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2-LOCAL DERIVATIONS ON ALGEBRAS OF MATRIX-VALUED FUNCTIONS ON A COMPACTUM

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This paper is dedicated to the memory
of Professor Inomjon Gulomjonovich Ganiev

The present paper is devoted to 2-local derivations. In 1997, P. Ŝemrl introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. After this, a number of paper were devoted to 2-local maps on different types of rings, algebras, Banach algebras and Banach spaces. A similar description for the finite-dimensional case appeared later in the paper of S. O. Kim and J. S. Kim. Y. Lin and T. Wong described 2-local derivations on matrix algebras over a finite-dimensional division ring. Sh. A. Ayupov and K. K. Kudaybergenov suggested a new technique and have generalized the above mentioned results for arbitrary Hilbert spaces. Namely they considered 2-local derivations on the algebra $B(H)$ of all linear bounded operators on an arbitrary Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation. Then there appeared several papers dealing with 2-local derivations on associative algebras. In the present paper 2-local derivations on various algebras of infinite dimensional matrix-valued functions on a compactum are described. We develop an algebraic approach to investigation of derivations and 2-local derivations on algebras of infinite dimensional matrix-valued functions on a compactum and prove that every such 2-local derivation is a derivation. As the main result of the paper it is established that every 2-local derivation on a $*$-algebra $C(Q, M_n(F))$ or $C(Q, \mathcal{N}_n(F))$, where Q is a compactum, $M_n(F)$ is the $*$-algebra of infinite dimensional matrices over complex numbers (real numbers or quaternoins) defined in section 1, $\mathcal{N}_n(F)$ is the $*$-subalgebra of $M_n(F)$ defined in section 2, is a derivation. Also we explain that the method developed in the paper can be applied to Jordan and Lie algebras of infinite dimensional matrix-valued functions on a compactum.

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Introduction

The present paper is devoted to 2-local derivations on algebras. Recall that a 2-local derivation is defined as follows: given an algebra A, a map $\Delta : A \to A$ (not linear in general) is called a 2-local derivation if for every x, $y \in A$, there exists a derivation $D_{x,y} : A \to A$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

In 1997, P. Šemrl introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. A similar description for the finite-dimensional case appeared later in [7]. In the paper [8] 2-local derivations have been described on matrix algebras over finite-dimensional division rings.

In [5] the authors suggested a new technique and have generalized the above mentioned results of [10] and [7] for arbitrary Hilbert spaces. Namely they considered 2-local derivations on the algebra $B(H)$ of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation. After it is also published a number of papers devoted to 2-local derivations on associative algebras.

In the present paper we also suggest another technique and apply to various associative algebras of infinite dimensional matrix-valued functions on a compactum. As a result we will have that every 2-local derivation on such an algebra is a derivation. As the main result of the paper it is established that every 2-local derivation on a $*$-algebra $C(Q, M_n(F))$ or $C(Q, \mathcal{N}_n(F))$, where Q is a compactum, $M_n(F)$ is the $*$-algebra of infinite dimensional matrices over complex numbers (real numbers or quaternoins) defined in Section 1, $\mathcal{N}_n(F)$ is the $*$-subalgebra of $M_n(F)$ defined in Section 2, is a derivation. Also we explain that the method developed in the paper can be applied to Jordan and Lie algebras of infinite dimensional matrix-valued functions on a compactum.

We conclude that there are a number of various associative algebras of infinite dimensional matrix-valued functions on a compactum every 2-local derivation of which is a derivation. The main results of this paper are new. The method of proving of these results presented in this paper is universal and can be applied to associative, Lie and Jordan algebras. Its respective modification allows to prove similar problem for Jordan and Lie algebras of infinite dimensional matrix-valued functions on a compactum. In this sense our method is useful.

1. Preliminaries

Let M be an associative algebra.

Definition. A linear map $D : M \to M$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for every two elements $x, y \in M$.

A map $\Delta : M \to M$ is called a 2-local derivation, if for every two elements $x, y \in M$ there exists a derivation $D_{x,y} : M \to M$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

It is known that each derivation D on a von Neumann algebra M is an inner derivation, that is there exists an element $a \in M$ such that

$$D(x) = ax - xa, \quad x \in M.$$

Therefore for a von Neumann algebra M the above definition is equivalent to the following one: A map $\Delta : M \to M$ is called a 2-local derivation, if for every two elements $x, y \in M$ there exists an element $a \in M$ such that $\Delta(x) = ax - xa$, $\Delta(y) = ay - ya$.

Let throughout the paper n be an arbitrary infinite cardinal number, Ξ be a set of indices of the cardinality n. Let $\{e_{ij}\}$ be a set of matrix units such that e_{ij} is a $n \times n$-dimensional matrix, i. e. $e_{ij} = (a^{\alpha\beta})_{\alpha\beta\in\Xi}$, the (i, j)-th component of which is 1, i. e. $a_{ij} = 1$, and the rest components are zeros.

Let $\{m_\xi\}$ be a set of $n \times n$-dimensional matrixes and $m_\xi = (m_\xi^{\alpha\beta})_{\alpha\beta\in\Xi}$ for every ξ. Then by $\sum_\xi m_\xi$ we denote the matrix whose components are sums of the corresponding components of matrixes of the set $\{m_\xi\}$, i. e.

$$\sum_\xi m_\xi = \left(\sum_\xi m_\xi^{\alpha\beta}\right)_{\alpha\beta\in\Xi}.$$

Here, the maximal quantity of nonzero summands of the sum $\sum_\xi m_\xi^{\alpha\beta}$ is countable.

