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FUNCTIONS WITH UNIFORM SUBLEVEL SETS ON CONES

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Abstract. Extended real-valued functions on a real vector space with uniform sublevel sets are important in optimization theory. Weidner studied these functions in [1]. In the present paper, we study the class of these functions, which coincides with the class of Gerstewitz functionals, on cones. These cone are not necessarily embeddable in vector spaces. Almost any Weidner's results are not true on cones without extra conditions. We show that the mentioned conditions are necessary, by nontrivial examples. Specially for element k from the cone $\mathscr{P}$, we define k-directional closed subsets of the cone and prove some properties of them. For a subset A of the cone $\mathscr{P}$, we characterize domain of the $\varphi_{A,k}$ (function with uniform sublevel set) and show that this function is k-transitive. One of the important conditions for satisfying the results, is that k has the symmetric element in the cone. Also, we prove that, under some conditions, the class of Gerstewitz functionals coincides with the class of k-translative functions on $\mathscr{P}$.

Keywords: cone, sublevel set.

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Introduction

The Gerstewitz functionals are suitable tools for studying in optimization, decision theory, mathematical finance for certain targets. The formula for the functionals introduced under more restrictive assumptions in [2]. The functionals in [3] and [4] were used in separation theorems for nonconvex sets and applied to scalarization in vector optimization. A descent method for the solution of set-valued optimization problems by means of the Gerstewitz functionals was presented in [5].

Some important mathematical setting, however, while close to the structure of vector spaces do not allow subtraction of their elements or multiplication by negative scalars. For example, the collection of convex subsets of vector spaces which are of interest in various contexts, do not form vector spaces. The classes of functions that may take infinite values or characterized through inequalities rather than equalities can be another example of this kind. These type of examples may not even be embedded into larger vector spaces. The cones includes most of these settings. These cones developed in [6] and [7] as the theory of locally convex cones by means of an order theoretic structure or a convex quasiuniform structure. Many of properties in functional analysis studied in locally convex cones (for examples, see [8–11].

A *cone* is defined to be a commutative monoid $\mathscr{P}$ together with a scalar multiplication by nonnegative real numbers satisfying the same axioms as for vector spaces; that is, $\mathscr{P}$ is endowed with an addition $(x, y) \mapsto x + y : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$ which is associative, commutative and admits a neutral element 0, i. e. it satisfies:

$$x + (y + z) = (x + y) + z,$$
$$x + y = y + x,$$
$$x + 0 = x,$$

for all $x, y, z \in \mathscr{P}$, and with a scalar multiplication $(r, x) \mapsto r \cdot x : \mathbb{R}_+ \times \mathscr{P} \to \mathscr{P}$ satisfying for all $x, y \in \mathscr{P}$ and $r, s \in \mathbb{R}_+$:

$$r \cdot (x + y) = r \cdot x + r \cdot y,$$
$$(r + s) \cdot x = r \cdot x + s \cdot x,$$
$$(rs) \cdot x = r \cdot (s \cdot x),$$
$$1 \cdot x = x,$$
$$0 \cdot x = 0.$$

The extended real numbers system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + (+\infty) = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$.

For cones $\mathscr{P}$ and $\mathscr{Q}$ a mapping $t : \mathscr{P} \to \mathscr{Q}$ is called a *linear operator* if $t(a+b) = t(a)+t(b)$ and $t(\alpha a) = \alpha t(a)$ hold for $a, b \in \mathscr{P}$ and $\alpha \geqslant 0$.

A *linear functional* on a cone $\mathscr{P}$ is a linear mapping $\mu : \mathscr{P} \to \overline{\mathbb{R}}$.

In this paper, we study the class of these functions, which coincides with the class of the Gerstewitz functionals, on cones. Almost any weidner's results are not true on cones without extra conditions. We show that the mentioned conditions are necessary, by examples.

1. Functions with Uniform Sublevel Sets

Let $\mathscr{P}$ be a cone, $A \subseteq \mathscr{P}$, $k \in \mathscr{P}$, $n \in \mathbb{R}_+$ and $M \subseteq \mathbb{R}$. We define

