Владикавказский математический журнал
2017, Том 19, Выпуск 4, С. 50–57

A NUMERICAL METHOD FOR THE SOLUTION OF FIFTH ORDER BOUNDARY VALUE PROBLEM IN ORDINARY DIFFERENTIAL EQUATIONS

P. K. Pandey

In this article we have proposed a technique for solving the fifth order boundary value problem as a coupled pair of boundary value problems. We have considered fifth order boundary value problem in ordinary differential equation for the development of the numerical technique. There are many techniques for the numerical solution of the problem considered in this article. Thus we considered the application of the finite difference method for the numerical solution of the problem. In this article we transformed fifth order differential problem into system of differential equations of lower order namely one and four. We discretized the system of differential equations into considered domain of the problem. Thus we got a system of algebraic equations. For the numerical solution of the problem, we have the system of algebraic equations. The solution of the algebraic equations is an approximate solution of the problem considered. Moreover we get numerical approximation of first and second derivative as a byproduct of the proposed method. We have shown that proposed method is convergent and order of accuracy of the proposed method is at lease quadratic. The numerical results obtained in computational experiment on the test problems approve the efficiency and accuracy of the method.

Mathematics Subject Classification (2010): 65L10, 65L12.

Key words: boundary value problem, cubic order convergence, difference method, fifth order differential equation, odd order problems, odd-even order problems.

1. Introduction

The presences of differential equations in mathematical modelling of physical phenomena in natural sciences are common. A fifth order differential equation and corresponding boundary value problem arise in the study of dynamics of the fluid in rheology [1, 2]. In this article we consider a method for the numerical solution of the fifth order boundary value problems of the following form:

$$u^{(5)}(x) = f(x,u), \quad a < x < b, \tag{1.1}$$

where function $f(x,u)$ is regular and differentiable in $[a,b]$ and subject to the boundary conditions

$$u(a) = \alpha_1, \quad u'(a) = \alpha_2, \quad u''(a) = \alpha_3, \quad u(b) = \beta_1 \quad \text{and} \quad u'(b) = \beta_2$$

where α_1, α_2, α_3, β_1 and β_2 are real constant.

A literature on the theoretical concepts of existence and uniqueness of the solution of problem (1.1) in detail can be found in [3]. Thus the existence and uniqueness of the solution to problem (1.1) is assumed. The emphasis in this article will be on the development of a numerical method for the approximate numerical solution of the fifth order boundary value problem.

In the literature little work reported on the solution of odd higher order boundary value problems. Some work reported in the literature specially on fifth order boundary value problems are finite difference method [4], spline method [5, 6], adomian decomposition method [7], spectral Galerkin and collocation method [2, 8, 9], differential transformation [10] and references therein. Recently problem (1.1) solved by reproducing kernel method and some literary work reported in [11].

Hence, the purpose of this article is to develop numerical method for solution of fifth order boundary value problems (1.1). An odd third order boundary value problem solved and boundary conditions incorporated in natural way in [12]. Motivated by that work, we developed finite difference method for numerical solution of fifth order boundary value problem by reducing to system of odd-even order boundary value problems. To the best of our knowledge, in the literature no method similar to proposed method for the numerical solution of problem (1.1) has been reported. We hope that others may find the proposed method an improvement to those existing finite difference methods for fifth order boundary value problems.

We have presented our work in this article as follows: In Section 2 the finite difference method, in Section 3 we derived a finite difference method. In Section 4, we have discussed convergence of the proposed method under appropriate condition. The application of the proposed method on the test problems and numerical results in Section 5. A discussion and conclusion on the overall performance of the proposed method are presented in Section 6.