Let throughout the paper F is the field of complex numbers $\mathbb{C}$ (real numbers $\mathbb{R}$ or quaternion body $\mathbb{H}$) and

$$M_n(F) = \Bigg\{ \sum_{i,j\in\Xi} \lambda^{ij} e_{ij} : (\forall\, i,j : \lambda^{ij} \in F)(\exists\, K \in \mathbb{R}) \\ (\forall\, n \in \mathbb{N})\Big(\forall\, \{e_{kl}\}_{kl=1}^n \subseteq \{e_{ij}\}\Big) \quad \Bigg\| \sum_{kl=1}^n \lambda^{kl} e_{kl} \Bigg\| \leqslant K \Bigg\},$$

where $\big\| \sum_{kl=1}^n \lambda^{kl} e_{kl} \big\|$ the norm of the matrix $\sum_{kl=1}^n \lambda^{kl} e_{kl}$ in the finite dimensional C^*-algebra, generated by $\{e_{kl}\}_{kl=1}^n$. It is easy to see that $M_n(F)$ is a vector space over F.

In $M_n(F)$ we introduce an associative multiplication as follows: if

$$x = \sum_{i,j\in\Xi} \lambda^{ij} e_{ij}, \quad y = \sum_{i,j\in\Xi} \mu^{ij} e_{ij}$$

are elements of $M_n(F)$ then

$$xy = \sum_{i,j\in\Xi} \Bigg[\sum_{\xi\in\Xi} \lambda^{i\xi} \mu^{\xi j} e_{ij} \Bigg].$$

With respect to this operation $M_n(F)$ becomes an associative algebra and $M_n(F) \cong B(l_2(\Xi))$, where $l_2(\Xi)$ is a Hilbert space over F with elements $\{x_i\}_{i\in\Xi}$, $x_i \in F$ for all $i \in \Xi$, $B(l_2(\Xi))$ is the associative algebra of all bounded linear operators on the Hilbert space $l_2(\Xi)$. Then $M_n(F)$ is a von Neumann algebra of infinite $n \times n$-dimensional matrices over F if $F = \mathbb{C}$ (see [2]) and $M_n(F)$ is a real von Neumann algebra if $F = \mathbb{R}$ or $\mathbb{H}$.

Recall that a Hilbert space H is an infinite dimensional inner product space which is a complete metric space with respect to the metric generated by the inner product [1, Section 1.5].

Similarly, if we take the algebra $B(H)$ of all bounded linear operators on an arbitrary Hilbert space H and if $\{q_i\}$ is an arbitrary maximal orthogonal set of minimal projections of $B(H)$, then $B(H) = \sum_{ij}^{\oplus} q_i B(H) q_j$ (see [4]).

Let throughout the paper X be a hyperstonean compactum, $C(X)$ denote the algebra of all F-valued continuous functions on X and

$$\mathscr{M} = \Bigg\{ \sum_{i,j\in\Xi} \lambda^{ij}(x) e_{ij} : \big(\forall\, i,j : \lambda^{ij}(x) \in C(X)\big)\big(\exists\, K \in \mathbb{R}\big) \\ \big(\forall\, m \in \mathbb{N}\big)\big(\forall\, \{e_{kl}\}_{kl=1}^m \subseteq \{e_{ij}\}\big) \Bigg\| \sum_{kl=1,\dots,m} \lambda^{kl}(x) e_{kl} \Bigg\| \leqslant K \Bigg\},$$

where $\big\| \sum_{kl=1,\dots,m} \lambda^{kl}(x) e_{kl} \big\|$ is the norm of $\sum_{kl=1,\dots,m} \lambda^{kl}(x) e_{kl}$ in the C^*-algebra $C(X, \overline{M_n(\mathbb{C})})$, where $\overline{M_n(\mathbb{C})}$ the finite dimensional C^*-algebra, generated by $\{e_{kl}\}_{kl=1}^m$. It is clear that $\overline{M_n(\mathbb{C})} \cong M_n(\mathbb{C})$ and

$$C\big(X, \overline{M_n(\mathbb{C})}\big) = C(X) \otimes \overline{M_n(\mathbb{C})}.$$

The set $\mathscr{M}$ is a vector space with point-wise algebraic operations. The map $\|\cdot\| : \mathscr{M} \to \mathbb{R}_+$ defined as

$$\|a\| = \sup_{\{e_{kl}\}_{kl=1}^n \subseteq \{e_{ij}\}} \Bigg\| \sum_{kl=1}^n \lambda^{kl}(x) e_{kl} \Bigg\|,$$

is a norm on the vector space $\mathscr{M}$, where $a \in \mathscr{M}$ and $a = \sum_{i,j\in\Xi} \lambda^{ij}(x)e_{ij}$.

In $\mathscr{M}$ we introduce an associative multiplication as follows: if

$$x = \sum_{i,j\in\Xi} \lambda^{ij}(x)e_{ij}, \quad y = \sum_{i,j\in\Xi} \mu^{ij}(x)e_{ij}$$

are elements of $\mathscr{M}$ then

$$xy = \sum_{i,j\in\Xi} \left[\sum_{\xi} \lambda^{i\xi}(x)\mu^{\xi j}(x)e_{ij}\right] \in \mathscr{M}.$$

With respect to this multiplication $\mathscr{M}$ becomes an associative algebra and $\mathscr{M}$ is a real or complex von Neumann algebra of type I_n by Theorem 5 in [4].

