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ON JANOWSKI TYPE HARMONIC FUNCTIONS ASSOCIATED WITH THE WRIGHT HYPERGEOMETRIC FUNCTIONS

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Abstract. In our present study we consider Janowski type harmonic functions class introduced and studied by Dziok, whose members are given by $h(z) = z + \sum_{n=2}^{\infty} h_n z^n$ and $g(z) = \sum_{n=1}^{\infty} g_n z^n$, such that $\mathscr{S}\mathscr{T}_H(F,G) = \left\{f = h + \bar{g} \in H : \frac{\mathfrak{D}_H f(z)}{f(z)} \prec \frac{1+Fz}{1+Gz};\ (-G \leqslant F < G \leqslant 1,\ \text{with } g_1 = 0)\right\}$, where $\mathfrak{D}_H f(z) = zh'(z) - \overline{zg'(z)}$ and $z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We investigate an association between these subclasses of harmonic univalent functions by applying certain convolution operator concerning Wright's generalized hypergeometric functions and several special cases are given as a corollary. Moreover we pointed out certain connections between Janowski-type harmonic functions class involving the generalized Mittag–Leffler functions. Relevant connections of the results presented herewith various well-known results are briefly indicated.

Keywords: harmonic functions, univalent functions, Wright's generalized hypergeometric functions.

AMS Subject Classification: 30C45, 30C55.

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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply-connected domain $\mathbb{D}$, we can write $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$ and are commonly denoted by H. In 1984 Clunie and Sheil-Small [1] introduced a class $\mathscr{S}_H$ of complex — valued harmonic maps f which are univalent and sense — preserving in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. The function $f \in \mathscr{S}_H$ can be represented by $f = h + \bar{g}$, is given by

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} h_n z^n + \overline{\sum_{n=1}^{\infty} g_n z^n},$$

where

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \quad g(z) = \sum_{n=1}^{\infty} g_n z^n, \quad |g_1| < 1 \tag{1.1}$$

are analytic in the open unit disk in $\mathbb{U}$. They also proved that the function $f = h + \bar{g} \in \mathscr{S}_H$ is locally univalent and sense preserving in $\mathbb{U}$, if and only if $|h'(z)| > |g'(z)|$, $\forall\, z \in \mathbb{U}$. For more basic study one may refer Duren [2] and Ahuja [3]. It is worthy to note that if $g(z) \equiv 0$ in (1.1), then the class $\mathscr{S}_H$ reduces to the familiar class $\mathscr{S}$ of analytic functions. For this class $f(z)$ may be expressed as of the form

$$f(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{1.2}$$

Further, we suppose that $\mathscr{S}_H^0$ subclass of $\mathscr{S}_H$ consisting of function $f \in \mathscr{S}_H$ of the form (1.1) with $g_1 = 0$ and is given by

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} h_n z^n + \overline{\sum_{n=2}^{\infty} g_n z^n},$$

where

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \quad g(z) = \sum_{n=2}^{\infty} g_n z^n, \quad g_1 = 0.$$

Now, we let $\mathscr{K}_H^0$, $\mathscr{S}\mathscr{T}_H^0$ and $\mathscr{C}_H^0$ denote the subclasses of $\mathscr{S}_H^0$ of harmonic functions which are respectively convex, starlike and close-to-convex in $\mathbb{U}$. Also let $\mathscr{T}_H^0$ be the class of sense preserving, typically real harmonic functions $f = h + \bar{g}$ in $\mathscr{S}_H$. For detailed study of these classes one may refer to [1, 2].

Now, we recall the subclass $\mathscr{S}_{\overline{H}}$ of $\mathscr{S}_H$ consisting of functions $f = h + \bar{g}$, so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |h_n| z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} |g_n| z^n, \tag{1.3}$$

which has been introduced and studied extensively by Silverman [4].

Let $a_i \in \mathbb{C}$, $((a_i/A_i) \neq 0, -1, -2, \ldots;\ i = 1, 2, \ldots, p)$ and $((b_i/B_i) \neq 0, -1, -2, \ldots;\ i = 1, 2, \ldots, q)$, for $A_i > 0$ $(i = 1, \ldots, p)$, $B_i > 0$ $(i = 1, \ldots, q)$ with $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geqslant 0$ the Wright's generalized hypergeometric functions [5] is defined by

$${}_p\Psi_q \left[\begin{array}{c} (a_i, A_i)_{1,p} \\ (b_i, B_i)_{1,q} \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + nA_i) z^n}{\prod_{i=1}^{q} \Gamma(b_i + nB_i) n!}, \tag{1.4}$$

which is analytic for suitable bounded values of $|z|$ (see also [6, 7]). The generalized Mittag-Leffler, Bessel–Maitland and generalized hypergeometric functions are some of the important special cases of Wright's generalized hypergeometric functions and for their details one may refer to [7–9].

