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GENERALIZATION OF THE OSTROWSKI INEQUALITIES ON TIME SCALES

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Abstract. The idea of time scales calculus' theory was initiated and introduced by Hilger (1988) in his PhD thesis order to unify discret and continuous analysis and to expend the discrete and continous theories to cases "in between". Since then, mathematical research in this field has exceeded more than 1000 publications and a lot of applications in the fields of science, i. e., operations research, economics, physics, engineering, statistics, finance and biology. Ostrowski proved an inequality to estimate the absolute deviation of a differentiable function from its integral mean. This result was obtained by applying the Montgomery identity. In the present paper we derive a generalization of the Montgomery identity to the various time scale versions such as discrete case, continuous case and the case of quantum calculus, by obtaining this generalization of Montgomery identity we would prove our results about the generalization of the Ostrowski inequalities (without weighted case) to the several time scales such as discrete case, continuous case and the case of quantum calculus and recapture the several published results of different authors of various papers and thus unify corresponding discrete version and continuous version. Similarly we would also derive our results about the generalization of the Ostrowski inequalities (weighted case) to the different time scales such as discrete case and continuous case and recapture the different published results of several authors of various papers and thus unify corresponding discrete version and continuous version. Moreover, we would use our obtained results (without weighted case) to the case of quantum calculus.

Keywords: the Ostrowski inequality, the Hölder inequality, the Montgomery identity, time scales, quantum calculus.

AMS Subject Classification: 26A15, 26D15, 26D20, 81Q99.

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1. Introduction

Ostrowski proved an inequality to estimate the absolute deviation of a differentiable function from its integral mean. The below inequality is called the Ostrowski inequality which is extracted from [1]. For more study about the Ostrowski inequality, we refer to [2–7].

$$\left|g(\theta)-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g(\tau)\,d\tau\right| \leqslant \sup_{\kappa<\theta<\ell}\left|g'(\theta)\right|(\ell-\kappa)\left[\frac{1}{4}+\frac{\left(\theta-\frac{\kappa+\ell}{2}\right)^2}{(\ell-\kappa)^2}\right] \tag{1.1}$$

holds and this result had obtained by applying the Montgomery identity [8]. These properties would be derived for general time scales, which recapture discrete, continuous and many other cases. The pattern of current paper is consist of four sections. In the Section 1 and 2, we would present some preliminaries about time scales that are needed in the remainder of the paper. In the Section 3, we would get time scales versions of the generalized Montgomery identity and of the generalized Ostrowski inequality. While in the section 4 we would give several weighted time scales of the Ostrowski inequality. Throughout, we use our established results for the especial cases of discrete, continuous and quantum time scale. In the Section 5, we would give conclusion of the paper.

2. Time Scales Essentials

The idea of time scales calculus' theory was initiated and introduced by Hilger (1988) in his PhD thesis [9] (supervized by Aulbach) in order to unify discret and continuous analysis and to expend the discrete and continous theories to cases "in between". Since then, mathematical research in this field has exceeded more than 1000 publications and a lot of applications in the fields of science, i. e., operations research, economics, physics, engineering, statistics, finance and biology [10]. Even the time scale calculus theory may be used in most of the branches of science in which dynamic processes are explained by discrete-time/continuous-time models. We prefer the researcher to the book [11] written by Bohner and Peterson about the introduction to the singled variable time scale calculus and its implementations.

In 2004, Bohner introduced the variations' calculus on time scale, he used the delta derivative and delta integral [12], and it has since then been further developed by several different authors in several different publications (see [13–18]). Many classical results of calculus of variations as necessary or sufficient conditions of optimality have been generalised to arbitrary time scale.

Definition 2.1. A time scale is an arbitrary nonempty closed subset of the real numbers. The most important examples of time scales are $\mathbb{R}$, $\mathbb{Z}$ and $q^{\mathbb{N}_0} := \{q^\ell | \ell \in \mathbb{N}_0\}$.

Definition 2.2. If $\mathbb{T}$ is a time scale, then we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(\theta) := \inf\{\tau \in \mathbb{T} | \tau > \theta\}$ for all $\theta \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(\theta) := \sup\{\tau \in \mathbb{T} | \tau < \theta\}$ for all $\theta \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ by $\mu(\theta) := \sigma(\theta) - \theta$ for all $\theta \in \mathbb{T}$. Furthermore for a function $g : \mathbb{T} \to \mathbb{R}$, we define $g^\sigma(\theta) = g(\sigma(\theta))$ for all $\theta \in \mathbb{T}$ and $g^\rho(\theta) = g(\rho(\theta))$ for all $\theta \in \mathbb{T}$. In this definition we use $\inf \varnothing = \sup \mathbb{T}$ (i. e., $\rho(\theta) = \theta$ if θ is the maximum of $\mathbb{T}$) and $\sup \varnothing = \inf \mathbb{T}$ (i. e., $\rho(\theta) = \theta$ if θ is the minimum of $\mathbb{T}$).