Let M be a C^*-algebra, $\Delta : M \to M$ be a 2-local derivation. Now let us show that Δ is homogeneous. Indeed, for each $x \in M$, and for $\lambda \in \mathbb{C}$ there exists a derivation $D_{x,\lambda x}$ such that $\Delta(x) = D_{x,\lambda x}(x)$ and $\Delta(\lambda x) = D_{x,\lambda x}(\lambda x)$. Then

$$\Delta(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda\Delta(x).$$

Hence, Δ is homogeneous. At the same time, for each $x \in M$, there exists a derivation D_{x,x^2} such that $\Delta(x) = D_{x,x^2}(x)$ and $\Delta(x^2) = D_{x,x^2}(x^2)$. Then

$$\Delta\big(x^2\big) = D_{x,x^2}\big(x^2\big) = D_{x,x^2}(x)x + xD_{x,x^2}(x) = \Delta(x)x + x\Delta(x).$$

In [6] it is proved that every Jordan derivation on a semi-prime algebra is a derivation. Since M is semi-prime (i. e. $aMa = \{0\}$ implies that $a = \{0\}$), the map Δ is a derivation if it is additive. Therefore, to prove that the 2-local derivation $\Delta : M \to M$ is a derivation it is sufficient to prove that $\Delta : M \to M$ is additive in the proof of Theorem 1.

2. 2-local derivations on some associative algebras of matrix-valued functions

Let Q be a compactum. Then the algebra $C(Q)$ of all continuous complex number-valued functions on Q is a C^*-algebra and by Theorem 1.17.2 in [9] the second dual space $C(Q)^{**}$ is a commutative von Neumann algebra. Hence there exists a hyperstonean compactum X such that $C(Q)^{**} \cong C(X)$. If we take the $*$-algebra $C(Q, M_n(\mathbb{C}))$ of all continuous maps of Q to $M_n(\mathbb{C})$, then we may assume that $C(Q, M_n(\mathbb{C})) \subseteq \mathscr{M}$. In this case the set $\{e_{ij}\}$ of constant functions belongs to $C(Q, M_n(\mathbb{C}))$ and the weak closure of $C(Q, M_n(\mathbb{C}))$ in $\mathscr{M}$ coincides with $\mathscr{M}$. Hence by separately weakly continuity of multiplication every derivation of $C(Q, M_n(\mathbb{C}))$ has a unique extension to a derivation on $\mathscr{M}$ [9, Lemma 4.1.4]. Therefore, if Δ is a 2-local derivation on $C(Q, M_n(\mathbb{C}))$, then for every two elements x, $y \in C(Q, M_n(\mathbb{C}))$ there exists a derivation $D_{x,y} : \mathscr{M} \to \mathscr{M}$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$, i. e. $D_{x,y}$ is a derivation of $\mathscr{M}$ (not only of $C(Q, M_n(\mathbb{C}))$). The following theorem is the key result of this section.

Theorem 1. *Let Δ be a 2-local derivation on $C(Q, M_n(\mathbb{C}))$. Then Δ is a derivation.*

We first prove some lemmas necessary for the proof of Theorem 1.

By the above arguments for every 2-local derivation Δ on $C(Q, M_n(F))$ and for each $x \in C(Q, M_n(F))$ there exist $a \in \mathscr{M}$ such that

$$\Delta(x) = ax - xa.$$

Put

$$e_{ij} := \sum_{\xi,\eta\in\Xi} \lambda^{\xi\eta} e_{\xi\eta},$$

where for all ξ, η, if $\xi = i$, $\eta = j$ then $\lambda^{\xi\eta} = \mathbf{1}$, otherwise $\lambda^{\xi\eta} = 0$, and $\mathbf{1}$ is unit of the algebra $C(Q)$. Let $\{a(ij)\} \subset \mathscr{M}$ be a subset such that

$$\Delta(e_{ij}) = a(ij)e_{ij} - e_{ij}a(ij).$$

for all i, j, let $a^{ij}e_{ij}, a^{ij} \in C(Q)$ be the (i, j)-th component of the element $e_{ii}a(ji)e_{jj}$ of $\mathscr{M}$ for all pairs of different indices i, j and let $\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}$ be the matrix with all such components, the diagonal components of which are zeros.

Lemma 1. *For each pair i, j of different indices the following equality is valid*

$$\triangle(e_{ij}) = \sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta} + a(ij)_{ii}\, e_{ij} - e_{ij}a(ij)_{jj}, \tag{1}$$

where $a(ij)^{ii}$, $a(ij)^{jj}$ are functions in $C(Q)$ which are the coefficients of the Peirce components $e_{ii}a(ij)e_{ii}$, $e_{jj}a(ij)e_{jj}$.

$\triangleleft$ Let k be an arbitrary index different from i, j and let $a(ij, ik) \in \mathscr{M}$ be an element such that

$$\Delta(e_{ik}) = a(ij, ik)e_{ik} - e_{ik}a(ij, ik) \quad \text{and} \quad \Delta(e_{ij}) = a(ij, ik)e_{ij} - e_{ij}a(ij, ik).$$

Then

$$\begin{aligned} e_{kk}\Delta(e_{ij})e_{jj} &= e_{kk}\big(a(ij, ik)e_{ij} - e_{ij}a(ij, ik)\big)e_{jj} = e_{kk}a(ij, ik)e_{ij} - 0 \\ &= e_{kk}a(ik)e_{ij} - e_{kk}e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{jj} = e_{kk}a_{ki}e_{ij} - e_{kk}e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{jj} \\ &= e_{kk}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{kk}e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{jj} = e_{kk}\left(\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}\right)e_{jj}. \end{aligned}$$

Similarly,

$$e_{kk}\Delta(e_{ij})e_{ii} = e_{kk}\left(\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}\right)e_{ii}.$$

Let $a(ij, kj) \in \mathscr{M}$ be an element such that

$$\Delta(e_{kj}) = a(ij, kj)e_{kj} - e_{kj}a(ij, kj) \quad \text{and} \quad \Delta(e_{ij}) = a(ij, kj)e_{ij} - e_{ij}a(ij, kj).$$

