_k x^j_k}{\rho},z\right\|\right)\leqslant1,\ z\in X\right\}\leqslant\varepsilon$$

$$\Longrightarrow\lim_{j\to\infty}\inf\left\{\rho^{\frac{p_k}{H}}:\sup_{k\geqslant1}M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x^i_k-\Delta^n_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\leqslant1,\ z\in X\right\}\leqslant\varepsilon\quad(\forall\, i\geqslant n_0).$$

It follows that $(x^i-x)\in\ell_\infty(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p)$.

Since $(x^i)\in\ell_\infty\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big)$ and $\ell_\infty\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big)$ is a linear space, so we have $x=x^i-(x^i-x)\in\ell_\infty\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big)$. This completes the proof. $\rhd$

Theorem 4. *If $0<p_k\leqslant q_k<\infty$ for each k, then $Z\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big)\subseteq Z\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,q\big)$, for $Z=c_0$ and c.*

$\lhd$ We prove the result for the case $Z=c_0$ and for the other case it will follow on applying similar arguments.

Let $(x_k)\in c_0(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p)$. Then there exist some $\rho>0$ such that

$$\lim_{k\to\infty}\left(M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\right)^{p_k}=0.$$

This implies that $\left(M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\right)^{p_k}<\varepsilon$ $(0<\varepsilon\leqslant1)$ for sufficiently large k. Hence we get

$$\lim_{k\to\infty}\left(M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\right)^{q_k}\leqslant\lim_{k\to\infty}\left(M_k\left(\left\|\frac{\Delta^n_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\right)^{p_k}=0$$
$$\Longrightarrow(x_k)\in c_0\left(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,q\right).$$

Thus $c_0\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big)\subseteq c_0\big(\|M,\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,q\big)$.
Similarly, $c\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big)\subseteq c\big(\|M,\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,q\big)$. This completes the proof. $\rhd$

The following result is a consequence of Theorem 6.

Corollary 5. (a) *If $0<\inf p_k\leqslant p_k\leqslant1$, for each k, then*

$$Z\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big)\subseteq Z\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda\big),\quad Z=c_0,c.$$

(b) *If $1\leqslant p_k\leqslant\sup p_k<\infty$, for each k, then*

$$Z\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda\big)\subseteq Z\big(M,\|\cdot,\cdot\|,\Delta^n_{(m)},\Lambda,p\big),\quad Z=c_0,c.$$

Theorem 6. *$Z(M,\|\cdot,\cdot\|,\Delta^{n-1}_{(m)},\Lambda,p) \subset Z(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p)$ (in general for $i = 1,2,\dots,n-1$) $Z(M,\|\cdot,\cdot\|,\Delta^{i}_{(m)},\Lambda,p) \subset Z(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p)$), for $Z=\ell_\infty$, c and c_0.*

$\lhd$ Here we prove the result for $Z=c_0$ and for the other cases it will follow on applying similar arguments.

Let $x=(x_k)\in c_0\big(M,\|\cdot,\cdot\|,\Delta^{n-1}_{(m)},\Lambda,p\big)$. Then there exist $\rho>0$ such that

$$\lim_{k\to\infty}\left(M_k\left(\left\|\frac{\Delta^{n-1}_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\right)^{p_k}=0. \tag{3}$$

On considering 2ρ, by the convexity of Orlicz functions, we have

$$\left(M_k\left(\left\|\frac{\Delta^{n}_{(m)}\lambda_k x_k}{2\rho},z\right\|\right)\right)\leqslant\frac12\left(M_k\left(\left\|\frac{\Delta^{n-1}_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\right)+\frac12\left(M_k\left(\left\|\frac{\Delta^{n-1}_{(m)}\lambda_{k-m} x_{k-m}}{\rho},z\right\|\right)\right).$$

Hence we have

$$\left(M_k\left(\left\|\frac{\Delta^{n}_{(m)}\lambda_k x_k}{2\rho},z\right\|\right)\right)^{p_k}$$
$$\leqslant D\left\{\left(\frac12\left(M_k\left(\left\|\frac{\Delta^{n-1}_{(m)}\lambda_k x_k}{\rho},z\right\|\right)\right)\right)^{p_k}+\left(\frac12\left(M_k\left(\left\|\frac{\Delta^{n-1}_{(m)}\lambda_{k-m} x_{k-m}}{\rho},z\right\|\right)\right)\right)^{p_k}\right\}.$$

Then using (3), we get

$$\lim_{k\to\infty}\left(M_k\left(\left\|\frac{\Delta^{n}_{(m)}\lambda_k x_k}{2\rho},z\right\|\right)\right)^{p_k}=0.$$

Thus $c_0\big(M,\|\cdot,\cdot\|,\Delta^{n-1}_{(m)},\Lambda,p\big)\subset c_0\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$. $\rhd$

The inclusion is strict as follows from the following example.