These definitions allow us to characterize every point in a time scale as following classifcation of points:

(i) θ right-scattered $\Longrightarrow \theta < \sigma(\theta)$,
(ii) θ right-dense $\Longrightarrow \theta = \sigma(\theta)$,
(iii) θ left-scattered $\Longrightarrow \rho(\theta) < \theta$,
(iv) θ left-dense $\Longrightarrow \rho(\theta) = \theta$,
(v) θ isolated $\Longrightarrow \rho(\theta) < \theta < \sigma(\theta)$,
(vi) θ dense $\Longrightarrow \rho(\theta) = \theta = \sigma(\theta)$.

Definition 2.3. A function $g : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by C_{rd}) if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$.

Theorem 2.1 (existence of antiderivatives). *Let g be rd-continuous. Then g has an antiderivative G satisfying $G^{\Delta} = g$.*

◁ See Theorem 1.74 of paper [11]. ▷

Definition 2.4. If g is *rd-continuous* and $\theta_0 \in \mathbb{T}$, then we define the integral

$$G(\theta) = \int_{\theta_0}^{\theta} g(s)\Delta s \quad \text{for} \quad \theta \in \mathbb{T}. \tag{2.1}$$

Therefore for $g \in C_{rd}$ we have $\int_{\kappa}^{\ell} g(s)\Delta s = G(\ell) - G(\kappa)$, where $G^{\Delta} = g$.

Theorem 2.2. *Let g, h be rd-continuous, $\kappa, \ell, l \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(i) $\int_{\kappa}^{\ell} [\alpha g(\theta) + \beta h(\theta)]\Delta\theta = \alpha \int_{\kappa}^{\ell} g(\theta)\Delta\theta + \beta \int_{\kappa}^{\ell} h(\theta)\Delta\theta$,

(ii) $\int_{\kappa}^{\ell} g(\theta)\Delta\theta = -\int_{\ell}^{\kappa} g(\theta)\Delta\theta$,

(iii) $\int_{\kappa}^{\ell} g(\theta)\Delta\theta = \int_{\kappa}^{l} g(\theta)\Delta\theta + \int_{l}^{\ell} g(\theta)\Delta\theta$,

(iv) $\int_{\kappa}^{\ell} g(\theta)h^{\Delta}(\theta)\Delta\theta = (gh)(\ell) - (gh)(\kappa) - \int_{\kappa}^{\ell} g^{\Delta}(\theta)h(\sigma(\theta))\Delta\theta$,

(v) $\int_{\kappa}^{\kappa} g(\theta)\Delta\theta = 0$.

◁ See Theorem 1.77 of paper [11]. ▷

Definition 2.5. Let $h_c, f_c : \mathbb{T}^2 \to \mathbb{R}$, $c \in \mathbb{N}_0$, be defined by

$$h_0(\theta, \tau) = f_0(\theta, \tau) = 1 \quad (\forall\, \tau, \theta \in \mathbb{T})$$

and then recursively by

$$h_{c+1}(\theta, \tau) = \int_{\tau}^{\theta} h_c(\sigma(s), \tau)\Delta s \quad (\forall\, \tau, \theta \in \mathbb{T}) \quad \text{and} \quad f_{c+1}(\theta, \tau) = \int_{\tau}^{\theta} f_c(s, \tau)\Delta s \quad (\forall\, \tau, \theta \in \mathbb{T}).$$

Theorem 2.3 (Hölder's inequality). *Let $\kappa, \ell \in \mathbb{T}$ and $g, h : [\kappa, \ell] \to \mathbb{R}$ be rd-continuous. Then*

$$\int_{\kappa}^{\ell} \left|g(\theta)h(\theta)\right|\Delta\theta \leqslant \left(\int_{\kappa}^{\ell} \left|g(\theta)\right|^p \Delta\theta\right)^{\frac{1}{p}} \left(\int_{\kappa}^{\ell} \left|h(\theta)\right|^q \Delta\theta\right)^{\frac{1}{q}}, \tag{2.2}$$

where $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$.

◁ See Theorem 6.13 of [11]. ▷

In 2008, Bohner et. al. proved the Ostrowski inequalities on time scales and they obtained unified and extended results to the literature and they also gave results to the quantum calculus case. In the current paper, we would obtain generalization of the Ostrowski inequalities on time scales and recapture the results of [19] for discrete and continuous versions and also recapture some results of papers [1, 8, 11, 20–22]. Moreover, we will use our parametric results to the case of quantum calculus.

