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Domain Identification for Semilinear Elliptic Equations in the Plane: the Zero Flux Case

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Abstract. We consider the problem of identifying the domain $\Omega \subset \mathbb{R}^2$ of a semilinear elliptic equation subject to given Cauchy data on part of the known outer boundary Γ and to the zero flux condition on the unknown inner boundary γ, where it is assumed that Γ is a piecewise C^1 curve and that γ is the boundary of a finite disjoint union of simply connected domains, each bounded by a piecewise C^1 Jordan curve. It is shown that, under appropriate smoothness conditions, the domain Ω is uniquely determined. The problem of existence of solution for given data is not considered since it is usually of lesser importance in view of measurement errors giving data for which no solution exists.

Keywords: *Domain identification, semilinear elliptic equations, finitely many holes, zero flux*

AMS subject classification: 35 R 30, 35 J 60

1. Introduction

Let Ω be a domain in $\mathbb{R}^2$ bounded by a known outer boundary Γ and containing finitely many internal holes represented by simply connected domains. Let γ be the (unknown) inner boundary of Ω. Consider the equation

$$A(x,u)u(x) = 0 \qquad (x \in \Omega)$$

where $A(x,u)$ is a (possibly semilinear) elliptic differential operator.

We address the problem of identifying the domain Ω of the equation $A(x,u)u = 0$, subject to given Cauchy conditions on an open portion Γ_0 of Γ and to the condition of zero $A(x,u)$-conormal derivative on γ. We note at once that the problem of existence of a solution is not considered here. In fact, as is often the case with inverse problems, the question of existence for given data is less important than that of uniqueness. Indeed, data given by measurements are usually affected with errors leading to a problem without solution. As a consequence, one works with a problem that may not have a solution and hence has to resort to a regularization. The particular case that $A(x,u)$ is the Laplacian is of special interest. The corresponding problem is related to

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the detection of cracks by the electric method and has been considered in [6] for the special case of one internal hole, the boundary of which is star-shaped and with the condition of zero normal derivative prescribed on γ. In physical terms, the function u is the electric potential, the normal derivative $\frac{\partial u}{\partial n}$ on Γ_0 is the electric flux through Γ_0 (the conductivity being assumed to be 1), and the condition $\frac{\partial u}{\partial n} = 0$ on γ means that the interior of the crack is filled with a non-conducting material (like air). The use of the electric method in crack detection was initiated in the pioneering work of Friedman and Vogelius [10]. The method was applied by Kubo [13], McIver [11], Alessandrini [1, 2] and others (see also [14]). It is noted that a variant of the problem corresponding to the condition $u = 0$ on γ is considered in [9] and in [17]. It was the case of the Laplace equation that motivated the present study since except for our recent paper [6], the published literature on crack detection by the electric method, to our knowledge, has been limited to infinitely thin cracks. However, the full strength our result is not utilized. In fact the result could find applications in heat conduction with temperature dependent conductivity and other problems in the nonlinear realm. These and related problems will be the object of a future study.

Let Ω be as above and let the internal holes in Ω be represented by the simply connected domains $\omega_1, ..., \omega_m$ with disjoint closures, i.e.,

$$\bar{\omega}_i \cap \bar{\omega}_j = \emptyset \qquad (i \neq j). \tag{1}$$

By our definition of γ, we have

$$\gamma = \bigcup_{i=1}^{m} \partial \omega_i.$$

It is assumed that γ is piecewise of C^1 type. We have in mind cracks in a solid, and thus, the discontinuities of $\partial \omega_i$ would correspond to crack edges.

Before giving a precise formulation of the problem, several remarks are in order.

First, the assumption of simply connected cracks excludes rectilinear cracks for which uniqueness does not hold with one boundary measurement (although it does hold with two boundary measurements).