For $A_i > 0$ $(i = 1, \ldots, p)$, $B_i > 0$, $b_i > 0$ $(i = 1, \ldots, q)$ with $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geqslant 0$ and $C_i > 0$ $(i = 1, \ldots, r)$, $D_i > 0$, $d_i > 0$ $(i = 1, \ldots, s)$ with $1 + \sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i \geqslant 0$, we define Wright's generalized hypergeometric functions

$${}_p\Psi_q \left[\begin{array}{c} (a_i, A_i)_{1,p} \\ (b_i, B_i)_{1,q} \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + nA_i) z^n}{\prod_{i=1}^{q} \Gamma(b_i + nB_i) n!}$$

and

$$ {}_r\Psi_s \left[\begin{array}{c} (c_i, C_i)_{1,r} \\ (d_i, D_i)_{1,s} \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(c_i + nC_i) z^n}{\prod_{i=1}^{s} \Gamma(d_i + nD_i) n!} \tag{1.5} $$

with

$$ \frac{\prod_{i=1}^{r} \Gamma(|c_i| + nC_i)/\Gamma|c_i|}{\prod_{i=1}^{s} \Gamma(d_i + nD_i)/\Gamma(d_i)} < 1. $$

We consider a harmonic univalent function

$$ \mathfrak{F}(z) = \mathfrak{H}(z) + \overline{\mathfrak{G}(z)} \in \mathscr{S}_H, \tag{1.6} $$

where

$$ \mathfrak{H}(z) = z \frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(a_i)} {}_p\Psi_q \left[\begin{array}{c} (a_i, A_i)_{1,p} \\ (b_i, B_i)_{1,q} \end{array} ; z \right] = z + \sum_{n=2}^{\infty} \theta_n \, z^n \tag{1.7} $$

and

$$ \mathfrak{G}(z) = \sigma z \frac{\prod_{i=1}^{s} \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(c_i)} {}_r\Psi_s \left[\begin{array}{c} (c_i, C_i)_{1,r} \\ (d_i, D_i)_{1,s} \end{array} ; z \right] = \sigma \sum_{n=1}^{\infty} \zeta_n \, z^n, \quad |\sigma| < 1, \tag{1.8} $$

and θ_n and ζ_n are given by

$$ \theta_n = \frac{\prod_{i=1}^{p} \Gamma(a_i + (n-1)A_i)/\Gamma(a_i)}{\prod_{i=1}^{q} (\Gamma(b_i + (n-1)B_i)/\Gamma(b_i))(n-1)!} \tag{1.9} $$

and

$$ \zeta_n = \frac{\prod_{i=1}^{r} \Gamma(c_i + (n-1)C_i)/\Gamma(c_i)}{\prod_{i=1}^{s} (\Gamma(d_i + (n-1)D_i)/\Gamma(d_i))(n-1)!} . \tag{1.10} $$

From (1.9) and (1.10), we have for $n \in \mathbb{N} = \{1, 2, \dots\}$

$$ |\theta_n| \leqslant \frac{\prod_{i=1}^{p} \Gamma(|a_i| + (n-1)A_i)/\Gamma(|a_i|)}{\prod_{i=1}^{q} (\Gamma(b_i + (n-1)B_i)/\Gamma(b_i))(n-1)!} = \nu_n \tag{1.11} $$

and

$$ |\zeta_n| \leqslant \frac{\prod_{i=1}^{r} \Gamma(|c_i| + (n-1)C_i)/\Gamma(c_i)}{\prod_{i=1}^{s} (\Gamma(d_i + (n-1)D_i)/\Gamma(d_i))(n-1)!} = \eta_n. \tag{1.12} $$