2. The Difference Method

Let us consider the following initial value problem,

$$\frac{du}{dx} = v(x), \quad a < x < b, \tag{2.1}$$

with the initial condition

$$u(a) = \alpha_1,$$

where $v(x)$ is some differentiable function in $[a, b]$. Then equation (1.1) transformed into the following form,

$$\frac{d^4 v}{dx^4} = f(x, u), \quad a < x < b, \tag{2.2}$$

with the boundary conditions

$$v(a) = \alpha_2, \quad v'(a) = \alpha_3, \quad v(b) = \beta_2 \quad \text{and} \quad v'(b) = \gamma_1,$$

where γ_1 is an approximate value and equal to $\frac{2(\alpha_1+(N+1)h\beta_2-\beta_1)}{((N+1)h)^2}$. Thus the fifth order boundary value problem (1.1) has been reduced to a system of lower order boundary value problems (2.1)–(2.2).

We define N finite numbers of nodal points $a \leqslant x_0 < x_1 < x_2 < \ldots < x_{N+1} \leqslant b$ using uniform step length h such that $x_i = a + ih$, $i = 0, 1, 2, \ldots, N + 1$, in $[a, b]$, the domain in which the solution of the problem (1.1) is desired. Suppose we wish to determine the numerical approximation of solution $u(x)$ of the problem (1.1) at the nodal point x_i and let u_i denotes the numerical approximation of $u(x)$ at node $x = x_i$, $i = 1, 2, \ldots, N$. Furthe let us denote f_i as the approximation of the value of the source function $f(x, u(x))$ at node $x = x_i$,

$i = 0, 1, 2, \ldots, N+1$. Thus the system boundary value problem (2.1)–(2.2), a transformed boundary value problem (1.1) at node $x = x_i$ may be written as,

$$v_i^{(4)} = f_i,$$
$$u_i' = v_i, \quad i = 1, 2, \ldots, N.$$

Following the ideas in [13, 14], we propose our finite difference method for $v(x)$, $v'(x)$ and $u(x)$ a numerical solution of problem (2.3),

$$-2(v_{i+1} - 2v_i + v_{i-1}) + h(v'_{i+1} - v'_{i-1}) = \frac{h^4}{90}(f_{i+1} + 13f_i + f_{i-1});$$
$$-3(v_{i+1} - v_{i-1}) + h(v'_{i+1} + 4v'_i + v'_{i-1}) = \frac{h^4}{60}(f_{i+1} - f_{i-1});$$
$$-hv_i + \frac{h^2}{2}v'_i + u_i - u_{i-1} = 0.$$

If the forcing function $f(x, u)$ in problem (1.1) is linear then the system of equations (2.4) will be linear otherwise we will obtain nonlinear system of equations.

3. Derivation of the Difference Method

In this section we out line the derivation of the proposed method, we have followed the same approach as given in [13, 14]. Let us write a linear combination of solution $v(x)$, $v'(x)$ and source function $f(x, u)$ at nodes $x_{x\pm1}$, x_i

$$a_1(v_{i+1} + v_{i-1}) + a_0 v_i + hb_1(v'_{i+1} - v'_{i-1}) + h^4\big(c_1(f_{i+1} + f_{i-1}) + c_0 f_i\big) = 0, \tag{3.1}$$

where a_0, a_1, b_1, c_0 and c_1 are constants to be determined. Using Taylor series expansion about the point x_i and method of undetermine coefficients, we obtain

$$(a_0, a_1, b_1, c_0, c_1) = \left(4, -2, 1, -\frac{13}{90}, -\frac{1}{90}\right). \tag{3.2}$$

Thus from (3.1)–(3.2), we have

$$-2(v_{i+1} - 2v_i + v_{i-1}) + h(v'_{i+1} - v'_{i-1}) - \frac{h^4}{90}\big(f_{i+1} + 13f_i + f_{i-1}\big) + T_i = 0, \tag{3.3}$$

where T_i is truncation error and equal to $\frac{h^8}{560}v_i^{(8)}$. Similarly we can derive the following equations

$$\begin{gathered} -3(v_{i+1} - v_{i-1}) + h(v'_{i+1} + 4v'_i + v'_{i-1}) - \frac{h^4}{60}(f_{i+1} - f_{i-1}) + T'_i = 0, \\ -hv_i + \frac{h^2}{2}v'_i + u_i - u_{i-1} + T_i^0 = 0, \end{gathered} \tag{3.4}$$

where T'_i, T_i^o are truncation errors and respectively equal to $\frac{h^7}{432}v_i^{(7)}$, $-\frac{h^3}{6}u_i^{(3)}$. Thus from (3.3)–(3.4), we conclude that the order of the proposed finite difference method (2.4) will be at least $O(h^2)$.