Second, the condition of disjoint closures (1) can be weakened. We still need the $\omega_i' s$ to be mutually disjoint, while, two distinct $\partial \omega_i$ and $\partial \omega_j$ can have an intersection consisting of a finite set of points. Note that if $\partial \omega_i \cap \partial \omega_j \ \ (i \neq j)$ contains a segment, then it can be shown that there is no uniqueness with one measurement.

As a third remark, we note that the present approach does not apply to the 3-dimensional case. For the latter case, we require the crack surfaces to be piecewise analytic (cf. Ang, Mennicken and Trong [5]).

Finally, we remark that in the case of elastic solids, stresses and displacements measured on the outer boundary uniquely determine the locations and shapes of internal cracks (Ang, Trong and Yamamoto [7, 8]). We also refer to the paper of Andrieux, Abda and Bui [4] dealing with the problem of rectilinear or planar cracks in elastic bodies from boundary measurements in terms of a functional introduced by the authors.

We now give a precise formulation of our problem. Let $A(x,u)$ be a differential operator of the form

$$A(x,u)u \equiv \sum_{i,j=1}^{2} D_j(a_{ij}(x,u)D_iu) - c(x)F(u) \tag{2}$$

where $F \in C^1(\mathbb{R})$, $a_{ij} \in C^1(\mathbb{R}^2 \times \mathbb{R})$ and (a_{ij}) satisfies the ellipticity condition, i.e. there exists a constant $c_0 > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x,\sigma)\xi_i\xi_j \geq c_0\,(\xi_1^2 + \xi_2^2)$$

for all $\xi = (\xi_1, \xi_2),\ (x,\sigma) \in \mathbb{R}^2 \times \mathbb{R}$. Consider the equation

$$A(x,u)u = 0 \tag{3}$$

subject to the boundary conditions

$$u|_{\Gamma_0} = f \tag{4}$$

$$\sum_{i,j=1}^{2} a_{ij}(x,u(x))\,n_j(x)D_iu(x) = g(x) \quad (x \in \Gamma_0) \tag{5}$$

where Γ_0 is an open subset of Γ, $n(x) = (n_1(x), n_2(x))$ is the unit vector normal to $\Gamma \cup \gamma$ at x and

$$\sum_{i,j=1}^{2} a_{ij}(x,u(x))\,n_j(x)D_iu(x) = 0 \qquad \big(x \in \gamma_* = \gamma \setminus \{y_1,\ldots,y_k\}\big), \tag{6}$$

where $\{y_1,\ldots,y_k\}$ is a finite subset of γ such that $\gamma_* = \gamma \setminus \{y_1,\ldots,y_k\}$ is of C^1 type. The functions c and F are assumed to satisfy

$$c \in C(\mathbb{R}^2),\ c(x) > 0 \text{ a.e. or } c \equiv 0 \text{ in } \Omega \tag{7}$$

$$f, g \in C(\bar{\Gamma}_0),\ F(0) = 0,\ F'(v) > 0 \text{ for all } v \neq 0. \tag{8}$$

The remainder of the paper consists of two sections. In Section 2, we state the main result of our paper and give a counterexample. Section 3 is devoted to the proof of the main result.

2. Main result, counterexample, and preliminary lemmas

We assume that Ω is in a family of plane domains with outer boundary Γ, and containing finitely many holes ω_i $(i = 1, ..., m)$ such that

$$\Gamma, \partial\omega_i \text{ are piecewise } C^1 \text{ Jordan curves in } \mathbb{R}^2 \tag{9}$$
$$\Gamma \cap \partial\omega_i = \emptyset,\ \bar{\omega_i} \cap \bar{\omega_j} = \emptyset \quad (i \neq j). \tag{10}$$

Then we have

Theorem 1. *Let* $(1), (2)$ *and* $(7) - (10)$ *hold. If we have either* $f \not\equiv \text{const}$ *or* $g \not\equiv 0$*, then there exists at most one pair* (Ω, u)*,* $u \in C(\bar{\Omega}) \cap C^2(\Omega \cup \Gamma_0 \cup \gamma_*) \cap H^1(\Omega)$ *for which* $(3) - (6)$ *hold.*

We shall give a counterexample to show that condition (7) on the coefficient c is essential. Indeed, concerning the coefficient c in equation (3), it would intituitively seem that the condition $c(x) \geq 0$ a.e. alone would ensure uniqueness. However, it is a classical result that this is not true. For the reader's convenience, we nevertheless include a counterexample.