3. Generalization of the Ostrowski Inequality on Time Scales

To prove the our main Theorem 3.1, we require the below generalized Montgomery identity.

Lemma 3.1 (the generalized Montgomery identity)**.** *Let $\kappa, \ell, \tau, \theta \in \mathbb{T}$, $\kappa < \ell$ and $g : [\kappa, \ell] \to \mathbb{R}$ be differentiable and parameter $\lambda \in [0,1]$. Then*

$$(1-\lambda)g(\theta) + \frac{\lambda}{2}\,(g(\kappa)+g(\ell)) = \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau + \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau, \qquad (3.1)$$

where

$$p(\theta,\tau) = \begin{cases} \tau - \Big(\kappa + \lambda\dfrac{\ell-\kappa}{2}\Big), & \kappa \leqslant \tau < \theta, \\ \tau - \Big(\ell - \lambda\dfrac{\ell-\kappa}{2}\Big), & \theta \leqslant \tau \leqslant \ell. \end{cases}$$

$\lhd$ Applying Theorem 2.2 (iv) we have

$$\int_{\kappa}^{\theta}\Big(\tau - \Big(\kappa + \lambda\,\frac{\ell-\kappa}{2}\Big)\Big)g^{\Delta}(\tau)\Delta\tau = \Big(\theta - \Big(\kappa + \lambda\,\frac{\ell-\kappa}{2}\Big)\Big)g(\theta) + \lambda\,\frac{\ell-\kappa}{2}\,g(\kappa) - \int_{\kappa}^{\theta} g^{\sigma}(\theta)\Delta\tau.$$

$$\int_{\theta}^{\ell}\Big(\tau - \Big(\ell - \lambda\,\frac{\ell-\kappa}{2}\Big)\Big)g^{\Delta}(\tau)\Delta\tau = -\Big(\theta - \Big(\ell - \lambda\,\frac{\ell-\kappa}{2}\Big)\Big)g(\theta) + \lambda\,\frac{\ell-\kappa}{2}\,g(\ell) - \int_{\theta}^{\ell} g^{\sigma}(\theta)\Delta\tau.$$

Therefore

$$\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau + \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau = \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau + \frac{1}{\ell-\kappa}\Bigg[(\ell-\kappa)(1-\lambda)g(\theta)$$

$$+\lambda\,\frac{\ell-\kappa}{2}\,(g(\kappa)+g(\ell)) - \int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau\Bigg] = (1-\lambda)g(\theta) + \frac{\lambda}{2}\,(g(\kappa)+g(\ell)),$$

i. e., (3.1) holds. $\rhd$

Remark 3.1. If we put $\lambda = 0$ in Lemma 3.1, then we recapture Lemma 3.1 of [19].

If we implement the generalized Montgomery identity to several time scales, we obtain some well-known and new results.

Discrete Case:

Corollary 3.1. *We suppose $\mathbb{T} = \mathbb{Z}$. Let $\kappa = 0$, $\ell = n$, $\tau = b$, $\theta = a$ and $g(c) = y_c$. Then*

$$(1-\lambda)y_a + \frac{\lambda}{2}\,(y_0 + y_n) = \frac{1}{n}\sum_{b=1}^{n} y_b + \frac{1}{n}\sum_{b=0}^{n-1} p(a,b)\Delta y_b,$$

where

$$p(a,0) = 0,$$

$$p(a,b) = \begin{cases} b - \dfrac{\lambda n}{2}, & 0 \leqslant b \leqslant a-1, \\ b - n\Big(1 - \dfrac{\lambda}{2}\Big), & a \leqslant b \leqslant n-1. \end{cases}$$

as we just require $1 \leqslant a \leqslant n$, $0 \leqslant b \leqslant n-1$.

Remark 3.2. If we put $\lambda = 0$ in Corollary 3.1, then we recapture Corollary 3.2 of paper [19] and Theorem 2.1 of paper [20].

Continuous Case:

Corollary 3.2. *We let* $\mathbb{T}=\mathbb{R}$. *Then*

$$(1-\lambda)g(\theta)+\frac{\lambda}{2}\left(g(\kappa)+g(\ell)\right)=\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g(\tau)\,d\tau+\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g'(\tau)\,d\tau.$$

Remark 3.3. If we put $\lambda=0$ in Corollary 3.2, then we recapture the Montgomery identity in the continuous case which may be seen in [8, p. 565] and Theorem 2.1 of paper [20].