$$A + nk = \{x + nk : \ x \in A\}, \tag{1}$$

$$A - nk = \{x \in \mathscr{P} : \ x + nk \in A\}, \tag{2}$$

$$A + Mk = \bigcup_{\substack{m \in M, \\ a \in A}} \{a\} + mk. \tag{3}$$

In the definition of $A - nk$, we note that a member of a cone does not necessarily have an additive symmetry. It is easy to see that $A+nk$ is just the set A shifted by nk and $A-nk$ is the largest set which, if it is shifted by nk, is contained in A, i. e. for each $B \subseteq \mathscr{P}$ that $B+nk \subseteq A$, we have $B \subseteq (A - nk)$. Also $0^+A := \{u \in \mathscr{P} : \ a + tu \in A \text{ for all } a \in A, \ t \in \mathbb{R}_+\}$ denotes the recession cone of A. For a functional $\varphi : \mathscr{P} \to \underline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we use its effective domain dom $\varphi := \{y \in \mathscr{P} : \varphi(y) \in \mathbb{R} \cup \{-\infty\}\}$ and its epigraph epi $\varphi := \{(y, t) \in \mathscr{P} \times \mathbb{R} : \varphi(y) \leqslant t\}$.

The sublevel sets of φ are $\text{sublev}_\varphi(t) := \{y \in \mathscr{P} : \varphi(y) \leqslant t\}$ with$t \in \mathbb{R}$. Also, we set $\inf \varnothing = +\infty$.

Definition 1.1. Let $\mathscr{P}$ be a cone and A a subset of $\mathscr{P}$. The k-directional closure $cl_k(A)$ of A consists of all elements $y \in \mathscr{P}$ with the following property: For each $\lambda \in \mathbb{R}_> := \{x \in \mathbb{R} \ : \ x > 0\}$, there exists some $t \in \mathbb{R}_+$ with $t < \lambda$ such that $y \in A + tk$.

A is said to be k-directionally closed if $A = \mathrm{cl}_k(A)$.

The following three lemmas are in the state of vector spaces in [1].

Lemma 1.1. *Let A be a subset of the cone $\mathscr{P}$. A is k-directionally closed if and only if, for each $y \in \mathscr{P}$, we have $y \in A$ whenever there exists some sequence $(t_n)_n \in \mathbb{N}$ of real numbers with $t_n \searrow 0$ and $y \in A + t_n k$.*

Lemma 1.2. *Let A be a subset of the cone $\mathscr{P}$.*
(a) $A \subseteq \mathrm{cl}_k(A)$.
(b) $\mathrm{cl}_k(A) \subseteq \mathrm{cl}_k(B)$ if $A \subseteq B \subseteq \mathscr{P}$.
(c) $\mathrm{cl}_k(A + y) = \mathrm{cl}_k(A) + y$ for each $y \in \mathscr{P}$

$\lhd$ (a) is clear $(t = 0)$. For (b), let $a \in \mathrm{cl}_k(A)$ and $\lambda \in \mathbb{R}_>$ be arbitrary. Then there exists some $t \in \mathbb{R}_+$ with $t < \lambda$ such that $a \in A + tk \subseteq B + tk$. Then $a \in \mathrm{cl}_k(B)$. For (c), we have $x \in \mathrm{cl}_k(A + y)$ if and only if for each $\lambda \in \mathbb{R}_>$, there exists some $t \in \mathbb{R}_+$ with $t < \lambda$ such that $x \in A + y + tk = A + tk + y$ if and only if $x \in \mathrm{cl}_k(A) + y$. $\rhd$

Lemma 1.3. *Let A be a subset of the cone $\mathscr{P}$. If $k \in 0^+A$, and there exists $-k \in \mathscr{P}$ such that $k + (-k) = 0$, then $\mathrm{cl}_{-k}(A)$ consists of all elements $y \in \mathscr{P}$ for which $y + tk \in A$ for all $t \in \mathbb{R}_>$.*

$\lhd$ Let $y \in \mathrm{cl}_{-k}(A)$. Then for each $t \in \mathbb{R}_>$, there exists some $t' \in \mathbb{R}_+$ with $t' < t$ such that $y \in A + t'(-k)$ and so $y + t'k \in A$. This yields $y + t'k + (t - t')k \in A$ since $k \in 0^+A$. Hence $y + tk \in A$ for all $t \in \mathbb{R}_>$.

On the other hand, if $y + tk \in A$ for all $t \in \mathbb{R}_>$, then $y \in A + t(-k)$ for all $t \in \mathbb{R}_>$ and so $y \in \mathrm{cl}_{-k}(A)$.

Proposition 1.1. *Let $\mathscr{P}$ be a cone and $A \subseteq \mathscr{P}$. Then*

$$(A - nk) + mk \subseteq A + (m - n)k \subseteq (A + mk) - nk$$

for all $k \in \mathscr{P}$ and $n, m \in \mathbb{R}_+$. Furthermore if $mn \geqslant 0$ or there exists $-k \in \mathscr{P}$ such that $k + (-k) = 0$, then

$$(A + nk) + mk = A + (m + n)k,$$

for all $n, m \in \mathbb{R}$.