4. Convergence Analysis

We will consider following linear test equation for convergence analysis of the proposed method (2.4).

$$u^{(5)}(x) = f(x), \quad a < x < b. \tag{4.1}$$

$$u(a) = \alpha_1, \quad u'(a) = \alpha_2, \quad u''(a) = \alpha_3, \quad u(b) = \beta_1 \quad \text{and} \quad u'(b) = \beta_2.$$

Let $\mathbf{u}$ be the solution obtained by method (2.4) of the problem (4.1), we can write in the matrix form

$$\mathbf{Ju} = \mathbf{RH}. \tag{4.2}$$

Let $\mathbf{U}$ be the exact solution of problem (4.1). Thus finite difference method (2.4) may be written in matrix

$$\mathbf{JU} = \mathbf{RH} + \mathbf{T}. \tag{4.3}$$

Let us define an error matrix a difference between approximate and exact solution of problem (4.1), i. e. $\mathbf{E} = \mathbf{u} - \mathbf{U}$ and subtract (4.3) from (4.2), we have

$$\mathbf{E} = -\mathbf{J}^{-1}\mathbf{T}, \tag{4.4}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{A}_{11} & h\mathbf{A}_{12} & \mathbf{A}_{13} \\ -3\mathbf{A}_{21} & h\mathbf{A}_{22} & \mathbf{A}_{23} \\ -h\mathbf{A}_{31} & \frac{h^2}{2}\mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}_{3N\times 3N},$$

and

$$\mathbf{A}_{11} = \begin{pmatrix} 4 & -2 & & & 0 \\ -2 & 4 & -2 & & \\ & \ddots & \ddots & \ddots & \\ & & -2 & 4 & -2 \\ 0 & & & -2 & 4 \end{pmatrix}_{N\times N}, \qquad \mathbf{A}_{12} = \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix}_{N\times N},$$

$$\mathbf{A}_{22} = \begin{pmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ 0 & & & 1 & 4 \end{pmatrix}_{N\times N}, \qquad \mathbf{A}_{31} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}_{N\times N},$$

$$\mathbf{A}_{33} = \begin{pmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ 0 & & & -1 & 1 \end{pmatrix}_{N\times N},$$

$\mathbf{A}_{12} = \mathbf{A}_{21}$, $\mathbf{A}_{31} = \mathbf{A}_{32}$ and $\mathbf{A}_{13} = \mathbf{A}_{23} = \mathbf{0}_{N\times N}$,

$$\mathbf{u} = [v_1, \ldots, v_N, v'_1, \ldots, v'_N, u_1, \ldots, u_N]^T, \quad \mathbf{U} = [V_1, \ldots, V_N, V'_1, \ldots, V'_N, U_1, \ldots, U_N]^T,$$

$\mathbf{RH} = (d_i)_{3N\times 1}$,

$$d_i = \begin{cases} 2\alpha_2 + h\alpha_3 + \frac{h^4}{90}(f_{i+1} + 13f_i + f_{i-1}), & i = 1; \\ \frac{h^4}{90}(f_{i+1} + 13f_i + f_{i-1}), & 2 \leqslant i \leqslant N-1; \\ 2\beta_2 + h\gamma_1 + \frac{h^4}{90}(f_{i+1} + 13f_i + f_{i-1}), & i = N; \\ -3\alpha_2 - h\alpha_3 + \frac{h^4}{60}(f_{i+1} - f_{i-1}), & i = N+1; \\ \frac{h^4}{60}(f_{i+1} - f_{i-1}), & N+2 \leqslant i \leqslant 2N-1; \\ 3\beta_2 + h\gamma_1 + \frac{h^4}{60}(f_{i+1} - f_{i-1}), & i = 2N; \\ \alpha_1, & i = 2N+1; \\ 0, & 2N+2 \leqslant i \leqslant 3N \end{cases}$$