Example 5. Let $m=3$, $n=2$, $M_k(x)=x^{10}$, for all $k\geqslant 1$ and $x\in[0,\infty)$ and $p_k=2$ for all k odd and $p_k=3$ for all k even. Consider the 2-normed space as defined in Example 2 and let the sequences $\Lambda=\left(\frac1k\right)$ and $x=(x_k)=(k^2,k^2)$. Then $\Delta^2_{(3)}\lambda_k x_k=0$, for all $k\in N$. Then $x\in c_0\big(M,\|\cdot,\cdot\|,\Delta^2_{(3)},\Lambda,p\big)$. Again we have $\Delta^1_{(3)}\lambda_k x_k=-3$, for all $k\in N$. Hence $x\notin c_0\big(M,\|\cdot,\cdot\|,\Delta^1_{(3)},\Lambda,p\big)$. Thus the inclusion is strict.

Theorem 7. *The following spaces $c_0\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$, $c\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$ and $\ell_\infty\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$ are not monotone and as such are not solid in general.*

$\lhd$ The proof follows from the following example. $\rhd$

Example 6. Let $n=2$, $m=3$, $p_k=1$ for all k odd and $p_k=2$ for all k even and $M_k(x)=x^2$, for all $k\geqslant 1$ and for all $x\in[0,\infty)$. consider the 2-normed space as defined in Example 1. Then $\Delta^2_{(3)}\lambda_k x_k=\lambda_k x_k-2\lambda_{k-3}x_{k-3}+\lambda_{k-6}x_{k-6}$, for all $k\in N$. Consider the J^{th} step space of a sequence space E defined as, for (x_k), $(y_k)\in E^J$ implies that $y_k=x_k$ for k odd and $y_k=0$ for k even. Consider the sequences $\Lambda=(k^3)$ and $x=\left(\frac{1}{k^2}\right)$. Then $x\in\ Z\big(M,\|\cdot,\cdot\|,\Delta^2_{(3)},\Lambda,p\big)$ for $Z=\ell_\infty$, c and c_o, but its J^{th} canonical pre-image does not belong to $Z\big(\|M,\cdot,\cdot\|,\Delta^2_{(3)},\Lambda,p\big)$ for $Z=\ell_\infty$, c and c_o. Hence the spaces $Z\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$ for $Z=\ell_\infty$, c and c_o are not monotone and as such are not solid in general.

Theorem 8. *The following spaces are not symmetric in general: $c_0\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$, $c\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$, $\ell_\infty\big(M,\|\cdot,\cdot\|,\Delta^{n}_{(m)},\Lambda,p\big)$.*

$\lhd$ The proof follows from the following example. $\rhd$

Example 7. Let $n = 2$, $m = 2$, $p_k = 2$ for all k odd and $p_k = 3$ for all k even and $M_k(x) = x^2$, for all $x \in [0,\infty)$ and for all $k \geqslant 1$. Consider the 2-normed space as defined in Example 1. Then $\Delta^2_{(2)}\lambda_k x_k = \lambda_k x_k - 2\lambda_{k-2}x_{k-2} + \lambda_{k-4}x_{k-4}$, for all $k \in N$. Consider the sequences $\Lambda = (1,1,1,\dots)$ and $x = (x_k)$ defined as $x_k = k$ for k odd and $x_k = 0$ for k even. Then $\Delta^2_{(2)}\lambda_k x_k = 0$, for all $k \in N$. Hence $(x_k) \in Z\bigl(M, \|\cdot,\cdot\|, \Delta^2_{(2)}, \Lambda, p\bigr)$, for $Z = \ell_\infty$, c and c_o. Consider the rearranged sequence, (y_k) of (x_k) defined as $(y_k) = (x_1, x_3, x_2, x_4, x_5, x_7, x_6, x_8, x_9, x_{11}, x_{10}, x_{12}, \dots)$. Then $(y_k) \notin Z(M, \|\cdot,\cdot\|, \Delta^2_{(2)}, \Lambda, p)$, for $Z = \ell_\infty$, c and c_o. Hence the spaces $Z\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$, for $Z = \ell_\infty$, c and c_o are not symmetric in general.

Theorem 9. *The following spaces are not convergence free in general:* $c_0\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$, $c\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$, $\ell_\infty\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$.