$\lhd$ Let $y \in (A - nk) + mk$.

$$y \in (A - nk) + mk \quad \Leftrightarrow \quad \exists\, x \in A - nk,\ y = x + mk \quad \Leftrightarrow \quad x + nk \in A, y = x + mk. \tag{4}$$

We have two cases:

Case I. Let $n \geqslant m \geqslant 0$. We have $x + nk = x + mk + (n - m)k$. Now (4) implies $y + (n - m)k \in A$ which is equivalent to $y \in A + (m - n)k$. This yields that

$$(A - nk) + mk \subseteq A + (m - n)k.$$

Let $y \in A + (m - n)k$. This means that $y \in A - (n - m)k$. We have $y + (n - m)k \in A$ by (2). Adding mk to both sides, it follows that $y + nk \in A + mk$. This yields that $y \in (A + mk) - nk$. We have

$$A + (m - n)k \subseteq (A + mk) - nk.$$

Case II. Let $m > n \geqslant 0$. Then (4) is equivalent to $y = x + nk + (m - n)k$, which implies that $y \in A + (m - n)k$. This yields

$$(A - nk) + mk \subseteq A + (m - n)k.$$

By the similar proof to case I, we have the second inclusion, i. e.

$$(A - nk) + mk \subseteq A + (m - n)k.$$

For the second part, first let $n, m \geqslant 0$. We have

$$\begin{aligned} y \in (A + nk) + mk &\Leftrightarrow \exists\, x \in A + nk,\ y = x + mk \\ &\Leftrightarrow \exists\, z \in A,\ x = z + nk \\ &\Leftrightarrow y = z + nk + mk \\ &\Leftrightarrow y = z + (n + m)k \\ &\Leftrightarrow y \in A + (n + m)k. \end{aligned}$$

Also

$$\begin{aligned} y \in (A - nk) - mk &\Leftrightarrow y + mk \in A - nk \\ &\Leftrightarrow y + mk + nk \in A \\ &\Leftrightarrow y + (n + m)k \in A \\ &\Leftrightarrow y \in A + (-n - m)k. \end{aligned}$$

Now, let there exists $-k \in \mathscr{P}$ such that $k + (-k)=0$. Let $t \geqslant 0$ be arbitrary. We have

$$\begin{aligned} y \in A - tk &\Leftrightarrow y + tk \in A \\ &\Leftrightarrow y + tk + t(-k) \in A + t(-k) \\ &\Leftrightarrow y \in A + t(-k). \end{aligned}$$

Then $A - tk = A + t(-k)$, for all $t \geqslant 0$. Therefore

$$\begin{aligned} y \in (A + mk) - nk &\Leftrightarrow \exists\, x \in A,\ y + nk = x + mk \\ &\Leftrightarrow y + m(-k) = x + n(-k) \in A + n(-k) \\ &\Leftrightarrow y \in (A + n(-k)) - m(-k) \\ &\Leftrightarrow y \in (A - nk) + mk. \end{aligned}$$

Then

$$y \in (A + mk) - nk = (A - nk) + mk. \ \triangleright$$

The following example shows that if $-k \notin \mathscr{P}$ and $mn < 0$, we do not necessarily have $(A + nk) + mk = A + (m + n)k$.

Example 1.1. The cone $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, with the usual algebraic operations (especially $0 \cdot (+\infty) = 0$), is a cone which is not embedded in any vector space. Let $A = [0, +\infty]$, $k = +\infty$ and $-n = m > 0$. So $A + nk = A - (-nk) = \{x \in \bar{\mathbb{R}} \ : \ x + (-nk) \in A\} = \{x \in \bar{\mathbb{R}} \ : \ x + (+\infty) \in A\} = \bar{\mathbb{R}}$ and then $(A+nk)+mk = \{+\infty\}$. But $A+(m+n)k = A+0\cdot(+\infty) = A$. So $(A + nk) + mk \subsetneqq A + (m + n)k$.

In the following proposition, which is similar to Proposition 3.1 in [1], we construct functions on cones with the following properties.

Proposition 1.2. *Let $\mathscr{P}$ be a cone. Consider a functional $\varphi : \mathscr{P} \to \overline{\mathbb{R}}$.*

(1) $\operatorname{sublev}_\varphi(t) = \operatorname{sublev}_\varphi(0) + tk$ *for all* $t \in \mathbb{R}$.