Quantum Calculus Case:

Corollary 3.3. *We let* $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $\kappa=q^m$, $\ell=q^n$ *and* $\tau=q^c$ *with* $m<n$. *Then*

$$(1-\lambda)g(\theta)+\frac{\lambda}{2}\left(g(q^n)+g(q^m)\right)=\frac{1}{q^n-q^m}\sum_{c=m}^{n-1} g\left(q^{c+1}\right)+\frac{1}{q^n-q^m}\sum_{c=m}^{n-1}\left[g(q^{c+1})-g(q^c)\right]p(\theta,q^c),$$

where

$$p(\theta,q^c)=\begin{cases} q^c-\left(q^m+\lambda\dfrac{q^n-q^m}{2}\right), & q^m\leqslant q^c<\theta,\\ q^c-\left(q^n-\lambda\dfrac{q^n-q^m}{2}\right), & \theta\leqslant q^c\leqslant q^n.\end{cases}$$

Remark 3.4. If we put $\lambda=0$ in Corollary 3.3, then we recapture Corollary 3.4 of [19].

Theorem 3.1 (the generalized Ostrowski inequality)**.** *Let* $\kappa,\ell,\tau,\theta\in\mathbb{T}$, $\kappa<\ell$ *and* $g:[\kappa,\ell]\to\mathbb{R}$ *be differentiable and parameter* $\lambda\in[0,1]$. *Then*

$$\left|(1-\lambda)g(\theta)+\frac{\lambda}{2}\left(g(\kappa)+g(\ell)\right)-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau\right|\leqslant\frac{M}{\ell-\kappa}\left(f_2(\theta,\kappa)+f_2(\theta,\ell)\right),\tag{3.2}$$

where

$$M=\sup_{\kappa<\theta<\ell}\left|g^{\Delta}(\theta)\right|.$$

This inequality is sharp in the sense that the right-hand side of (3.2) *cannot be replaced by a smaller one.*

◁ Using Lemma 3.1 with $p(\theta,\tau)$, we have

$$\left|(1-\lambda)g(\theta)+\frac{\lambda}{2}\left(g(\kappa)+g(\ell)\right)-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau\right|=\left|\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau\right|$$

$$\leqslant\frac{M}{\ell-\kappa}\left(\int_{\kappa}^{\theta}\left|\tau-\left(\kappa+\lambda\frac{\ell-\kappa}{2}\right)\right|\Delta\tau+\int_{\theta}^{\ell}\left|\tau-\left(\ell-\lambda\frac{\ell-\kappa}{2}\right)\right|\Delta\tau\right)$$

$$=\frac{M}{\ell-\kappa}\left[\int_{\kappa}^{\theta}\left(\tau-\left(\kappa+\lambda\frac{\ell-\kappa}{2}\right)\right)\Delta\tau+\int_{\theta}^{\ell}\left(\left(\ell-\lambda\frac{\ell-\kappa}{2}\right)-\tau\right)\Delta\tau\right]$$

$$=\frac{M}{\ell-\kappa}\left(f_2(\theta,\kappa)+f_2(\theta,\ell)\right).\quad ▷$$

REMARK 3.5. If we put $\lambda = 0$ in Theorem 3.1, then we recapture Theorem 3.5 of [19] and which is stated in the following as:

Corollary 3.4. *Suppose the assumptions of Theorem* 3.1 *is true. Then*

$$\left|g(\theta) - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right| = \left|\frac{1}{\ell-\kappa}\int_\kappa^\ell p(\theta,\tau)g^\Delta(\tau)\Delta\tau\right|$$

$$\leqslant \frac{M}{\ell-\kappa}\left(\int_\kappa^\theta |\tau-\kappa|\Delta\tau + \int_\theta^\ell |\tau-\ell|\Delta\tau\right) = \frac{M}{\ell-\kappa}\left(\int_\kappa^\theta (\tau-\kappa)\Delta\tau + \int_\theta^\ell (\ell-\tau)\Delta\tau\right) \tag{3.3}$$

$$= \frac{M}{\ell-\kappa}\left(f_2(\theta,\kappa) + f_2(\theta,\ell)\right).$$

Note that, since $p(\theta,\kappa) = 0$, the smallest value attaining the supremum in M is greater than κ. To prove the sharpness of inequality (3.3), let $g(\theta) = \theta$, $\kappa = T_1$, $\ell = T_2$ and $\theta = T_2$. It follows that $g^\Delta(\theta) = 1$ and $M = 1$. Beginning with the left-hand side of (3.2), we have

$$\left|g(\theta) - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right| = \left|T_2 - \frac{1}{T_2-T_1}\int_{T_1}^{T_2} \sigma(\tau)\Delta\tau\right|$$

$$= \left|T_2 - \frac{1}{T_2-T_1}\left(\int_{T_1}^{T_2} (\sigma(\tau)+\tau)\Delta\tau - \int_{T_1}^{T_2} \tau\Delta\tau\right)\right|$$

$$= \left|T_2 - \frac{1}{T_2-T_1}\left(\int_{T_1}^{T_2} (\tau^2)^\Delta\Delta\tau - \int_{T_1}^{T_2} \tau\Delta\tau\right)\right| = \left|-T_1 + \frac{1}{T_2-T_1}\int_{T_1}^{T_2} \tau\Delta\tau\right|.$$