For some fixed value of $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for

$$\prod_{i=1}^{q} B_i^{B_i} \geqslant \prod_{i=1}^{p} A_i^{A_i}; \quad \prod_{i=1}^{s} D_i^{D_i} \geqslant \prod_{i=1}^{r} C_i^{C_i},$$

we denote

$$ {}_p\Psi_q \left[\begin{array}{c} (|a_i| + jA_i, A_i)_{1,p} \\ (|b_i| + jB_i, B_i)_{1,q} \end{array} ; 1 \right] = {}_p\Psi_q^j ; \quad {}_r\Psi_s \left[\begin{array}{c} (|c_i| + jC_i, C_i)_{1,r} \\ (|d_i| + jD_i, D_i)_{1s} \end{array} ; 1 \right] = {}_r\Psi_s^j \tag{1.13}$$

provided that

$$\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + \frac{p-q}{2} > \frac{1}{2} + j; \quad \sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + \frac{r-s}{2} > \frac{1}{2} + j. \tag{1.14}$$

Making use of (1.11), (1.12) and (1.13), we have

$$\sum_{n=1+j}^{\infty} (n-j)_j \, \nu_n = \frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(|a_i|)} \, {}_p\Psi_q^j \tag{1.15}$$

and

$$\sum_{n=1+j}^{\infty} (n-j)_j \, \eta_n = \frac{\prod_{i=1}^{s} \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(|c_i|)} \, {}_r\Psi_s^j \tag{1.16}$$

provided that (1.14) holds true.

The convolution of two functions $f(z)$ of the form (1.1) and $\mathscr{F}(z)$ of the form

$$\mathscr{F}(z) = z + \sum_{n=2}^{\infty} \mathbf{h}_n z^n + \overline{\sum_{n=1}^{\infty} \mathbf{g}_n z^n} \tag{1.17}$$

be given by

$$(f * \mathscr{F})(z) = f(z) * \mathscr{F}(z) = z + \sum_{n=2}^{\infty} h_n \mathbf{h}_n z^n + \overline{\sum_{n=1}^{\infty} g_n \mathbf{g}_n z^n}. \tag{1.18}$$

Now, we introduce a convolution operator $\Omega(p, q, r, s)$ as

$$\Omega(p, q, r, s) f(z) = f(z) * \mathfrak{F}(z) = h(z) * \mathfrak{H}(z) + \overline{g(z) * \mathfrak{G}(z)}, \tag{1.19}$$

where $f = h + \overline{g}$ and $\mathfrak{F}(z) = \mathfrak{H}(z) + \overline{\mathfrak{G}(z)}$ given by (1.1) and (1.6) respectively. Hence

$$\Omega(p, q, r, s) f(z) = z + \sum_{n=2}^{\infty} \theta_n h_n z^n + \overline{\sum_{n=1}^{\infty} \zeta_n g_n z^n}. \tag{1.20}$$

We consider the following classes of functions due to Dziok [10]. Let $\mathscr{S}\mathscr{T}_{\mathscr{H}}(F, G)$ denote the class of functions $f \in \mathscr{S}_H$ whose members are given by (1.1), such that

$$\mathscr{S}\mathscr{T}_H(F, G) = \left\{ f = h + \bar{g} \in H : \frac{\mathfrak{D}_H f(z)}{f(z)} \prec \frac{1+Fz}{1+Gz}; \ (-G \leqslant F < G \leqslant 1, \text{ with } g_1 = 0) \right\}, \tag{1.21}$$

where $\mathfrak{D}_H f(z) = zh'(z) - \overline{zg'(z)}$ and $z \in \mathbb{U}$, where H denote the class of harmonic functions in the unit disc $\mathbb{U}$.

Moreover, let us define

$$\mathscr{CV}_H(F,G) := \{f \in \mathscr{ST}_H : \mathfrak{D}_H f \in \mathscr{ST}_H(F,G)\}.$$

We should notice that the class

$$\mathscr{ST}(F,G) := \mathscr{ST}_H(F,G) \cap \mathscr{A}$$

was introduced by Janowski [11]. The classes

$$\mathscr{ST}_H(\alpha) := \mathscr{ST}_H(2\alpha - 1, 1) \quad \text{and} \quad \mathscr{CV}_H(\alpha) := \mathscr{CV}_H(2\alpha - 1, 1)$$

were investigated by Jahangiri [12, 13] also see [4]. Finally, the classes

$$\mathscr{ST}_H := \mathscr{ST}_H(0) \quad \text{and} \quad \mathscr{CV}_H := \mathscr{CV}_H(0)$$

are the classes of functions $f \in \mathscr{S}_H$ which are starlike in $\mathbb{U}$ and $\mathscr{ST}_H(F,G) \subset \mathscr{ST}_H$, $\mathscr{CV}_H(F,G) \subset \mathscr{CV}_H$.