(2) $\operatorname{epi}\ \varphi = \{(y, t) \in \mathscr{P} \times \mathbb{R} \ : \ y \in \operatorname{sublev}_\varphi(0) + tk\}$.

(3) $\varphi(y) = \inf\{t \in \mathbb{R} \ : \ y \in \operatorname{sublev}_\varphi(0) + tk\}$ *for all* $y \in \mathscr{P}$.

(4) $\varphi(x) = \varphi(y) + \lambda$ *for all* $\lambda \in \mathbb{R}$, $y \in \mathscr{P}$ *and* $x \in \{\ y\} + \lambda k$.
We call that (4) *is the k-transitivity for the mapping* φ.

Then we have (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3).

Moreover, if there exists $-k \in \mathscr{P}$, *such that* $k + (-k) = 0$, *then* (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1) *and so all statements are equivalent.*

$\lhd$ (1) $\Longleftrightarrow$ (2) is straightforward.

(2) $\Longrightarrow$ (3) is similar to Proposition 3.1 from [1].

Now, let there exists $-k \in \mathscr{P}$ such that $k + (-k) = 0$.

For (3) $\Longrightarrow$ (4), let $\lambda \in \mathbb{R}$ be arbitrary, $y \in \mathscr{P}$ and $x \in \{y\} + \lambda k$. If $\lambda \leqslant 0$, then by the definition $x + |\lambda| k \in \{y\}$. Then $x + |\lambda| k = y$ and so (3) implies

$$\begin{aligned}\varphi(x) &= \inf\{t \in \mathbb{R} :\ x \in \operatorname{sublev}_\varphi(0) + tk\} \\ &= \inf\{t \in \mathbb{R} :\ x + |\lambda| k \in \operatorname{sublev}_\varphi(0) + tk + |\lambda| k\} \\ &= \inf\{t \in \mathbb{R} :\ y \in \operatorname{sublev}_\varphi(0) + (t + |\lambda|)k\} \\ &= \varphi(y) - |\lambda| = \varphi(y) + \lambda.\end{aligned}$$

If $\lambda \geqslant 0$, then $x \in \{y\} + \lambda k$ implies $x = y + \lambda k$. We have $y \in \{y + \lambda k\} + (-\lambda)k$ and so $y \in \{x\} + (-\lambda)k$. Now by the previews case, we conclude that $\varphi(y) = \varphi(x) + (-\lambda)$ and so $\varphi(x) = \varphi(y) + \lambda$.

For (4) $\Longrightarrow$ (1) we have

$$\begin{aligned}\operatorname{sublev}_\varphi(0) + tk &= \{y + tk :\ \varphi(y) \leqslant 0,\ y \in \mathscr{P}\} \\ &= \{x \in \mathscr{P} :\ \varphi(x + (-tk)) \leqslant 0\}\ \ (x := y + tk \Longleftrightarrow y = x + (-tk)) \\ &= \{x \in \mathscr{P} :\ \varphi(x) - t \leqslant 0\}\ \ (\text{by } (4)) \\ &= \{x \in \mathscr{P} :\ \varphi(x) \leqslant t\} = \operatorname{sublev}_\varphi(t).\end{aligned}$$

Then (4) $\Longrightarrow$ (1) was proved. $\rhd$

Remark 1.1. We note that (4) implies $\operatorname{sublev}_\varphi(0) + tk \subseteq \operatorname{sublev}_\varphi(t)$ generally. Indeed, for all $t \in \mathbb{R}$, we have:

$$\begin{aligned}\operatorname{sublev}_\varphi(0) + tk &= \{y + tk :\ \varphi(y) \leqslant 0,\ y \in \mathscr{P}\} \\ &= \{x \in \mathscr{P} :\ \exists\, y \in \{x\} - tk,\ \varphi(x - tk) \leqslant 0\}\ \ (x := y + tk \Longleftrightarrow y \in \{x\} - tk) \\ &= \{x \in \mathscr{P} :\ \exists\, y \in \{x\} - tk,\ \varphi(x) - t \leqslant 0\} \\ &= \{x \in \mathscr{P} :\ \exists\, y \in \{x\} - tk,\ \varphi(x) \leqslant t\} \subseteq \operatorname{sublev}_\varphi(t).\end{aligned}$$

The following example shows that (4) does not necessarily imply (1) even the cancelation law holds for k ($a + k = b + k \Longleftrightarrow a = b$ for all $a, b \in \mathscr{P}$).