$\lhd$ The proof follows from the following example. $\rhd$

Example 8. Let $m = 3$, $n = 1$, $p_k = 6$ for all k and $M_k(x) = x^5$, for k is even and $M_k(x) = |x|$, for k is odd, for all $x \in [0,\infty)$. Then $\Delta^1_{(3)}\lambda_k x_k = \lambda_k x_k - \lambda_{k-3}x_{k-3}$, for all $k \in \mathbb{N}$. Consider the 2-normed space as defined in Example 2. Let $\Lambda = \bigl(\frac{5}{k}\bigr)$ and consider the sequences (x_k) and (y_k) defined as $x_k = \bigl(\frac{4}{5}k, \frac{4}{5}k\bigr)$ for all $k \in \mathbb{N}$ and $y_k = \bigl(\frac{1}{5}k^3, \frac{1}{5}k^3\bigr)$ for all $k \in \mathbb{N}$. Then $(x_k) \in Z\bigl(M, \|\cdot,\cdot\|, \Delta^1_{(3)}, \Lambda, p\bigr)$, but $(y_k) \notin Z\bigl(M, \|\cdot,\cdot\|, \Delta^1_{(3)}, \Lambda, p\bigr)$, for $Z = \ell_\infty$, c and c_o. Hence the spaces $Z\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$, for $Z = \ell_\infty$, c and c_o are not convergence free in general.

Theorem 10. *The following spaces are not sequence algebra in general:* $c_0\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$, $c\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$, $\ell_\infty\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$.

$\lhd$ The proof follows from the following example. $\rhd$

Example 9. Let $n = 2$, $m = 1$, $p_k = 1$ for all k and $M_k(x) = x^2$, for each $k \in \mathbb{N}$ and $x \in [0,\infty)$. Then $\Delta^2_{(1)}\lambda_k x_k = \lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}$, for all $k \in \mathbb{N}$. Consider the 2-normed space as defined in Example 2. Consider $\Lambda = \bigl(\frac{1}{k^4}\bigr)$ and let $x = (k^5, k^5)$ and $y = (k^6, k^6)$. Then $x, y \in Z\bigl(M, \|\cdot,\cdot\|, \Delta^2_{(1)}, \Lambda, p\bigr)$, $Z = \ell_\infty$ and c, but $x, y \notin Z\bigl(M, \|\cdot,\cdot\|, \Delta^2_{(1)}, \Lambda, p\bigr)$, for $Z = c_o$ Hence the spaces $c\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$, $\ell_\infty\bigl(M, \|\cdot,\cdot\|, \Delta^n_{(m)}, \Lambda, p\bigr)$ are not sequence algebra in general.

Example 10. Let $n = 2$, $m = 1$, $p_k = 3$ for all k and $M_k(x) = x^7$, for each $k \in \mathbb{N}$ and $x \in [0,\infty)$. Then $\Delta^2_{(1)}\lambda_k x_k = \lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}$, for all $k \in \mathbb{N}$. Consider the 2-normed space as defined in Example 1. Consider $\Lambda = \bigl(\frac{1}{k^6}\bigr)$ and let $x = (k^7)$ and $y = (k^7)$. Then $x, y \in c_0\bigl(M, \|\cdot,\cdot\|, \Delta^2_{(1)}, \Lambda, p\bigr)$, but $x, y \notin Z\bigl(M, \|\cdot,\cdot\|, \Delta^2_{(1)}, \Lambda, p\bigr)$, for $Z = \ell_\infty$, c. Hence the space $c_0\bigl(M, \|\cdot,\cdot\|, \Delta^2_{(1)}, \Lambda, p\bigr)$ is not sequence algebra in general.

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ВЕСОВЫЕ ПРОСТРАНСТВА ВЕКТОРНОЗНАЧНЫХ РАЗНОСТНЫХ ПОСЛЕДОВАТЕЛЬНОСТЕЙ, ОПРЕДЕЛЯЕМЫЕ ПОСЛЕДОВАТЕЛЬНОСТЬЮ ФУНКЦИЙ ОРЛИЧА

Дутта Х.

Вводятся новые классы разностных последовательностей со значениями в 2-нормированном векторном пространстве с помощью последовательности функций Орлича, ограниченной последовательности положительных чисел и весовой последовательности ненулевых вещественных чисел. Устанавливается, что эти классы являются полными паранормированными пространствами и изучаются некоторые их свойства.

Ключевые слова: разностная последовательность, 2-норма, паранорма, функция Орлича, полнота, солидность, симметричность, монотонность.