Then

$$\begin{aligned} e_{ii}\Delta(e_{ij})e_{kk} &= e_{ii}\big(a(ij, kj)e_{ij} - e_{ij}a(ij, kj)\big)e_{kk} \\ &= 0 - e_{ij}a(ij, kj)e_{kk} = 0 - e_{ij}a(kj)e_{kk} = 0 - e_{ij}a_{jk}e_{kk} \\ &= e_{ii}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij}e_{kk} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{kk} = e_{ii}\left(\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}\right)e_{kk}. \end{aligned}$$

Also similarly we have

$$e_{jj}\Delta(e_{ij})e_{kk} = e_{jj}\left(\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}\right)e_{kk},$$

$$e_{ii}\Delta(e_{ij})e_{ii} = e_{ii}\left(\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}\right)e_{ii},$$

$$e_{jj}\Delta(e_{ij})e_{jj} = e_{jj}\left(\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}\right)e_{jj}.$$

Hence the equality (1) is valid. $\triangleright$

We take elements of the sets $\{\{e_{i\xi}\}_\xi\}_i$ and $\{\{e_{\xi j}\}_\xi\}_j$ in pairs $(\{e_{\alpha\xi}\}_\xi, \{e_{\xi\beta}\}_\xi)$ such that $\alpha \neq \beta$. Then using the set $\{(\{e_{\alpha\xi}\}_\xi, \{e_{\xi\beta}\}_\xi)\}$ of such pairs we get the set $\{e_{\alpha\beta}\}$.

Let $x_o = \{e_{\alpha\beta}\}$ be a set $\{v^{ij}e_{ij}\}_{ij}$ such that for all i, j if $(\alpha,\beta) \neq (i,j)$ then $v_{ij} = 0 \in C(Q)$ else $v_{ij} = \mathbf{1} \in C(Q)$. Then $x_o \in C(Q, M_n(\mathbb{C}))$. Fix different indices i_o, j_o. Let $c \in \mathscr{M}$ be an element such that

$$\Delta(e_{i_oj_o}) = ce_{i_oj_o} - e_{i_oj_o}c \quad \text{and} \quad \Delta(x_o) = cx_o - x_oc.$$

Put $c = \sum_{i,j\in\Xi} c^{ij}e_{ij} \in \mathscr{M}$ and $\bar{a} = \sum_{i\neq j} a^{ij}e_{ij} + \sum_{i\in\Xi} a^{ii}e_{ii}$, where $a^{ii}e_{ii} = c^{ii}e_{ii}$ for every $i \in \Xi$.

Lemma 2. *Let ξ, η be arbitrary different indices, and let $b = \sum_{i,j\in\Xi} b^{ij}e_{ij} \in \mathscr{M}$ be an element such that*

$$\Delta(e_{\xi\eta}) = be_{\xi\eta} - e_{\xi\eta}b \quad \textit{and} \quad \Delta(x_o) = bx_o - x_ob.$$

Then $c^{\xi\xi} - c^{\eta\eta} = b^{\xi\xi} - b^{\eta\eta}$.

$\triangleleft$ We have that there exist $\bar{\alpha}$, $\bar{\beta}$ such that $e_{\xi\bar{\alpha}}, e_{\bar{\beta}\eta} \in \{e_{\alpha\beta}\}$ (or $e_{\bar{\alpha}\eta}, e_{\xi\bar{\beta}} \in \{e_{\alpha\beta}\}$, or $e_{\bar{\alpha},\bar{\beta}} \in \{e_{\alpha\beta}\}$), and there exists a chain of pairs of indices $(\hat{\alpha},\hat{\beta})$ in Ω, where $\Omega = \{(\check{\alpha},\check{\beta}) : e_{\check{\alpha},\check{\beta}} \in \{e_{\alpha\beta}\}\}$, connecting pairs $(\xi,\bar{\alpha})$, $(\bar{\beta},\eta)$, i. e.

$$(\xi,\bar{\alpha}),\ (\bar{\alpha},\ \xi_1),\ (\xi_1,\eta_1),\ \ldots,\ (\eta_2,\bar{\beta}),\ (\bar{\beta},\eta).$$

Then

$$c^{\xi\xi} - c^{\bar{\alpha}\bar{\alpha}} = b^{\xi\xi} - b^{\bar{\alpha}\bar{\alpha}}, \quad c^{\bar{\alpha}\bar{\alpha}} - c^{\xi_1\xi_1} = b^{\bar{\alpha}\bar{\alpha}} - b^{\xi_1\xi_1},$$
$$c^{\xi_1\xi_1} - c^{\eta_1\eta_1} = b^{\xi_1\xi_1} - b^{\eta_1\eta_1}, \quad \ldots, \quad c^{\eta_2\eta_2} - c^{\bar{\beta}\bar{\beta}} = b^{\eta_2\eta_2} - b^{\bar{\beta}\bar{\beta}}, \quad c^{\bar{\beta}\bar{\beta}} - c^{\eta\eta} = b^{\bar{\beta}\bar{\beta}} - b^{\eta\eta}.$$

Hence

$$c^{\xi\xi} - b^{\xi\xi} = c^{\bar{\alpha}\bar{\alpha}} - b^{\bar{\alpha}\bar{\alpha}}, \quad c^{\bar{\alpha}\bar{\alpha}} - b^{\bar{\alpha}\bar{\alpha}} = c^{\xi_1\xi_1} - b^{\xi_1\xi_1},$$
$$c^{\xi_1\xi_1} - b^{\xi_1\xi_1} = c^{\eta_1\eta_1} - b^{\eta_1\eta_1}, \quad \ldots, \quad c^{\eta_2\eta_2} - b^{\eta_2\eta_2} = c^{\bar{\beta}\bar{\beta}} - b^{\bar{\beta}\bar{\beta}}, \quad c^{\bar{\beta}\bar{\beta}} - b^{\bar{\beta}\bar{\beta}} = c^{\eta\eta} - b^{\eta\eta}.$$

and $c^{\xi\xi} - b^{\xi\xi} = c^{\eta\eta} - b^{\eta\eta}$, $c^{\xi\xi} - c^{\eta\eta} = b^{\xi\xi} - b^{\eta\eta}$.