Example 1.2. Consider the cone $\mathbb{R}_+$. Let $\varphi(y) = y$ (the identity mapping). By assuming $k = 1$, we have $a + k = b + k$ implies $a = b$ for all $a, b \in \mathbb{R}_+$. Let $\lambda \in \mathbb{R}$ be arbitrary. For each $x, y \in \mathbb{R}_+$ with $x \in \{\ y\} + \lambda k$, we have $\varphi(x) = \varphi(y) + \lambda$. This shows that (4) holds. On the other hand, $\operatorname{sublev}_\varphi(0) = \{0\}$. Then $\operatorname{sublev}_\varphi(0) + \lambda k = \{\lambda\}$, but $\operatorname{sublev}_\varphi(\lambda) = [0, \lambda]$ for $\lambda > 0$. This conclude that (1) dose not hold for $\lambda > 0$.

We note that $\operatorname{sublev}_\varphi(0) + \lambda k = \operatorname{sublev}_\varphi(\lambda) = \emptyset$ for $\lambda \leqslant 0$.

Now, we define functional with uniform sublevel sets is of the following type, which is often referred to in the literature as the Gerstewitz functional.

Definition 1.2. Let A be a subset of the cone $\mathscr{P}$. The function $\varphi_{A,k} : \mathscr{P} \to \overline{\mathbb{R}}$ is defined by

$$\varphi_{A,k}(y) := \inf\{t \in \mathbb{R} : y \in A + tk\}.$$

Proposition 1.3. *Let A be a subset of the cone $\mathscr{P}$. We have*
(1) dom $\varphi_{A,k} = A + \mathbb{R}k$. *Moreover, if there is $-k \in \mathscr{P}$ such that $k + (-k) = 0$, then*
(2) $\varphi_{A,k}$ *is k-translative.*
(3) $A + tk \subseteq \mathrm{sublev}_{\varphi_{A,k}}(t)$ *for all $t \in \mathbb{R}$.*
(4) *The condition*

$$\mathrm{sublev}_{\varphi_{A,k}}(t) = A + tk \quad \textit{for all} \quad t \in \mathbb{R}$$

holds if and only if $-k \in 0^+A$ and A is k-directionally closed.

$\lhd$ (1) follows immediately from the definition of $\varphi_{A,k}$.

Now, let $-k \in \mathscr{P}$, such that $k + (-k) = 0$. For (2) let $\lambda \in \mathbb{R}$ be arbitrary, $y \in \mathscr{P}$ and $x \in \{y\} + \lambda k$. If $\lambda \leqslant 0$, then by the definition $x + |\lambda|k = y$ and

$$\begin{aligned}\varphi_{A,k}(x + |\lambda|k) &= \inf\{t \in \mathbb{R} : \ x + |\lambda|k \in A + tk\} \\ &= \inf\{t \in \mathbb{R} : \ x + |\lambda|k + |\lambda|(-k) \in (A + tk) + |\lambda|(-k)\} \\ &= \inf\{t \in \mathbb{R} : \ x \in A + (t - |\lambda|)k\} \quad \text{(by Proposition 1.1)} \\ &= \varphi_{A,k}(x) + |\lambda|.\end{aligned}$$

If $\lambda \geqslant 0$, then $x \in \{y\} + \lambda k$ implies $x = y + \lambda k$. We have $y \in \{y + \lambda k\} + (-\lambda)k$ and so $y \in \{x\} + (-\lambda)k$. Now by the previous case, we conclude that $\varphi_{A,k}(y) = \varphi_{A,k}(x) + (-\lambda)$ and so $\varphi_{A,k}(x) = \varphi_{A,k}(y) + \lambda$.

For (3), it is easy to see that $A \subseteq \mathrm{sublev}_{\varphi_{A,k}}(0)$. We have

$$\begin{aligned}A + tk &\subseteq \mathrm{sublev}_{\varphi_{A,k}}(0) + tk \\ &= \{y + tk : \ \varphi_{A,k}(y) \leqslant 0, \ y \in \mathscr{P}\} \\ &= \{x \in \mathscr{P} : \ \varphi_{A,k}(x - tk) \leqslant 0\} \\ &= \{x \in \mathscr{P} : \ \varphi_{A,k}(x) - t \leqslant 0\} \quad \text{(by (2))} \\ &= \{x \in \mathscr{P} : \ \varphi_{A,k}(x) \leqslant t\} = \mathrm{sublev}_{\varphi_{A,k}}(t).\end{aligned} \tag{5}$$