Lately, Dziok [10] gave the following necessary and sufficient coefficient condition for $f \in \mathscr{ST}_H(F,G)$.

Lemma 1 [10]. *Let $f \in H$ be assumed as in* (1.1), *then $f \in \mathscr{ST}_H(F,G)$ if*

$$\sum_{n=2}^{\infty} \big([n(1+G) - (1+F)]\,|h_n| + [n(1+G) + (1+F)]\,|g_n|\big) \leqslant G - F, \tag{1.22}$$

where $h_1 = 1$, $g_1 = 0$ and $(-G \leqslant F < G \leqslant 1)$.

Lemma 2 [10]. *Let $f \in H$ be assumed as in* (1.3) *and $f \in \mathscr{ST}_{\overline{H}}(F,G)$ if and only if*

$$\sum_{n=2}^{\infty} \big([n(1+G) - (1+F)]\,|h_n| + [n(1+G) + (1+F)]\,|g_n|\big) \leqslant G - F, \tag{1.23}$$

where $h_1 = 1$, $g_1 = 0$ and $(-G \leqslant F < G \leqslant 1)$.

Remark 1. In [10], it is also shown that $f = h + \bar{g}$ be given by (1.3) is in the family $\mathscr{ST}_{\overline{H}}(F,G)$, if and only if

$$\sum_{n=2}^{\infty}[n(1+G) - (1+F)]\,|h_n| + \sum_{n=1}^{\infty}[n(1+G) + (1+F)]\,|g_n| \leqslant G - F, \tag{1.24}$$

where $h_1 = 1$, $|g_1| < 1$ and $(-G \leqslant F < G \leqslant 1)$.

Moreover we note that, if $f \in \mathscr{ST}_H(F,G)$, then

$$|h_n| \leqslant \frac{G-F}{n(1+G)-(1+F)}, \; n \geqslant 2 \quad \text{and} \quad |g_n| \leqslant \frac{G-F}{n(1+G)+(1+F)}, \; n \geqslant 1.$$

The application of the special functions on Geometric Function Theory always attracts researchers various kinds of special functions for example hypergeometric functions [14–16], confluent hypergeometric functions [17], generalized hypergeometric functions [5, 18]. Wright functions [19–22], Fox–Wright functions [5, 23], Mittag-Leffler functions [24] generalized Bessel functions [25], have rich applications in analytic and harmonic univalent functions. Motivated with the work of [21, 26, 27], we obtain some inclusion relation between the classes $\mathscr{ST}_H(F,G)$, $\mathscr{K}_H^0$, and $\mathscr{ST}_H^0$, or $\mathscr{C}_H^0$ by applying he convolution operator Ω.

2. Mapping Properties of H Related with Convolution Operator Ω

In order to establish our main results we shall require the following lemmas.

Lemma 3 [1]. *If $f = h + \bar{g} \in \mathscr{K}_H^0$, where h and g are given by (1.1) with $g_1 = 0$, then*

$$|h_n| \leqslant \frac{n+1}{2}, \quad |g_n| \leqslant \frac{n-1}{2}.$$

Lemma 4 [1]. *Let $f = h + \bar{g} \in \mathscr{S}\mathscr{T}_H^0$ or $\mathscr{C}_H^0$, where h and g are given by (1.1) with $g_1 = 0$. Then*

$$|h_n| \leqslant \frac{(2n+1)(n+1)}{6} \quad \text{and} \quad |g_n| \leqslant \frac{(2n-1)(n-1)}{6}.$$

Theorem 1. *Let $\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + \frac{p-q}{2} > \frac{5}{2}$ and $\sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + \frac{r-s}{2} > \frac{5}{2}$, the inequality*

$$\begin{aligned}&\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)}\left\{(1+G)\,{}_p\Psi_q^2 + (3+4G-F)\,{}_p\Psi_q^1 + 2\,(G-F)\left({}_p\Psi_q^0 - 1\right)\right\}\\ &\quad + |\sigma|\,\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)}\left\{(1+G)\,{}_r\Psi_s^2 + (3+2G+F)\,{}_r\Psi_s^1\right\} \leqslant 2\,(G-F)\end{aligned} \tag{2.1}$$

holds. Then $\Omega(\mathscr{K}_H^0) \subset \mathscr{S}\mathscr{T}_H(F,G)$.