Therefore $c^{\xi\xi} - c^{\eta\eta} = b^{\xi\xi} - b^{\eta\eta}$. $\triangleright$

Lemma 3. *Let x be an element of the algebra $C(Q, M_n(\mathbb{C}))$. Then*

$$\Delta(x) = \bar{a}x - x\bar{a},$$

where $\bar{a}$ is defined as above.

$\triangleleft$ Let $d(ij) \in \mathscr{M}$ be an element such that

$$\Delta(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) \quad \text{and} \quad \Delta(x) = d(ij)x - xd(ij)$$

and $i \neq j$. Then

$$\Delta(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) = e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} + (1-e_{ii})d(ij)e_{ij} - e_{ij}d(ij)(1-e_{jj})$$
$$= a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj} + \sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}$$

for all i, j by Lemma 1.

Since

$$e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} = a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj}$$

we have

$$(1-e_{ii})d(ij)e_{ii} = \sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii}, \quad e_{jj}d(ij)(1-e_{jj}) = e_{jj}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}$$

for all different i and j.

Let $b = \sum_{i,j\in\Xi} b^{ij}e_{ij} \in \mathscr{M}$ be an element such that

$$\Delta(e_{ij}) = be_{ij} - e_{ij}b \quad \text{and} \quad \Delta(x_o) = bx_o - x_ob.$$

Then $b^{ii} - b^{jj} = c^{ii} - c^{jj}$ by Lemma 2. We have $b^{ii} - b^{jj} = d(ij)^{ii} - d(ij)^{jj}$ since

$$be_{ij} - e_{ij}b = d(ij)e_{ij} - e_{ij}d(ij).$$

Hence

$$c^{ii} - c^{jj} = d(ij)^{ii} - d(ij)^{jj}, \quad c^{jj} - c^{ii} = d(ij)^{jj} - d(ij)^{ii}.$$

Therefore we have

$$e_{jj}\Delta(x)e_{ii} = e_{jj}(d(ij)x - xd(ij))e_{ii}$$
$$= e_{jj}d(ij)(1-e_{jj})xe_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}x(1-e_{ii})d(ij)e_{ii} - e_{jj}xe_{ii}d(ij)e_{ii}$$
$$= e_{jj}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{jj}x\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}xe_{ii}d(ij)e_{ii}$$
$$= e_{jj}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{jj}x\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii} + c^{jj}e_{jj}xe_{ii} - e_{jj}xe_{ii}c^{ii}e_{ii}$$
$$= e_{jj}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{jj}x\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii} + e_{jj}\left(\sum_{\xi} a^{\xi\xi}e_{\xi\xi}\right)xe_{ii} - e_{jj}x\left(\sum_{\xi} a^{\xi\xi}e_{\xi\xi}\right)e_{ii}$$
$$= e_{jj}\sum_{\xi,\eta\in\Xi} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{jj}x\sum_{\xi,\eta\in\Xi} a^{\xi\eta}e_{\xi\eta}e_{ii} = e_{jj}(\bar{a}x - x\bar{a})e_{ii}.$$

Let $d(ii), v, w \in \mathscr{M}$ be elements such that

$$\Delta(e_{ii}) = d(ii)e_{ii} - e_{ii}d(ii) \quad \text{and} \quad \Delta(x) = d(ii)x - xd(ii),$$

$$\Delta(e_{ii}) = ve_{ii} - e_{ii}v, \quad \Delta(e_{ij}) = ve_{ij} - e_{ij}v,$$

$$\Delta(e_{ii}) = we_{ii} - e_{ii}w, \quad \Delta(e_{ji}) = we_{ji} - e_{ji}w.$$

Then we deduce

$$(1-e_{ii})a(ij)e_{ii} = (1-e_{ii})ve_{ii} = (1-e_{ii})d(ii)e_{ii},$$

$$e_{ii}a(ji)(1-e_{ii}) = e_{ii}w(1-e_{ii}) = e_{ii}d(ii)(1-e_{ii}).$$

By Lemma 1

$$\Delta(e_{ij}) = a(ij)e_{ij} - e_{ij}a(ij) = \sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ij} - e_{ij}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta} + a(ij)^{ii}e_{ij} - e_{ij}a(ij)^{jj}$$

and

$$(1 - e_{ii})a(ij)e_{ii} = \sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii}.$$

Similarly

$$e_{ii}a(ji)(1 - e_{ii}) = e_{ii}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}.$$