and $\mathbf{T} = (t_i)_{3N\times 1}$,

$$t_i = \begin{cases} -\frac{h^8}{560} v_i^{(8)}, & 1 \leqslant i \leqslant N; \\ -\frac{h^7}{432} v_i^{(7)}, & N+1 \leqslant i \leqslant 2N; \\ \frac{h^3}{6} u_i^{(3)}, & 2N+1 \leqslant i \leqslant 3N. \end{cases}$$

It is easy to prove that matrices $\mathbf{A}_{11}$, $\mathbf{A}_{22}$ and $\mathbf{A}_{33}$ are invertible [15, 16]. Let us define

$$v_k^{up} = \max_{j=1,2,\ldots,k-1} \|A_{jk}A_{kk}^{-1}\|,\ k = 2,3, \qquad v_k^{low} = \max_{j=k+1,3} \|A_{jk}A_{kk}^{-1}\|,\ k = 1,2,$$

$$M^* = \prod_{2\leqslant k\leqslant 3} \left(1 + v_k^{up}\right) \quad \text{and} \quad M_* = \prod_{1\leqslant k\leqslant 2} \left(1 + v_k^{low}\right).$$

Let us assume

$$M_* M^* < M_* + M^*$$

then matrix $\mathbf{J}$ is invertible [17] and

$$\|\mathbf{J}^{-1}\| \leqslant \frac{\max_k \|A_{kk}^{-1}\| M_* M^*}{M_* + M^* - M_* M^*}. \tag{4.5}$$

Thus from (4.4) and (4.5), we have

$$\|\mathbf{E}\| = \|\mathbf{J}^{-1}\mathbf{T}\| \leqslant \|T\| \frac{\max_k \|A_{KK}^{-1}\| M_* M^*}{M_* + M^* - M_* M^*}. \tag{4.6}$$

Thus from equation (4.6) it follows that $\|\mathbf{E}\|$ is bounded and it will tends to zero as h approaches to zero. This established the convergence of the method (2.4) and the order of convergence of method (2.4) is at least $O(h^2)$.

5. Numerical Results

To demonstrate the computational efficiency of method (2.4), we have considered two model problems. In each model problem, we took uniform step size h. In Table 1 and Table 2, we have shown MAE_0, MAE_1 and MAE_2 the maximum absolute error in the solution $u(x)$, derivatives of solution $u'(x)$ and $u''(x)$ of the problems (1) for different values of N. We have used the following formulas in computation of MAE_0, MAE_1 and MAE_2:

$$MAE_0 = \max_{1\leqslant i\leqslant N} |u(x_i) - u_i|,$$

$$MAE_1 = \max_{1\leqslant i\leqslant N} \left|u'(x_i) - v_i\right|,$$

$$MAE_2 = \max_{1\leqslant i\leqslant N} \left|u''(x_i) - v_i'\right|.$$

We have used Gauss Seidel and Newton–Raphson iteration method to solve respectively linear and nonlinear system of equations arised from equation (2.4). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-6} or the number of iteration reached 10^3.

PROBLEM 1. The model linear problem given by

$$u^{(5)}(x) = -4u'(x), \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 2, \quad u(1) = \exp(1)\sin(1), \quad u'(1) = \exp(1)(\sin(1)+\cos(1)).$$

The analytical solution of the problem is $u(x) = \exp(x)\sin(x)$. The MAE_0, MAE_1 and MAE_2 computed by method (2.4) for different values of N are presented in Table 1.