$\lhd$ Let $f = h + \bar{g} \in \mathscr{K}_H^0$, where h and g are given by (1.1) with $g_1 = 0$. We have to prove that $\Omega(f) \in \mathscr{S}\mathscr{T}_H(F,G)$, where $\Omega(f)$ is defined by (1.20). To prove $\Omega(f) \in \mathscr{S}\mathscr{T}_H(F,G)$, in view of Lemma 1, it is sufficient to prove that $P_1 \leqslant G - F$, where

$$P_1 = \sum_{n=2}^{\infty}[n(1+G) - (1+F)]\,|\theta_n h_n| + \sum_{n=2}^{\infty}[n(1+G) + (1+F)]\,|\zeta_n g_n|. \tag{2.2}$$

By using Lemma 3

$$\begin{aligned}P_1 &\leqslant \frac{1}{2}\left[\sum_{n=2}^{\infty}(n+1)\,[n(1+G) - (1+F)]\,|\theta_n| + \sum_{n=2}^{\infty}(n-1)\,[n(1+G) + (1+F)]\,|\zeta_n|\right]\\ &= \frac{1}{2}\left[\sum_{n=2}^{\infty}\left\{(n-1)(n-2)(1+G) + (n-1)\,[3+4G-F] + 2\,(G-F)\right\}\nu_n\right]\\ &\qquad + \frac{|\sigma|}{2}\left[\sum_{n=2}^{\infty}\left\{(1+G)(n-2) + (3+2G+F)\right\}\eta_n\right]\\ &= \frac{1}{2}\left[\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)}\left\{(1+G)\,{}_p\Psi_q^2 + [3+4G-F]\,{}_p\Psi_q^1 + 2\,(G-F)\left({}_p\Psi_q^0 - 1\right)\right\}\right.\\ &\qquad \left. + |\sigma|\,\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)}\left\{(1+G)\,{}_r\Psi_s^2 + (3+2G+F)\,{}_r\Psi_s^1\right\}\right] \leqslant G - F\end{aligned}$$

by the given hypothesis. This completes the proof of the Theorem 1.

The result is sharp for the function

$$\Lambda(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right) z^n - \sum_{n=2}^{\infty} \left(\frac{n-1}{2}\right) \bar{z}^n. \quad \triangleright$$

Theorem 2. *Let* $\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + \frac{p-q}{2} > \frac{7}{2}$ *and* $\sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + \frac{r-s}{2} > \frac{7}{2}$. *If the inequality*

$$\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)} \big\{2\,(1+G)\,{}_p\Psi_q^3 + (15G+2F+13)\,{}_p\Psi_q^2 + (24G-9F+15)\,{}_p\Psi_q^1$$
$$+ 6\,(G-F)\big({}_p\Psi_q^0 - 1\big)\big\} + |\sigma|\,\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)} \big\{2\,(1+G)\,{}_r\Psi_s^3 + (9G+2F+11)\,{}_r\Psi_s^2 \tag{2.3}$$
$$+ 3\,(2G+F+3)_r\Psi_s^1\big\} \leqslant 6\,(G-F)$$

holds, then $\Omega(\mathscr{S}\mathscr{T}_H^0) \subset \mathscr{S}\mathscr{T}_{\mathscr{H}}(F,G)$ *and* $\Omega(\mathscr{C}_H^0) \subset \mathscr{S}\mathscr{T}_H(F,G)$.

$\triangleleft$ Let $f = h + \bar{g} \in \mathscr{S}\mathscr{T}_H^0(or, \mathscr{C}_H^0)$, where h and g are given by (1.1) with $g_1 = 0$, we need to prove that $\Omega(f) \in \mathscr{S}\mathscr{T}_H(F,G)$, where $\Omega(f)$ is defined by (1.20). In view of Lemma 1, it is sufficient to prove that $P_1 \leqslant 1-\gamma$, where P_1 is given by (2.2). Now using Lemma 4, we have