Taking all this into account, we derive the following chain of equalities:

$$\begin{aligned} e_{ii}\Delta(x)e_{ii} &= e_{ii}(d(ii)x - xd(ii))e_{ii} \\ &= e_{ii}d(ii)(1 - e_{ii})xe_{ii} + e_{ii}d(ii)e_{ii}xe_{ii} - e_{ii}x(1 - e_{ii})d(ii)e_{ii} - e_{ii}xe_{ii}d(ii)e_{ii} \\ &= e_{ii}a(ji)(1 - e_{ii})xe_{ii} + e_{ii}d(ii)e_{ii}xe_{ii} - e_{ii}x(1 - e_{ii})a(ij)e_{ii} - e_{ii}xe_{ii}d(ii)e_{ii} \\ &= e_{ii}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{ii}x\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii} + e_{ii}d(ii)e_{ii}xe_{ii} - e_{ii}xe_{ii}d(ii)e_{ii} \\ &= e_{ii}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{ii}x\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii} + c^{ii}e_{ii}xe_{ii} - e_{ii}xc^{ii}e_{ii} \\ &= e_{ii}\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{ii}x\sum_{\xi\neq\eta} a^{\xi\eta}e_{\xi\eta}e_{ii} + e_{ii}\left(\sum_{\xi} a^{\xi\xi}e_{\xi\xi}\right)xe_{ii} - e_{ii}x\left(\sum_{\xi} a^{\xi\xi}e_{\xi\xi}\right)e_{ii} \\ &= e_{ii}\sum_{\xi,\eta\in\Xi} a^{\xi\eta}e_{\xi\eta}xe_{ii} - e_{ii}x\sum_{\xi,\eta\in\Xi} a^{\xi\eta}e_{\xi\eta}e_{ii} = e_{ii}(\bar{a}x - x\bar{a})e_{ii}. \end{aligned}$$

It follows that

$$\Delta(x) = \bar{a}x - x\bar{a}$$

for all $x \in C(Q, M_n(\mathbb{C}))$. ▷

Proof of Theorem 1. By Lemma 3 $\Delta(e_{ii}) = \bar{a}e_{ii} - e_{ii}\bar{a} \in \mathscr{M}$. Hence

$$\sum_{\xi} a^{\xi i}e_{\xi i} - \sum_{\xi} a^{i\xi}e_{i\xi} \in \mathscr{M}.$$

Then

$$e_{ii}\left(\sum_{\xi} a^{\xi i}e_{\xi i} - \sum_{\xi} a^{i\xi}e_{i\xi}\right) = a^{ii}e_{ii} - \sum_{\xi} a^{i\xi}e_{i\xi} \in \mathscr{M}$$

and

$$\left(\sum_{\xi} a^{\xi i}e_{\xi i} - \sum_{\xi} a^{i\xi}e_{i\xi}\right)e_{ii} = \sum_{\xi} a^{\xi i}e_{\xi i} - a^{ii}e_{ii} \in \mathscr{M}.$$

Therefore, $\sum_{\xi} a^{\xi i}e_{\xi i}$, $\sum_{\xi} a^{i\xi}e_{i\xi} \in \mathscr{M}$, i. e. $\bar{a}e_{ii}, e_{ii}\bar{a} \in \mathscr{M}$. Hence $e_{ii}\bar{a}x, x\bar{a}e_{ii} \in \mathscr{M}$ for each i. Let

$$V = \left\{ \sum_{i,j\in\Xi} \lambda^{ij}e_{ij} : \{\lambda^{ij}\}_{i,j\in\Xi} \subset C(X) \right\}.$$

Then $\bar{a}x, x\bar{a} \in V$ for each element $x = \{x^{ij}e_{ij}\} \in C(Q, M_n(\mathbb{C}))$, i. e.

$$\sum_{\xi} a^{i\xi}x^{\xi j}e_{ij}, \sum_{\xi} x^{i\xi}a^{\xi j}e_{ij} \in C(Q)e_{ij}$$

for all i, j. Thus, for all $x, y \in C(Q, M_n(\mathbb{C}))$ we have that the elements $\bar{a}x$, $x\bar{a}$, $\bar{a}y$, $y\bar{a}$, $\bar{a}(x+y)$, $(x+y)\bar{a}$ belong to V. Hence

$$\Delta(x+y) = \Delta(x) + \Delta(y)$$

by Lemma 3.

Similarly for all x, $y \in C(Q, M_n(\mathbb{C}))$ we have

$$(\bar{a}x + x\bar{a})y = \bar{a}xy - x\bar{a}y \in \mathscr{M}, \quad \bar{a}xy = \bar{a}(xy) \in V.$$

Then $x\bar{a}y = \bar{a}xy - (\bar{a}x - x\bar{a})y$ and $x\bar{a}y \in V$. Therefore

$$\bar{a}(xy) - (xy)\bar{a} = \bar{a}xy - x\bar{a}y + x\bar{a}y - xy\bar{a} = (\bar{a}x - x\bar{a})y + x(\bar{a}y - y\bar{a}).$$

Now it can be easily seen that

$$\Delta(xy) = \Delta(x)y + x\Delta(y)$$

by Lemma 3. By Section 1 Δ is homogeneous. Hence, Δ is a linear operator and a derivation. The proof is complete. $\triangleright$

If we take the $*$-algebra $C(Q, M_n(F))$, $F = \mathbb{R}$ or $\mathbb{H}$, then we can similarly prove the following theorem.

Theorem 2. *Let Δ be a 2-local derivation on $C(Q, M_n(F))$. Then Δ is a derivation.*

To prove Theorem 2, we need to repeat the proof of Theorem 1 with very minor modification.

Let $\sum_{ij}^{o} Fe_{ij}$ be the following set

$$\Bigg\{ \sum_{i,j\in\Xi} \lambda^{ij}e_{ij} : (\forall\, i,j:\ \lambda^{ij} \in F)\, (\forall\, \varepsilon > 0)\, (\exists\, n_o \in \mathbb{N})$$

$$(\forall\, n \geqslant m \geqslant n_o) \quad \left\| \sum_{i=m}^{n} \left[\sum_{k=1,\dots,i-1} \left(\lambda^{ki}e_{ki} + \lambda^{ik}e_{ik}\right) + \lambda^{ii}e_{ii} \right] \right\| < \varepsilon \Bigg\}.$$

where $\|\cdot\|$ is a norm of a matrix. Then $\sum_{ij}^{o} Fe_{ij} \subset M_n(F)$.