Counterexample. Let $\tilde{c} \in C(\mathbb{R})$ be a function satisfying

$$\tilde{c}(x_1) \begin{cases} > 0 & \text{for all } x_1 > 0 \\ = 0 & \text{for all } x_1 \leq 0, \end{cases} \tag{11}$$

put $c(x_1, x_2) = \tilde{c}(x_1)$ for all $(x_1, x_2) \in \mathbb{R}^2$ and let z be the solution of the Volterra equation

$$z(x_1) = \alpha + \int_{-1}^{x_1} \int_{-1}^{\xi} \tilde{c}(\tau) z(\tau)\, d\tau d\xi \qquad (\alpha > 0). \tag{12}$$

Then clearly $z \in C^2(\mathbb{R}^2)$ and in view of (11), (12) we get

$$z(x_1) = \alpha \quad (x_1 \leq 0) \tag{13}$$
$$z"(x_1) = \tilde{c}(x_1) z(x_1). \tag{14}$$

Put $v(x_1, x_2) = z(x_1)$ and $c(x_1, x_2) = \tilde{c}(x_1)$. In view of (13) and (14) we have

$$\Delta v - cv = 0 \qquad \text{for all } x \in \mathbb{R}^2 \tag{15}$$
$$v(x_1, x_2) = \alpha \qquad \text{for all } x_1 \leq 0. \tag{16}$$

Now, let ω be any simply connected domain satisfying

(i) $\bar{\omega} \subset (-1, 0) \times (-1, 1)$

(ii) $\gamma = \partial\omega$ is a C^1 Jordan curve.

Put

$$\Omega_\gamma = (-1, 1) \times (-1, 1) \setminus \bar{\omega}$$
$$\Gamma_0 = \{(x_1, 1) :\ -1 \leq x_1 \leq 1\}.$$

By (15) and (16), v satisfies

$$\left.\begin{array}{r}\Delta v - cv = 0 \quad \text{on } \Omega_\gamma \\ v|_{\Gamma_0} = z(x_1), \left.\dfrac{\partial v}{\partial n}\right|_{\Gamma_0} = 0 \\ \left.\dfrac{\partial v}{\partial n}\right|_{\gamma} = 0.\end{array}\right\}$$

So (Ω_γ, v) is a solution of problem (3) - (6), but clearly uniqueness fails in this case.

We now state some lemmas that will be used in the proof of Theorem 1.

Lemma 1. *Let $\mathcal{C}$ be a piecewise C^1 Jordan curve. Then there is a homeomorphism $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $h(\mathcal{C}) = \partial B(0,1)$, $h(\mathcal{U}_1) = B(0,1)$ and $h(\mathcal{U}_2) = \mathbb{R}^2 \setminus \overline{B(0,1)}$ where $\mathcal{U}_1$ is the simply connected domain of $\mathbb{R}^2$ bounded by $\mathcal{C}$, $\mathcal{U}_2 = \mathbb{R}^2 \setminus \bar{\mathcal{U}}_1$ and B(0,r) is the ball of radius $r > 0$ centered at $0 \in \mathbb{R}^2$.*