$$P_1 \leqslant \frac{1}{6}\left[\sum_{n=2}^{\infty}(n+1)(2n+1)[n(1+G)-(1+F)]\,|\theta_n| + |\sigma|\sum_{n=2}^{\infty}(n-1)(2n-1)\right.$$
$$\left.\times\,[n(1+G)+(1+F)]\,|\zeta_n|\right] = \frac{1}{6}\left[\sum_{n=2}^{\infty}\big\{2(1+G)(n-1)(n-2)(n-3)\right.$$
$$\left.+\,(15G+2F+13)(n-1)(n-2) + (24G-9F+15)(n-1) + 6(G-F)\big\}\,\nu_n\right]$$
$$+\,\frac{|\sigma|}{6}\left[\sum_{n=2}^{\infty}\big\{2(1+G)(n-1)(n-2)(n-3) + (9G+2F+11)(n-1)(n-2)\right.$$
$$\left.+\,(6G+3F+9)(n-1)\big\}\,\eta_n\right] = \frac{1}{6}\left[\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(|a_i|)}\big\{2(1+G)\,{}_p\Psi_q^3 + (15G+2F+13)\,{}_p\Psi_q^2\right.$$
$$\left.+\,(24G-9F+15)\,{}_p\Psi_q^1 + 6\,(G-F)\big({}_p\Psi_q^0-1\big)\big\}\right] + \frac{|\sigma|}{6}\left[\frac{\prod_{i=1}^{s}\Gamma(d_i)}{\prod_{i=1}^{r}\Gamma(|c_i|)}\big\{2\,(1+G)\,{}_r\Psi_s^3\right.$$
$$\left.+\,(9G+2F+11)\,{}_r\Psi_s^2 + 3\,(2G+F+3)\,{}_r\Psi_s^1\big\}\right] \leqslant G-F$$

by the given hypothesis. Thus, the proof of Theorem 2 is established.

The result is sharp for the function

$$f(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}}. \quad \triangleright$$

In our next theorem, we establish connections between $\mathscr{S}\mathscr{T}_H(F,G)$.

Theorem 3. *Let* $\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} |a_i| + \frac{p-q}{2} > \frac{1}{2}$ *and* $\sum_{i=1}^{s} d_i - \sum_{i=1}^{r} |c_i| + \frac{r-s}{2} > \frac{1}{2}$. *If the inequality*

$$\frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(|a_i|)} \left({}_p\Psi_q^0 - 1\right) + |\sigma| \frac{\prod_{i=1}^{s} \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(|c_i|)} \, {}_r\Psi_s^0 \leqslant 1 \tag{2.4}$$

holds, then $\Omega(\mathscr{S}\mathscr{T}_H(F,G)) \subseteq \mathscr{S}\mathscr{T}_H(F,G)$.

$\triangleleft$ Let $f = h + \bar{g} \in \mathscr{S}\mathscr{T}_H(F,G)$ be given by (1.1) with $|g_1| < 1$. We have to prove that $P_2 \leqslant G - F$, where

$$P_2 = \sum_{n=2}^{\infty} [n(1+G) - (1+F)] \, |\theta_n h_n| + |\sigma| \sum_{n=1}^{\infty} [n(1+G) + (1+F)] \, |\zeta_n g_n|. \tag{2.5}$$

Now, using Remark 1, we have

$$P_2 \leqslant (G-F) \sum_{n=2}^{\infty} \nu_n + (G-F)\, \sigma \sum_{n=1}^{\infty} \eta_n$$

$$= (G-F) \left(\frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(|a_i|)} \left({}_p\Psi_q^0 - 1\right) + |\sigma| \frac{\prod_{i=1}^{s} \Gamma(d_i)}{\prod_{i=1}^{r} \Gamma(|c_i|)} \, {}_r\Psi_s^0 \right) \leqslant G - F$$

by the given hypothesis, this completes the proof of Theorem 3.