For (4), let A be a k-directionally closed set such that $-k \in 0^+A$. Let $y \in \mathrm{sublev}_{\varphi_{A,k}}(0)$. We have $\varphi_{A,k}(y) \leqslant 0$. If $\varphi_{A,k}(y) < 0$, then there exists $t > 0$, such that $y \in A - tk$ and so $y \in A + t(-k)$. Then $y \in A$ (since $-k \in 0^+A$). Now, if $\varphi_{A,k}(y) = 0$, then for each positive real λ there exists $0 \leqslant t < \lambda$ such that $y \in A + tk$. This yields that $y \in \mathrm{cl}_k(A)$. Since A is k-directionally closed, so $y \in A$. This shows that $\mathrm{sublev}_{\varphi_{A,k}}(0) \subseteq A$. Then $A = \mathrm{sublev}_{\varphi_{A,k}}(0)$. We conclude that $A + tk = \mathrm{sublev}_{\varphi}(0) + tk$. By considering (5), we have $\mathrm{sublev}_{\varphi_{A,k}}(t) = A + tk$.

Conversely, by the assumption, we have $A = \mathrm{sublev}_{\varphi_{A,k}}(0)$ (take $t = 0$). Let $y \in \mathrm{cl}_k(A)$ be arbitrary. Then for each $\lambda \in \mathbb{R}_>$ there exists $t \in \mathbb{R}_+$ with $t < \lambda$ such that $y \in \mathrm{sublev}_{\varphi_{A,k}}(0) + tk = A + tk$. Then $\varphi_{A,k}(y) \leqslant 0$ and so $y \in \mathrm{sublev}_{\varphi_{A,k}}(0) = A$. This shows that A is k-directionally closed. Now let $y \in A$. For each $t \in \mathbb{R}_+$, we have $y + t(-k) \in A + t(-k) = A - tk$ and so $\varphi_{A,k}(y + t(-k)) \leqslant 0$. Then $y + t(-k) \in \mathrm{sublev}_{\varphi_{A,k}}(0) = A$ for each $t \in \mathbb{R}_+$. This yields that $-k \in 0^+A$. $\rhd$

Corollary 1.1. *Assume $\varphi : \mathscr{P} \to \overline{\mathbb{R}}$ has uniform sublevel sets described by* (1) *in Proposition* 1.2. *If $-k \in \mathscr{P}$, then each of these sublevel sets is k-directionally closed, and $-k$ belongs to the recession cone of each of these sublevel sets.*

$\lhd$ By Proposition 1.2 we have $\varphi = \varphi_{A,k}$ for $A = \mathrm{sublev}_{\varphi}(0)$. Also the statement (1) in this proposition implies that $\mathrm{sublev}_{\varphi}(t) = \mathrm{sublev}_{\varphi_{A,k}}(t) = A + tk$ for all $t \in \mathbb{R}$. Thus, Proposition 1.3 yields $-k \in 0^+A$ and that A is k-directionally closed. Hence, for each $t \in \mathbb{R}$, $-k \in 0^+(A + tk)$ and $A + tk$ is k-directionally closed by Lemma 1.2 (c). $\rhd$

Theorem 1.1. *Let A be a subset of the cone $\mathscr{P}$. Consider $\tilde{A} := \text{sublev}_{\varphi_{A,k}}(0)$. If $-k \in \mathscr{P}$ such that $k + (-k) = 0$, then*

(1) *$\tilde{A}$ is the unique set for which*

$$\text{sublev}_{\varphi_{A,k}}(t) = \tilde{A} + tk \quad \textit{for all} \quad t \in \mathbb{R} \tag{6}$$

holds.

(2) *$\tilde{A}$ is the unique set with the following properties:*

(a) *$\tilde{A}$ is k-directionally closed,*

(b) *$-k \in 0^+\tilde{A}\backslash\{0\}$ and*

(c) *$\varphi_{A,k}$ coincides with $\varphi_{\tilde{A},k}$ on $\mathscr{P}$.*

(3) *$\tilde{A}$ is the k-closure of $A - \mathbb{R}_+k$. It consists of those points $y \in \mathscr{P}$ for which $\{y\} - tk \subseteq A - \mathbb{R}_+k$ holds for each $t \in \mathbb{R}_>$.*

$\lhd$ The proof is similar to the proof of Theorem 3.1 in [1]: Set $\tilde{A} := \text{sublev}_{\varphi_{A,k}}(0)$.