Table 1

Maximum absolute error (Problem 1)

N	Maximum absolute error		
	MAE_0	MAE_1	MAE_2
16	.62513351(-3)	.15091896(-3)	.54979324(-3)
32	.16224384(-3)	.56743622(-4)	.43153763(-3)
64	.44584274(-4)	.71525574(-6)	.29802322(-4)
128	.11444092(-4)	.47683716(-6)	.47683716(-4)

PROBLEM 2. The nonlinear model problem given by

$$u^{(5)}(x) = \frac{(u'(x))^2}{(5+x)^3} + \frac{23}{(5+x)^5}, \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = \ln(5), \quad u'(0) = \frac{1}{5}, \quad u''(0) = -\frac{1}{25}, \quad u(1) = \ln(6), \quad u'(1) = \frac{1}{6}.$$

The analytical solution of the problem is $u(x) = \ln(x+5)$. The MAE_0, MAE_1 and MAE_2 computed by method (2.4) for different values of N are presented in Table 2.

Table 2

Maximum absolute error (Problem 2)

N	Maximum absolute error		
	MAE_0	MAE_1	MAE_2
16	.19752979(-3)	.40866435(-3)	.24468731(-2)
32	.43869019(-4)	.20343065(-3)	.24441648(-2)
64	.11444092(-4)	.10140240(-3)	.24403818(-2)
128	.25033951(-5)	.50351024(-4)	.24273539(-2)

The accuracy in numerical approximation of solution in considered model problems increases as step size h. The order of accuracy in the numerical experiment can be estimated and it is quadratic in the numerical approximation of solution of problems. The advantage of the proposed method (2.4) is we get numerical approximation of derivatives as a byproduct. It is evident in numerical experiment that proposed method (2.4) is convergent.

6. Conclusion

To find the approximate numerical solution of fifth order boundary value problems using finite difference method has been developed. At nodal point $x = x_i$, $i = 1, 2, \ldots, N$, we have obtained a system of algebraic equations given by (2.4) which is system of linear equations if source function $f(x, u)$ is linear otherwise system of nonlinear equations. The propose method is computationally efficient and accurate; moreover we get numerical approximation of first and second derivative as a byproduct. In future work, we will deal with similar extension of the present idea to solve higher order boundary value problems. Work in this direction is in progress.

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Received November 30, 2016

Pramod Kumar Pandey
Dyal Singh College (University of Delhi),
Associate Professor
Lodhi Road, New Delhi-110003, India
E-mail: pramod_10p@hotmail.com

ЧИСЛЕННЫЙ МЕТОД РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ ПЯТОГО ПОРЯДКА ДЛЯ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Панди П. К.

В данной статье предложена методика решения граничной задачи пятого порядка как сопряженной пары граничных задач. Рассматривается граничная задача пятого порядка для обыкновенного дифференциального уравнения. Существуют различные методы численного решения этой задачи. Мы рассматриваем применение метода конечных разностей для численного решения задачи. В данной статье мы преобразовали дифференциальную задачу пятого порядка в систему дифференциальных уравнений более низкого порядка, а именно первого и четвертого. Далее, мы провели дискретизацию системы дифференциальных уравнений в рассматриваемой области и, тем самым, получили систему алгебраических уравнений. Теперь для численного решение задачи мы располагаем системой алгебраических уравнений, решение которой служит приближенным решением рассматриваемой задачи. Кроме того, мы получаем численное приближение первой и второй производных в качестве побочного продукта предлагаемого метода. Показано, что предлагаемый метод сходится и порядок точности предлагаемого метода, по меньшей мере, квадратичен. Численные результаты, полученные в ходе вычислительного эксперимента по тестовым задачам, подтверждают эффективность и точность метода.

Ключевые слова: краевая задача, сходимость кубического порядка, разностный метод, дифференциальное уравнение пятого порядка, задача нечетного порядка, задача четно-нечетного порядка.