The result is sharp for the function

$$f(z) = z - \sum_{n=2}^{\infty} \left(\frac{G-F}{n(1+G) - (1+F)} \right) |x_n| z^n + \sum_{n=1}^{\infty} \left(\frac{G-F}{n(1+G) + (1+F)} \right) |y_n| \bar{z}^n,$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1. \quad \triangleright$$

3. Some Consequences Related with Mittag-Leffler Functions

If we let $p = q = r = s = 1$ and $a_1 = A_1 = c_1 = C_1 = 1$ in (1.6), then $W(z)$ reduces to a harmonic univalent function $\mathbf{E}(z)$ involving the following generalized Mittag-Leffler functions as

$$\mathbf{E}(z) = z\Gamma(b_1)\, \mathbf{E}_{b_1,B_1}^{1,1}[z] + \overline{\sigma z \Gamma(d_1)\, \mathbf{E}_{d_1,D_1}^{1,1}[z]}, \tag{3.1}$$

where

$$\mathbf{E}_{b_1,B_1}^{1,1}[z] = {}_1\Psi_1 \left[\begin{array}{c} (1,1) \\ (b_1, B_1) \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(b_1 + nB_1)}$$

and

$$\mathbf{E}^{1,1}_{d_1,D_1}[z] = {}_1\Psi_1 \left[\begin{array}{c} (1,1) \\ (d_1,D_1) \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(d_1+nD_1)} .$$

With these specializations, the convolution operator $\Omega(p,q,r,s)$ reduces to the operator $\Phi(b_1;B_1;d_1;D_1)$, which is defined as

$$\Phi(b_1;B_1;d_1;D_1)f(z) = f(z) * \mathbf{E}(z) = h(z) * z\Gamma(b_1)\mathbf{E}^{1,1}_{b_1,B_1}[z] + \sigma\overline{g(z) * z\Gamma(d_1)\mathbf{E}^{1,1}_{d_1,D_1}[z]}. \quad (3.2)$$

For these specific values of $p=q=r=s=1$ and $a_1=A_1=c_1=C_1=1$, Theorems 1–3 yield the following results.

Corollary 1. *If the inequality*

$$\begin{aligned} &\Gamma(b_1)\Big\{(1+G)\,\mathbf{E}^{3,1}_{b_1+2B_1,B_1}(1) + (3+4G-F)\,\mathbf{E}^{2,1}_{b_1+B_1,B_1}(1) + 2\,(G-F)\big(\mathbf{E}^{1,1}_{b_1,B_1}-1\big)\Big\} \\ &\quad + |\sigma|\,\Gamma(d_1)\Big\{(1+G)\,\mathbf{E}^{3,1}_{d_1+2D_1,D_1}(1) + (3+2G+F)\,\mathbf{E}^{2,1}_{d_1+D_1,D_1}(1)\Big\} \leqslant 2\,(G-F) \end{aligned} \quad (3.3)$$

holds. Then $\Phi(\mathscr{K}^0_{\mathscr{H}}) \subset \mathscr{S}\mathscr{T}_{\mathscr{H}}(F,G)$.

Now, we state new inclusion results for Janowski-type harmonic functions due to Dziok [10] without proof.

Corollary 2. *If the inequality*

$$\begin{aligned} &\Gamma(b_1)\Big\{2\,(1+G)\,\mathbf{E}^{4,1}_{b_1+3B_1,B_1}(1) + (15G+2F+13)\,\mathbf{E}^{3,1}_{b_1+2B_1,B_1}(1) \\ &\qquad + (24G-9F+15)\,\mathbf{E}^{2,1}_{b_1+B_1,B_1}(1) + 6\,(G-F)\big(\mathbf{E}^{1,1}_{b_1,B_1}-1\big)\Big\} \\ &\quad + |\sigma|\,\Gamma(d_1)\Big\{2\,(1+G)\,\mathbf{E}^{4,1}_{d_1+3D_1,D_1}(1) + (9G+2F+11)\,\mathbf{E}^{3,1}_{d_1+2D_1,D_1}(1) \\ &\qquad + 3\,(2G+F+3)\,\mathbf{E}^{2,1}_{d_1+D_1,D_1}(1)\Big\} \leqslant 6\,(G-F) \end{aligned} \quad (3.4)$$

holds. Then $\Phi(\mathscr{S}\mathscr{T}^0_{H}) \subset \mathscr{S}\mathscr{T}_{H}(F,G)$*, and* $\Phi(C^0_{\mathscr{H}}) \subset \mathscr{S}\mathscr{T}_{H}(F,G)$.

Corollary 3. *If the inequality*

$$\Gamma(b_1)\Big\{\Big(\mathbf{E}^{1,1}_{b_1,B_1}-1\Big)\Big\} + |\sigma|\,\Gamma(d_1)\,\Big(\mathbf{E}^{1,1}_{d_1,D_1}\Big) \leqslant 1 \quad (3.5)$$

holds. Then $\Phi(\mathscr{S}\mathscr{T}_{\overline{H}}(F,G)) \subset \mathscr{S}\mathscr{T}_{H}(F,G)$.