Theorem 3. *$\sum_{ij}^{o} Fe_{ij}$ is a C^*-algebra with respect to the algebraic operations and the norm in $M_n(F)$ (see [3]).*

$\triangleleft$ We have $\sum_{ij}^{o} Fe_{ij}$ is a normed subspace of the algebra $M_n(F)$.

Let (a_n) be a sequence of elements in $\sum_{ij}^{o} Fe_{ij}$ such that (a_n) norm converges to some element $a \in M_n(F)$. We have $e_{ii}a_ne_{jj} \to e_{ii}ae_{jj}$ at $n \to \infty$ for all i and j. Hence $e_{ii}ae_{jj} \in e_{ii}M_n(F)e_{jj}$ for all i, j. Let

$$b^n = \sum_{i=n}^{\infty} \left[\sum_{k=1}^{i} (e_{i-1,i-1}ae_{kk} + e_{kk}ae_{i-1,i-1}) + e_{ii}ae_{ii} \right]$$

and

$$c_m^n = \sum_{i=n}^{\infty} \left[\sum_{k=1}^{i} (e_{i-1,i-1}a_me_{kk} + e_{kk}a_me_{i-1,i-1}) + e_{ii}a_me_{ii} \right],$$

for any n. Then $c_m^n \to b^n$ as $m \to \infty$. It should be proven that (b^n) is a fundamental sequence.

Let $\varepsilon \in \mathbb{R}_+$ and fix n. Then there exist m_o such that for all $m > m_o$

$$\|b^n - c_m^n\| < \frac{\varepsilon}{3}.$$

Hence for every $n_o > n$ and $m > m_o$

$$\left\| \left(\sum_{k=n_o}^{\infty} e_{kk} \right) (b^n - c_m^n) \left(\sum_{k=n_o}^{\infty} e_{kk} \right) \right\| = \|b^{n_o} - c_m^{n_o}\| < \frac{\varepsilon}{3}.$$

At the same time, since $a_m \in \sum_{ij}^o Fe_{ij}$, there exists $n_1 > n_o$ such that for all $l > p > n_1$ we have

$$\|c_m^l - c_m^p\| < \frac{\varepsilon}{3}.$$

Therefore for all $l > p > n_1$ the following relations hold:

$$\|b^l - b^p\| = \|b^l - c_m^l + c_m^l - c_m^p + c_m^p - b^p\|$$
$$\leqslant \|b^l - c_m^l\| + \|c_m^l - c_m^p\| + \|c_m^p - b^p\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since ε is arbitrarily chosen, (b^n) is fundamental. Therefore $a \in \sum_{ij}^o e_{ii} M_n(F) e_{jj}$. Since the sequence (a_n) is arbitrarily chosen, $\sum_{ij}^o Fe_{ij}$ is a Banach space.

Let $\sum_{i,j \in \Xi} a_{ij}$, $\sum_{i,j \in \Xi} b_{ij}$ be arbitrary elements of the Banach space $\sum_{ij}^o Fe_{ij}$. Let $a_m = \sum_{kl=1}^m a_{kl}$, $b_m = \sum_{kl=1}^m b_{kl}$ for all natural numbers m. We have the sequence (a_m) converges to $\sum_{i,j \in \Xi} a_{ij}$ and the sequence (b_m) converges to $\sum_{i,j \in \Xi} b_{ij}$ in $\sum_{ij}^o Fe_{ij}$. Also for all n and m $a_m b_n \in \sum_{ij}^o Fe_{ij}$. Then for any n the sequence $(a_m b_n)$ converges to $\sum_{i,j \in \Xi} a_{ij} b_n$ as $m \to \infty$. Hence $\sum_{i,j \in \Xi} a_{ij} b_n \in \sum_{ij}^o Fe_{ij}$. Note that $\sum_{ij}^o Fe_{ij} \subseteq M_n(F)$. Therefore, for any $\varepsilon \in \mathbb{R}_+$ there exists n_o such that

$$\left\| \sum_{i,j \in \Xi} a_{ij} b_{n+1} - \sum_{i,j \in \Xi} a_{ij} b_n \right\| \leqslant \left\| \sum_{i,j \in \Xi} a_{ij} \right\| \|b_{n+1} - b_n\| \leqslant \varepsilon$$

for any $n > n_o$. Hence the sequence $(\{a_{ij}\} b_n)$ converges to $\sum_{i,j \in \Xi} a_{ij} \sum_{i,j \in \Xi} b_{ij}$ as $n \to \infty$. Since $\sum_{ij}^o e_{ii} M_n(F) e_{jj}$ is a Banach space, $\sum_{i,j \in \Xi} a_{ij} \sum_{i,j \in \Xi} b_{ij} \in \sum_{ij}^o Fe_{ij}$. Now, the relation $\sum_{ij}^o Fe_{ij} \subseteq M_n(F)$, implies that $\sum_{ij}^o Fe_{ij}$ is a C*-algebra. $\triangleright$

Since Fe_{ii} is a simple C^*-algebra for all i, the proof of Theorem 8 in implies that [3] the C^*-algebra $\sum_{ij}^o Fe_{ij}$ is simple.

Let $\mathscr{N}_n(F) = \sum_{ij}^o Fe_{ij}$. Then $C(Q, \mathscr{N}_n(F))$ is a real or complex C^*-algebra, where ($F = \mathbb{C}$, $\mathbb{R}$ or $\mathbb{H}$) and $C(Q, \mathscr{N}_n(F)) \subseteq \mathscr{M}$. Hence similar to Theorems 1, 2 we can prove the following theorem.