By Proposition 1.3, $\varphi_{A,k}$ is k-translative. Thus, Proposition 1.2 implies (6) and

$$\varphi_{A,k}(y) := \inf\{t \in \mathbb{R} : y \in \tilde{A} + tk\}.$$

Hence $\varphi_{A,k} = \varphi_{\tilde{A},k}$, and (6) yields

$$\text{sublev}_{\varphi_{\tilde{A},k}}(t) = \tilde{A} + tk$$

for all $t \in \mathbb{R}$. By Proposition 1.3, (2a) and (2b) are satisfied.

For (3), we have $A \subseteq \tilde{A}$. This implies $A - \mathbb{R}_+k \subseteq \tilde{A}$ by (2b) and thus $\text{cl}_k(A - \mathbb{R}_+k) \subseteq \text{cl}_k(\tilde{A}) = \tilde{A}$. Let $y \in \tilde{A}$. By the definition of $\varphi_{A,k}$ there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \in \mathbb{R}_+$ and $t_n \to 0$ such that

$$y \in A + \varphi_{A,k}(y)k + t_nk \subseteq A - \mathbb{R}_+k + t_nk$$

holds for each $n \in \mathbb{N}$. Hence $y \in \text{cl}_k(A-\mathbb{R}_+k)$, and $\tilde{A} = \text{cl}_k(A-\mathbb{R}_+k)$. Since $(A-\mathbb{R}_+k)-\mathbb{R}_+k \subseteq A - \mathbb{R}_+k$, we have $-k \in 0^+(A - \mathbb{R}_+k)$. Thus, the second statement of (3) results from Lemma 1.3. The uniqueness of $\tilde{A}$ satisfying (1) is obvious.

For uniqueness of $\tilde{A}$ in (2), assume now that (2a)–(2c) hold for some set $B \subseteq \mathscr{P}$. Relations (2a) and (2b) imply

$$\text{sublev}_{\varphi_{B,k}}(t) = B + tk \quad \text{for all} \quad t \in \mathbb{R},$$

by Proposition 1.3. By (2c) for $t = 0$ in the above relation, we have

$$B = \text{sublev}_{\varphi_{B,k}}(0) = \text{sublev}_{\varphi_{A,k}}(0). \rhd$$

Consequently, the class of the Gerstewitz functionals turns out to be equivalent to the class of functions with uniform sublevel sets generated by a linear shift of some set into a specified direction. Theorem 1.1 yields together with Proposition 1.2 the following theorem.

Theorem 1.2. *Let A be a subset of the cone $\mathscr{P}$. For each $k \in \mathscr{P}\backslash\{0\}$ such that $-k \in \mathscr{P}$ with $k + (-k) = 0$, the class of the Gerstewitz functionals $\{\varphi_{A,k} : A \subseteq \mathscr{P}\}$ coincides with the class of k-translative functions on $\mathscr{P}$ and with the class of functions $\varphi : \mathscr{P} \to \overline{\mathbb{R}}$ having uniform sublevel sets describe by (1) in Proposition 1.2.*

In the following example we illustrate Theorem 1.1 and Theorem 1.2 on a cone which is not a vector space.

Example 1.3. Let E will be a real vector space and $\mathscr{P} = \mathrm{Conv}(E)$ be the cone of all non-empty convex subsets of E, endowed with the usual addition and multiplication of sets by non-negative scalars, that is $\alpha A = \{\alpha a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A,\ b \in B\}$ for $A, B \in \mathrm{Conv}(E)$ and $\alpha \geqslant 0$. We note that $\mathscr{P}$ is not a vector space. Let $A \subseteq E$ and $\mathscr{A} = \mathrm{Conv}(A)$. We have $\mathscr{A} \subseteq \mathrm{Conv}(E)$. Let $k \in E$ such that $-k \in 0^+A$. The sets $K = \{k\}$ and $-K = \{-k\}$ are elements of $\mathscr{P} = \mathrm{Conv}(E)$ such that $K + (-K) = \{0\}$. Also we have $-K \in 0^+\mathscr{A}$. Indeed, if $X \in \mathscr{A}$, then $x + t(-k) \in A$, for all $x \in X$ and $t \in \mathbb{R}_+$, since $-k \in 0^+A$. Then $X + t(-K) \in \mathscr{A}$. The Gerstewitz functional $\varphi_{\mathscr{A},K}$ is an extension of $\varphi_{A,k}$ to $\mathscr{P}$ and $\varphi_{\mathscr{A},K}$ satisfies Theorems 1.1 and 1.2.

In the following, we construct two examples by using Examples 3.1 and 3.2 from [1].