Beginning with the right-hand side of (3.2), we have

$$\frac{M}{\ell-\kappa}\left(f_2(\theta,\kappa) + f_2(\theta,\ell)\right) = \frac{1}{T_2-T_1}\left(\int_{T_1}^{T_2} (\tau-T_1)\Delta\tau - \int_{T_1}^{T_2} (\tau-T_2)\Delta\tau\right)$$

$$= \frac{1}{T_2-T_1}\left(-T_1T_2 + T_1^2 + \int_{T_1}^{T_2} \tau\Delta\tau\right) = -T_1 + \frac{1}{T_2-T_1}\int_{T_1}^{T_2} \tau\Delta\tau.$$

Therefore in this particular case

$$\left|g(\theta) - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right| \geqslant \frac{M}{\ell-\kappa}\left(f_2(\theta,\kappa) + f_2(\theta,\ell)\right)$$

and by (3.2) also

$$\left|g(\theta) - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right| \leqslant \frac{M}{\ell-\kappa}\left(f_2(\theta,\kappa) + f_2(\theta,\ell)\right).$$

So the sharpness of the Ostrowski inequality is shown.

The following are the different cases of the generalized Ostrowski inequality with $\lambda = 0$.

Discrete Case:

Corollary 3.5. *We suppose $\mathbb{T} = \mathbb{Z}$. Let $\kappa = 0$, $\ell = n$, $\tau = b$, $\theta = a$ and $g(c) = y_c$. Then*

$$\left| y_a - \frac{1}{n}\sum_{b=1}^{n} y_b \right| = \frac{M}{n}\left[\left| a - \frac{n+1}{2} \right|^2 + \frac{n^2-1}{4} \right], \tag{3.4}$$

where

$$M = \max_{1<a<n-1} |\Delta y_a|.$$

This is the discrete Ostrowski inequality (see Theorem 3.1 of [20]), where the constant 1/4 in the right-hand side of (3.4) is the best possible in the sense that it cannot be replaced by a smaller one.

Continuous Case:

Corollary 3.6. *We suppose $\mathbb{T} = \mathbb{R}$. Then*

$$\left| g(\theta) - \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g(\tau)\, d\tau \right| = M(\ell-\kappa)\left[\frac{1}{4} + \frac{\left(\theta - \frac{\kappa+\ell}{2}\right)^2}{(\ell-\kappa)^2} \right],$$

where

$$M = \sup_{\kappa<\theta<\ell} \left| g'(\theta) \right|.$$

This is the Ostrowski inequality in the continuous case [1], where again the constant 1/4 in the right-hand side is the best possible.

Quantum Calculus Case:

Corollary 3.7. *We suppose $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $\kappa = q^m$ and $\ell = q^n$ with $m < n$. Then*

$$\left| g(\theta) - \frac{1}{q^n - q^m}\int_{q^m}^{q^n} g^{\sigma}(\tau)\Delta\tau \right| \leqslant \frac{M}{q^n-q^m}\left[\frac{2}{1+q}\left(\left(\theta - \frac{\frac{1+q}{2}(q^m+q^n)}{2} \right)^2 \right.\right.$$
$$\left.\left. + \frac{-\left(\frac{1+q}{2}\right)^2 (q^m+q^n)^2 + (2(1+q)-2)(q^{2m}+q^{2n})}{4} \right) \right],$$

where

$$M = \sup_{q^m<\theta<q^n} \left| \frac{g(q\theta) - g(\theta)}{(q-1)\theta} \right|,$$

and 1/4 in the right-hand side is the best possible.

Corollary 3.8. *If put $\lambda = 1$ in Theorem 3.1. Then we obtain following average trapezoid type inequality on time scale*

$$\left| \frac{g(\kappa)+g(\ell)}{2} - \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau \right| \leqslant \frac{M}{\ell-\kappa}\left(f_2(\theta,\kappa) + f_2(\theta,\ell) \right), \tag{3.5}$$

where

$$M = \sup_{\kappa<\theta<\ell} \left| g^{\Delta}(\theta) \right|.$$

4. The Weighted Case

The below weighted Ostrowski inequality with parameter on time scale holds.