Figure 1

Lemma 2. *Let $\mathcal{C}$,$\mathcal{U}_1$,$\mathcal{U}_2$ be as in Lemma 1, $\mathcal{C}_1$ an arc of $\mathcal{C}$ and G an open neighborhood of $\overline{\mathcal{C}}_1$ in $\mathbb{R}^2$. Then there exists a neighborhood $\mathcal{U}$ of $\overline{\mathcal{C}}_1$ in $\mathbb{R}^2$, $\mathcal{U} \subset G$, such that $\mathcal{U}$ is of the form (see Figure 1)*

$$\mathcal{U} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup S$$

where $\mathcal{V}_1 = \mathcal{U} \cap \mathcal{U}_1$, $\mathcal{V}_2 = \mathcal{U} \cap \mathcal{U}_2$, $S = \mathcal{U} \cap \mathcal{C}$ are connected and furthermore $S \subset \partial\mathcal{V}_1 \cap \partial\mathcal{V}_2$.

The proof of Lemma 1 which relies on the Riemann mapping theorem is omitted. Lemma 2 is a direct consequence of Lemma 1.

3. Proof of Theorem 1

Let (Ω^1, u^1) and (Ω^2, u^2) be solutions of problem (3) - (6). We claim that $\Omega^1 = \Omega^2$ and $u^1 = u^2$.

We first set some notations. Let $\omega_1^i, ..., \omega_{m_i}^i$ $(i = 1, 2)$ be the holes in Ω^i and put

$$\gamma^i = \bigcup_{j=1}^{m_i} \partial\omega_j^i \qquad (i = 1, 2).$$

By assumption, γ^i is C^1-smooth except at a finite number of points $\{y_1^i, ..., y_{k_i}^i\} \subset \gamma^i$. Put

$$\gamma_*^i = \gamma^i \setminus \{y_1^i, ..., y_{k_i}^i\}. \tag{17}$$

We denote by W the (connected) component of $\Omega^1 \cap \Omega^2$ such that $\Gamma \subset \partial W$. For $x \in \gamma^2 \cap \Omega^1$, let m_x be the index in the set $\{1, \ldots, m_2\}$ such that $x \in \partial\omega_{m_x}^2$ and let C_x be the maximal arc of $\partial\omega_{m_x}^2$ such that $x \in C_x$ and $C_x \subset \partial\omega_{m_x}^2 \cap \Omega^1$.

With these notations, we begin the proof of Theorem 1. The proof is by contradiction. If $\Omega^1 \setminus \bar{\Omega}^2 = \emptyset$ and $\Omega^2 \setminus \bar{\Omega}^1 = \emptyset$, then $\Omega^1 = \Omega^2$. Thus we can assume, for example, that $\Omega^1 \setminus \bar{\Omega}^2 \neq \emptyset$. The proof is divided into several steps. The crucial one is Step 3 which establishes the existence of an open subset U_0 of Ω^1 such that ∂U_0 is piecewise of C^1 type and that $\partial U_0 \subset \gamma^1 \cup (\partial W \setminus \Gamma)$. Note that

$$\partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1 = \partial W \setminus (\gamma^1 \cup \Gamma). \tag{18}$$

Step 1. $u^1 = u^2$ *on* W.

Proof. We have

$$\sum_{i,j=1}^{2} D_j(a_{ij}(x, u^1)D_i u^1) - c(x)F(u^1) = 0 \tag{19}$$

$$\sum_{i,j=1}^{2} D_j(a_{ij}(x, u^2)D_i u^2) - c(x)F(u^2) = 0. \tag{20}$$

Letting $v \equiv u^2 - u^1$, one has

$$a_{ij}(x, u^2) - a_{ij}(x, u^1) = v \int_0^1 \frac{\partial a_{ij}}{\partial u}(x, u^1 + t(u^2 - u^1))\, dt$$

$$F(u^2) - F(u^1) = v \int_0^1 F'(u^1 + t(u^2 - u^1))\, dt.$$

Substracting (19) from (20) and using the latter relations, we get after some computations

$$\sum_{i,j=1}^{2} D_j(a_{ij}(x, u^1)D_i v) - \sum_{i,j=1}^{2} D_j(v\, b_{ij}) - c(x)\tilde{c}(x)v = 0 \tag{21}$$

where

$$\left.\begin{aligned} b_{ij}(x) &= b_{ij}(x, u_1(x), u_2(x)) \\ \tilde{c}(x) &= \tilde{c}(x, u_1(x), u_2(x)). \end{aligned}\right\}$$