Example 1.4. Let $\mathscr{P} = \mathrm{Conv}(\mathbb{R}^2)$, $C = \{(y_1, y_2) \in \mathbb{R}^2 \ :\ y_1 > 0\} \cup \{(y_1, y_2) \in \mathbb{R}^2 \ :\ y_1 = 0,\ y_2 \geqslant 0\}$ and $\mathscr{A} := \mathrm{Conv}(C)$. It is easy to see that C introduce the lexicographic order on $\mathbb{R}^2$. The recession cone $0^+\mathscr{A}$ of $\mathscr{A}$ is jast $\mathscr{A}$.

Consider the element $-\mathscr{K} = \{(1, 0)\}$ of the cone $0^+\mathscr{A}$. By considering $\mathscr{K} = \{(-1, 0)\}$, we have $\mathscr{K} + (-\mathscr{K}) = \{(0, 0)\}$. Let $B \in \mathscr{P}$. We define $y_{1,B} := \inf\{y_1 \ :\ (y_1, y_2) \in B\}$ and $y_{2,B} := \inf\{y_2 :\ (y_1, y_2) \in B\}$. Then

$$\varphi_{\mathscr{A},\mathscr{K}}(B) = \begin{cases} -y_{1,B}, & \text{if } y_{1,B} \neq -\infty, \\ +\infty, & \text{if } y_{1,B} = -\infty. \end{cases}$$

$\mathscr{A}$ is not $\mathscr{K}-$directionally closed, since $D = \{(0, -1)\}$ belongs to $\mathrm{cl}_{\mathscr{K}}(\mathscr{A})$, but not to $\mathscr{A}$. But the set $\widetilde{\mathscr{A}} := \mathrm{sublev}_{\varphi_{\mathscr{A},\mathscr{K}}}(0) = \{B \in \mathrm{Conv}(\mathbb{R}^2) \ :\ y_{1,B} \geqslant 0\}$ is $\mathscr{K}-$directionally closed. It is easy to verify that the statements of Theorem 1.1 hold for $\widetilde{\mathscr{A}}$.

Example 1.5. Let the cone $\mathscr{A}$ be as the same of Example 1.4 with $\mathscr{K} = \{(0, -1)\}$. Then

$$\mathrm{dom}\,\varphi_{\mathscr{A},\mathscr{K}} = \{B \in \mathscr{P} \ :\ y_{1,B} > 0\} \cap \{B \in \mathscr{P} \ :\ y_{1,B} = 0,\ y_{2,B} \neq -\infty\},$$

and

$$\varphi_{\mathscr{A},\mathscr{K}}(B) = \begin{cases} +\infty, & \text{if } y_{1,B} < 0, \\ +\infty, & \text{if } y_{1,B} = 0,\ y_{2,B} = -\infty, \\ -y_{2,B}, & \text{if } y_{1,B} = 0,\ y_{2,B} \neq -\infty, \\ -\infty, & \text{if } y_{1,B} > 0. \end{cases}$$

We see that $\mathscr{A}$ coincides with $\mathrm{sublev}_{\varphi_{\mathscr{A},\mathscr{K}}}(0)$ and $\mathscr{A}$ is $\mathscr{K}-$directionally closed.

Remark 1.2. If we consider $\mathbb{R}^2$ as a subcone of $\mathscr{P}$ in the Examples 1.4 and 1.5, the functions $\varphi_{\mathscr{A},\mathscr{K}}$ are extensions of the functions $\varphi_{A,k}$ in Example 3.1 and 3.2 in [1].

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ФУНКЦИИ С ОДНОРОДНЫМИ ПОДУРОВНЯМИ НА КОНУСАХ

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Аннотация. Расширенные вещественнозначные функции в вещественном векторном пространстве с однородными множествами подуровней важны в теории оптимизации. В настоящей работе изучается класс этих функций, совпадающий с классом функционалов Герштевица на конусах. эти конусы, вообще говоря, не вложимы в векторные пространства. Почти все результаты Вейднера из [1] неверны на конусах без дополнительных условий. На нетривиальных примерах показывается, что упомянутые условия необходимы. Для элемента k из конуса $\mathscr{P}$ определяются k-направленные замкнутые подмножества конуса и доказываются некоторые их свойства. Для подмножества A конуса $\mathscr{P}$ получена характеризация области определения $\varphi_{A,k}$ (функция с равномерным множеством подуровней) и показано, что эта функция k-транзитивна. Установлено также, что при некоторых условиях класс функционалов Герштевица совпадает с классом k-трансляционных функций на $\mathscr{P}$.

Ключевые слова: конус, набор подуровней.

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