Theorem 4. *Let Δ be a 2-local derivation on $C(Q, \mathscr{N}_n(F))$. Then Δ is a derivation.*

It is known that the set $\mathscr{M}_{sa}$ of all self-adjoint elements (i. e. $a^* = a$) of $\mathscr{M}$ forms a Jordan algebra with respect to the multiplication $a \cdot b = \frac{1}{2}(ab + ba)$. The following problem can be similarly solved.

Problem 1: Develop a Jordan analog of the method applied in the proof of Theorem 1 and prove that every 2-local derivation Δ on the Jordan algebra $\mathscr{M}_{sa}$ or $C(Q, M_n(F)_{sa})$ or $C(Q, \mathscr{N}_n(F)_{sa})$ is a derivation.

It is known that the set $\mathscr{M}_k = \{a \in \mathscr{M} : a^* = -a\}$ forms a Lie algebra with respect to the multiplication $[a, b] = ab - ba$. So it is natural to consider the following problem.

Problem 2: Develop a Lie analog of the method applied in the proof of Theorem 1 and prove that every 2-local derivation Δ on the Lie algebra $\mathscr{M}_k$ or $C(Q, M_n(F)_k)$ or $C(Q, \mathscr{N}_n(F)_k)$ is a derivation.

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References

1. *Akhiezer N. I., Glazman I. M.* Theory of Linear Operators in Hilbert Space.—N. Y.: Dover Publ., Inc., 1993.—[Transl. from the Russian].
2. *Arzikulov F. N.* Infinite order and norm decompositions of C^*-algebras // Int. J. Math. Anal.—2008.—Vol. 2, № 5.—P. 255–262.
3. *Arzikulov F. N.* Infinite norm decompositions of C^*-algebras // Oper. Theory Adv. Appl.—Basel: Springer AG, 2012.—Vol. 220.—P. 11–21. DOI: 10.1186/s40064-016-3468-7.
4. *Arzikulov F. N.* Infinite order decompositions of C^*-algebras // SpringerPlus.—2016.—Vol. 5 (1).—P. 1–13. DOI: 10.1186/s40064-016-3468-7.
5. *Ayupov Sh. A., Kudaybergenov K. K.* 2-local derivations and automorphisms on $B(H)$ // J. Math. Anal. Appl.—2012.—Vol. 395.—P. 15–18. DOI: 10.1016/j.jmaa.2012.04.064.
6. *Bresar M.* Jordan derivations on semiprime rings // Proc. Amer. Math. Soc.—1988.—Vol. 104.—P. 1003–1006. DOI: 10.2307/2047580.
7. *Kim S. O., Kim J. S.* Local automorphisms and derivations on M_n // Proc. Amer. Math. Soc.—2004.—Vol. 132.—P. 1389–1392. DOI: 10.1090/S0002-9939-03-07171-5.
8. *Lin Y., Wong T.* A note on 2-local maps // Proc. Edinb. Math. Soc.—2006.—Vol. 49.—P. 701–708. DOI: 10.1017/S0013091504001142.
9. *Sakai S.* C^*-Algebras and W^*-Algebras.—Berlin etc: Springer-Verlag, 1971.
10. *Šemrl P.* Local automorphisms and derivations on $B(H)$ // Proc. Amer. Math. Soc.—1997.—Vol. 125.—P. 2677–2680. DOI: 10.1090/S0002-9939-97-04073-2.

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2-ЛОКАЛЬНЫЕ ДИФФЕРЕНЦИРОВАНИЯ НА АЛГЕБРАХ МАТРИЧНО-ЗНАЧНЫХ ФУНКЦИЙ НА КОМПАКТЕ

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В 1997 г. P. Ŝemrl ввел понятие 2-локального дифференцирования и описал 2-локальные дифференцирования на алгебре $B(H)$ всех ограниченных линейных операторов в бесконечномерном сепарабельном гильбертовом пространстве H. После этого, ряд работ был посвящен 2-локальным дифференцированиям на разных типах колец, алгебр, банаховых алгебр и банаховых пространств. Аналогичное описание для конечномерного случая появилось позднее в работе С. О. Кима и Дж. С. Кима. Й. Лин и Т. Вонг описали 2-локальные дифференцирования на матричных алгебрах над конечномерным делимым кольцом. Ш. А. Аюпов и К. К. Кудайбергенов предложили новую технику и обобщили упомянутые выше результаты для произвольных гильбертовых пространств. А именно, они рассмотрели 2-локальные дифференцирования на алгебре $B(H)$ всех линейных ограниченных операторов

в произвольном гильбертовом пространстве H и доказали, что всякое 2-локальное дифференцирование на $B(H)$ является дифференцированием. После этого опубликован ряд работ, посвященных 2-локальным дифференцированиям на ассоциативных алгебрах.
В настоящей работе описаны 2-локальные дифференцирования на различных алгебрах бесконечномерных матрично-значных функций на компакте. Мы развиваем алгебраический подход к исследованию дифференцирований и 2-локальных дифференцирований на алгебрах бесконечномерных матрично-значных функций на компакте и доказываем, что каждое такое 2-локальное дифференцирование является дифференцированием. В качестве основного результата работы установлено, что каждое 2-локальное дифференцирование на $*$-алгебре $C(Q, M_n(F))$ или $C(Q, \mathcal{N}_n(F))$, где Q — компакт, $M_n(F)$ — $*$-алгебра бесконечномерных матриц над комплексными числами (вещественными числами или кватернионами), $\mathcal{N}_n(F)$ — $*$-подалгебра в $M_n(F)$ является дифференцированием. Также поясняется, что разработанный в данной работе метод может быть применен к йордановым и лиевым алгебрам бесконечномерных матрично-значных функций на компакте.

Ключевые слова: дифференцирование, 2-локальное дифференцирование, ассоциативная алгебра, C^*-алгебра, алгебра фон Неймана.