Remark 2. If we put $p=q=r=s=1$, $a_1=c_1=1$, $A_1=C_1=0$ and $\sigma=1$, then

$$\mathscr{W}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(b_1)}{\Gamma(b_1+B_1(n-1))(n-1)!}\, z^n + \overline{\sum_{n=1}^{\infty} \frac{\Gamma(d_1)}{\Gamma(d_1+D_1(n-1))(n-1)!}\, z^n}$$

and results of Theorems 1–3 gives to new inclusion results for Janowski-type harmonic functions due to Dziok [10].

Remark 3. If we put $p=r=2$, $q=s=1$ and $A_1=A_2=B_1=C_1=C_2=D_1=1$ and $\sigma=1$, then

$$\mathscr{W}(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\,(a_2)_{n-1}}{(b_1)_{n-1}\,(n-1)!}\, z^n + \overline{\sum_{n=2}^{\infty} \frac{(c_1)_{n-1}\,(c_2)_{n-1}}{(d_1)_{n-1}\,(n-1)!}\, z^n}$$

and results of Theorems 1–3 yields the new results for the subclasses of Janowski-type harmonic functions due to Dziok [10].

REMARK 4. By taking $F = 2\alpha - 1$ and $G = 1$ one can get the inclusion results for the subclasses $\mathscr{S}\mathscr{T}_H(2\alpha-1,1) \equiv \mathscr{S}\mathscr{T}_H(\alpha)$ and $\mathscr{C}\mathscr{V}_H(2\alpha-1,1) \equiv \mathscr{C}\mathscr{V}_H(\alpha)$ defined and studied by Jahangiri [12, 13].

CONCLUDING REMARK. By defining

$$\mathscr{C}\mathscr{V}_H(F,G) := \left\{ f \in \mathscr{S}\mathscr{T}_H : \mathfrak{D}_H f \in \mathscr{S}\mathscr{T}_H(F,G) \right\}$$

due to Dziok[10] as in Lemma 1 we state the following result: A function $f \in \mathscr{C}\mathscr{V}^0_H(F,G)$ if

$$\sum_{n=2}^{\infty} n \left(\frac{n(1+G)-(1+F)}{G-F} |h_n| + \frac{n(1+G)+(1+F)}{G-F} |g_n| \right) \leqslant 1, \tag{3.6}$$

where $h_1 = 1$; $g_1 = 0$ and $(-G \leqslant F < G \leqslant 1)$. Proceeding as in above results we can obtain analogous inclusion results for the function class $\mathscr{C}\mathscr{V}^0_H(F,G)$ we left this as an exercise to interested readers.

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О ГАРМОНИЧЕСКИХ ФУНКЦИЙ ТИПА ЯНОВСКОГО, СВЯЗАННЫХ С ГИПЕРГЕОМЕТРИЧЕСКИМИ ФУНКЦИЯМИ РАЙТА

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Аннотация. В настоящей работе мы рассматриваем класс гармонических функций типа Яновского, введенный и изученный Дзиоком, члены которого задаются формулой $h(z) = z + \sum_{n=2}^{\infty} h_n z^n$, $g(z) = \sum_{n=1}^{\infty} g_n z^n$ такой, что

$$\mathscr{S}\mathscr{T}_H(F,G) = \left\{ f = h + \overline{g} \in H : \frac{\mathfrak{D}_H f(z)}{f(z)} \prec \frac{1+Fz}{1+Gz}; \quad (-G \leqslant F < G \leqslant 1,\ g_1 = 0) \right\},$$

где $\mathfrak{D}_H f(z) = zh'(z) - \overline{zg'(z)}$, $z \in \mathbb{U} = \{z : \ z \in \mathbb{C}$ и $|z| < 1\}$. Мы изучаем связь между этими подклассами гармонических однолистных функций, применяя определенный оператор свертки, касающийся обобщенных гипергеометрических функций Райта, и в качестве следствия приводятся несколько частных случаев. Кроме того, мы указали на определенные связи между классом гармонических функций типа Яновского, включающими обобщенные функции Миттаг-Леффлера. Кратко указаны соответствующие связи представленных результатов с различными известными результатами.

Ключевые слова: гармонические функции, однолистные функции, обобщенные гипергеометрические функции Райта.

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