Theorem 4.1. *Suppose the assumptions of Theorem* 3.1 *is true and* $\xi \in \mathbb{T}$ *and* $q \in C_{rd}$. *Then*

$$\left|A + (1-\lambda)g(\theta) - \int_{\kappa}^{\ell} q^{\sigma}(\tau)g^{\sigma}(\tau)\Delta\tau\right| \leqslant A + \int_{\kappa}^{\theta} q^{\sigma}(\tau)\big|(1-\lambda)g(\theta) - g^{\sigma}(\tau)\big|\Delta\tau + \int_{\theta}^{\ell} q^{\sigma}(\tau)\big|(1-\lambda)g(\theta) - g^{\sigma}(\tau)\big|\Delta\tau \tag{4.1}$$

$$\leqslant \begin{cases} A + (1-\lambda)\left(\int_{\kappa}^{\ell} |g(\theta)|^{p}\Delta\tau\right)^{\frac{1}{p}}\left(\int_{\kappa}^{\ell} \big(q^{\sigma}(\tau)\big)^{q}\Delta\tau\right)^{\frac{1}{q}} \\ \qquad + \left(\int_{\kappa}^{\ell} \big|g^{\sigma}(\tau)\big|^{p}\Delta\tau\right)^{\frac{1}{p}}\left(\int_{\kappa}^{\ell} \big(q^{\sigma}(\tau)\big)^{q}\Delta\tau\right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1,\ p > 1, \\ A + \sup\limits_{\kappa \leqslant \tau < \ell} q^{\sigma}(\tau)\big[h_2(\kappa,\theta) + h_2(\ell,\theta)\big], \\ \dfrac{g(\sigma(\ell)) - g(\sigma(\kappa))}{2} + \left|(1-\lambda)g(\theta) - \dfrac{g(\sigma(\kappa)) + g(\sigma(\ell))}{2}\right|, \end{cases} \tag{4.2}$$

where

$$A = \frac{\lambda}{2}\,(g(\kappa) + g(\ell)), \qquad \int_{\kappa}^{\ell} q^{\sigma}(\tau)\Delta\tau = 1, \quad q(\tau) \geqslant 0.$$

$\lhd$ As from left side of (3.2) we have

$$\left|\frac{\lambda}{2}\,(g(\kappa) + g(\ell)) + (1-\lambda)g(\theta) - \int_{\kappa}^{\ell} q^{\sigma}(\tau)g^{\sigma}(\tau)\Delta\tau\right|$$

$$= \left|\frac{\lambda}{2}\,(g(\kappa) + g(\ell)) + \int_{\kappa}^{\ell} q^{\sigma}(\tau)\big((1-\lambda)g(\theta) - g^{\sigma}(\tau)\big)\Delta\tau\right|$$

$$\leqslant A + \int_{\kappa}^{\theta} q^{\sigma}(\tau)\big|(1-\lambda)g(\theta) - g^{\sigma}(\tau)\big|\Delta\tau + \int_{\theta}^{\ell} q^{\sigma}(\tau)\big|(1-\lambda)g(\theta) - g^{\sigma}(\tau)\big|\Delta\tau$$

and therefore (4.1) is shown.

The first part of (4.2) can be done easily by applying Hölder's inequality. By factoring

$\sup_{\kappa\leqslant\tau<\ell} q^\sigma(\tau)$, we have

$$A + \int_\kappa^\theta q^\sigma(\tau)\big|(1-\lambda)g(\theta) - g^\sigma(\tau)\big|\Delta\tau + \int_\theta^\ell q^\sigma(\tau)\big|(1-\lambda)g(\theta) - g^\sigma(\tau)\big|\Delta\tau$$
$$\leqslant A + \sup_{\kappa\leqslant\tau<\ell} q^\sigma(\tau)\Bigg(\int_\kappa^\theta \big(g^\sigma(\tau) - (1-\lambda)g(\theta)\big)\Delta\tau + \int_\theta^\ell \big((1-\lambda)g(\theta) - g^\sigma(\tau)\big)\Delta\tau\Bigg)$$
$$= A + \sup_{\kappa\leqslant\tau<\ell} q^\sigma(\tau)\Bigg(\int_\theta^\kappa \big((1-\lambda)g(\theta) - g^\sigma(\tau)\big)\Delta\tau + \int_\theta^\ell \big((1-\lambda)g(\theta) - g^\sigma(\tau)\big)\Delta\tau\Bigg)$$
$$= A + \sup_{\kappa\leqslant\tau<\ell} q^\sigma(\tau)\big[h_2(\kappa,\theta) + h_2(\ell,\theta)\big]$$

and therefore the 2nd part of (4.2) holds. Finally for deriving the 3rd inequality, we implement the fact that

$$\sup_{\kappa\leqslant\tau<\ell}\big\{|g(\sigma(\tau)) - (1-\lambda)g(\theta)|\big\} = \max\big\{g(\sigma(\ell)) - (1-\lambda)g(\theta), (1-\lambda)g(\theta) - g(\sigma(\kappa))\big\}$$
$$= \frac{g(\sigma(\ell)) - g(\sigma(\kappa))}{2} + \left|(1-\lambda)g(\theta) - \frac{g(\sigma(\kappa)) + g(\sigma(\ell))}{2}\right|.$$

Thus (4.2) is shown. ▷

Remark 4.1. If we put $q^\sigma(\tau) = \frac{1}{\ell-\kappa}$ in Theorem 4.1 then obtain the result without weights.

Remark 4.2. If we put $q^\sigma(\tau) = \frac{1}{\ell-\kappa}$ and $\lambda = 0$ in Theorem 4.1 then we recapture the Theorem 3.1 of [19].

Remark 4.3. Theorem 4.1 with $\lambda = 0$ states a similar result as shown in Theorem 3.1 of [21], if we consider the normalized isotonic functional $B(g) = \int_\kappa^\ell q^\sigma(\tau)g^\sigma(\tau)\Delta\tau$.

Remark 4.4. The second inequality of (4.2) with $\lambda = 0$ is comparable to the achievement in Theorem 3.1 of [22] for the continuous case (see Corollary 3.3 of [22]).

Discrete Case:

Corollary 4.1. *We suppose $\mathbb{T} = \mathbb{Z}$. Let $\kappa = 0$, $\ell = n$, $\tau = b$, $\theta = a$, $\xi = c$ and $g(c) = y_c$. Then $\sum_{a=1}^n q_a = 1$, $0 \leqslant q_a$, and*

$$\left|A + (1-\lambda)y_a - \sum_{b=1}^n q_b y_b\right| \leqslant A + (1-\lambda)\sum_{b=1}^n q_b|y_a| + \sum_{b=1}^n q_b|y_b|$$
$$\leqslant \begin{cases} A + (1-\lambda)\Bigg(\sum\limits_{b=1}^n |y_a|^p\Bigg)^{\frac{1}{p}}\Bigg(\sum\limits_{b=1}^n q_b^q\Bigg)^{\frac{1}{q}} + \Bigg(\sum\limits_{b=1}^n |y_b|^p\Bigg)^{\frac{1}{p}}\Bigg(\sum\limits_{b=1}^n q_b^q\Bigg)^{\frac{1}{q}}, & \frac{1}{p} + \frac{1}{q} = 1,\ p > 1, \\ A + \max_{b=1,\dots,n} q(b)\Bigg(\sum\limits_{b=1}^n y_b\Bigg), \\ \frac{y_n - y_0}{2} + \left|(1-\lambda)y_a - \frac{y_0+y_n}{2}\right|, \end{cases}$$

where $A = \frac{\lambda}{2}(y_0 + y_n)$.

Remark 4.5. If we put $\lambda = 0$ in Corollary 4.1 then we recapture the Theorem 4.1 of [20] and Corollary 4.3 of [19].

CONTINUOUS CASE:

Corollary 4.2. *We suppose* $\mathbb{T}=\mathbb{R}$. *Then* $\int_\kappa^\ell q(\tau)d\tau=1$, $0\leqslant q(\tau)$ *and*

$$\left|A+(1-\lambda)g(\theta)-\int_\kappa^\ell q(\tau)g(\tau)\,d\tau\right|$$

$$\leqslant A+\int_\kappa^\theta q(\tau)\big|(1-\lambda)g(\theta)-g(\tau)\big|\,d\tau+\int_\theta^\ell q(\tau)\big|(1-\lambda)g(\theta)-g(\tau)\big|\,d\tau$$

$$\leqslant\begin{cases} A+(1-\lambda)\left(\int_\kappa^\ell |g(\theta)|^p\,d\tau\right)^{\frac{1}{p}}\left(\int_\kappa^\ell (q(\tau))^q\,d\tau\right)^{\frac{1}{q}} \\ \qquad +\left(\int_\kappa^\ell |g(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\left(\int_\kappa^\ell (q(\tau))^q\,d\tau\right)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ A+\sup\limits_{\kappa\leqslant\tau<\ell} q(\tau)\big[h_2(\kappa,\theta)+h_2(\ell,\theta)\big], \\ \frac{g(\ell)-g(\kappa)}{2}+\left|(1-\lambda)g(\theta)-\frac{g(\kappa)+g(\ell)}{2}\right|. \end{cases}$$

where

$$A=\frac{\lambda}{2}\,(g(\kappa)+g(\ell)),\qquad \int_\kappa^\ell q(\tau)\,d\tau=1,\quad q(\tau)\geqslant 0.$$

REMARK 4.6. If we put $\lambda=0$ in Corollary 4.2 then we recapture the Corollary 4.4 of [19].

Corollary 4.3. *If we put* $\lambda=1$ *in Theorem* 3.1. *Then we obtain following average trapezoid type inequality with weights on time scale*

$$\left|\frac{g(\kappa)+g(\ell)}{2}-\int_\kappa^\ell q^\sigma(\tau)g^\sigma(\tau)\Delta\tau\right|\leqslant\frac{g(\kappa)+g(\ell)}{2}+\int_\kappa^\ell q^\sigma(\tau)|g^\sigma(\tau)|\Delta\tau$$

$$\leqslant\begin{cases} \frac{g(\kappa)+g(\ell)}{2}+\left(\int_\kappa^\ell |g^\sigma(\tau)|^p\Delta\tau\right)^{\frac{1}{p}}\left(\int_\kappa^\ell (q^\sigma(\tau))^q\Delta\tau\right)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ \frac{g(\kappa)+g(\ell)}{2}+\sup\limits_{\kappa\leqslant\tau<\ell} q^\sigma(\tau)\left(\int_\kappa^\ell q^\sigma(\tau)g^\sigma(\tau)\Delta\tau\right), \\ g(\sigma(\ell)). \end{cases}$$

Another interesting conclusion of Theorem 4.1 with $q^\sigma(\tau)=\frac{1}{\ell-\kappa}$ and $\lambda=0$ is the following corollary.

Corollary 4.4. *Suppose* $\kappa,\ell,\tau,\theta\in\mathbb{T}$, $\kappa<\ell$ *and* g *is differentiable. Then*

$$\left|g(\theta)-\frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right|\leqslant\frac{M}{\ell-\kappa}\,(f_2(\theta,\kappa)+f_2(\theta,\ell)), \tag{4.3}$$

where

$$M = \sup_{\sigma(\kappa)\leqslant\theta<\ell} \left|g^{\Delta}(\theta)\right|.$$

Note that this was shown in several ways in Theorem 3.1 In inequality (4.3) we implement the fact that the functions h_2 and f_2 satisfy $h_2(\tau,\theta) = (-1)^2 f_2(\theta,\tau)$, $\forall\,\theta \in \mathbb{T}$, $\tau \in \mathbb{T}^c$ (see Theorem 1.112 of [11]).

REMARK 4.7. Moreover note that there is a small difference of (4.2) in comparison to Theorem 3.1, as we have $\sup_{\sigma(\kappa)\leqslant\theta<\ell}$ instead of $\sup_{\kappa<\theta<\ell}$. This is just important if κ is right-dense, i. e., $\sigma(\kappa) = \kappa$. But in those cases the inequality does not change and is still sharp. Furthermore in the proof of Theorem 3.1 we could have picked $\sup_{\kappa\leqslant\theta<\ell}$ as explained before.

5. Conclusion

In this paper, the generalized Ostrowski inequalities (with weights and without weights) are proved on time scales and thus our results unified and extended corresponding to discrete and continuous versions of previously proved results of different researchers in various papers [1, 8, 11, 19–22]. Moreover, we have used our obtained results to the quantum calculus case.

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ОБОБЩЕНИЕ НЕРАВЕНСТВ ОСТРОВСКОГО НА ВРЕМЕННЫХ ШКАЛАХ

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Аннотация: Идея теории исчисления временных шкал была инициирована Хильгером (1988) в его докторской диссертации с целью унификации дискретного и непрерывного анализа и применить дискретную и континуальную теории к случаям «промежуточным». С тех пор математические исследования в этой области породили более 1000 публикаций с приложениями в различных науках, таких как исследование операций, экономика, физика, техника, статистика, финансы, биология. Островский доказал неравенство для оценки абсолютного отклонения дифференцируемой функции от ее интегрального среднего. Этот результат был получен с помощью тождества Монтгомери. В настоящей статье мы выводим обобщение тождества Монтгомери для различных временных шкал, таких как дискретный случай, непрерывный случай и случай квантового исчисления. Получив это обобщение тождества Монтгомери, мы докажем наши результаты об обобщении неравенства Островского (без весового случая) для упомянутых временных шкал. Таким образом, удается повторить несколько ранее опубликованных результатов разных авторов в различных статьях и унифицировать соответствующую дискретную версию и непрерывную версии. Точно так же мы также получим наши результаты об обобщении неравенств Островского (весовой случай) на разные временные шкалы, повторим ранее опубликованные результаты и, тем самым, унифицируем соответствующую дискретную версию и непрерывную версию. Более того, мы применим полученные нами результаты (без весового случая) к случаю квантового исчисления.

Ключевые слова: неравенство Островского, неравенство Гёльдера, тождество Монтгомери, шкалы времени, квантовое исчисление.

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