On the other hand, on Γ_0 one has

$$v|_{\Gamma_0} = 0 \tag{22}$$

$$\sum_{i,j=1}^{2} a_{ij}(x, u^1(x))n_j(x)D_i v(x) = 0 \quad (x \in \Gamma_0) \tag{23}$$

where we have used the condition $u^1 = u^2$ on Γ_0. By uniqueness of the Cauchy problem for elliptic equations [15] we have $v = 0$ on W, i.e. $u^1 = u^2$ on W, which completes the proof of Step 1 ∎

Step 2. *Let $x \in \partial W \setminus (\gamma^1 \cup \Gamma)$. Then $x \in \gamma^2 \cap \Omega^1$ and $C_x \subset \partial W \setminus (\gamma^1 \cup \Gamma) = \partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1$.*

Proof. Noting that $\partial\Omega^1 = \Gamma \cup \gamma^1$ and $W \subset \Omega^1$, we have $\partial W \setminus (\gamma^1 \cup \Gamma) \subset \overline{\Omega}^1 \setminus \partial\Omega^1 = \Omega^1$. On the other hand, since W is a component of $\Omega^1 \cap \Omega^2$, we have

$$\partial W \subset \partial(\Omega^1 \cap \Omega^2) \subset \partial\Omega^1 \cup \partial\Omega^2 \subset \gamma^1 \cup \gamma^2 \cup \Gamma. \tag{24}$$

It follows that $\partial W \setminus (\gamma^1 \cup \Gamma) \subset \gamma^2$. Hence $\partial W \setminus (\gamma^1 \cup \Gamma) \subset \gamma^2 \cap \Omega^1$.

We now show that

$$C_x \subset \partial W \setminus (\gamma^1 \cup \Gamma) \qquad \text{for } x \in \partial W \setminus (\gamma^1 \cup \Gamma). \tag{25}$$

Let C_1 be a subarc of C_x satisfying $\overline{C}_1 \subset C_x$. By Lemma 2, there exists a neighborhood ω of C_1 such that

$$\omega \subset \Omega^1 \qquad \text{and} \qquad \omega \cap \left(\bigcup_{j=1}^{m_2} \omega_j^2 \setminus \omega_{m_x}^2 \right) = \emptyset,$$

ω being of the form

$$\omega = U_1 \cup U_2 \cup S, \qquad S \subset \partial U_1 \cap \partial U_2 \tag{26}$$

where $U_1 \equiv \omega \cap \omega_{m_x}^2, U_2 \equiv \omega \cap \Omega^2, S \equiv \omega \cap C_x$ are connected.

Figure 2

From (24) one has $\omega \cap (\partial\Omega^1 \cup \partial W) \neq \emptyset$. The relation $\omega \subset \Omega^1$ implies $\omega \cap \partial\Omega^1 = \emptyset$. Hence $\omega \cap \partial W \neq \emptyset$, i.e.

$$\omega \cap W \neq \emptyset. \tag{27}$$

On the other hand, we have

$$U_1 = \omega \cap \omega_{m_x}^2 \subset \Omega^1 \setminus \bar{\Omega}^2 \subset \Omega^1 \setminus \overline{W}. \tag{28}$$

Hence (26) and (27) together imply that $U_2 \cap W \neq \emptyset$. The connectedness of U_2 and the maximality of W as a connected subset of $\Omega^1 \cap \Omega^2$ imply that

$$U_2 \subset W. \tag{29}$$

From (26) and (29) we get $C_1 \subset \partial W \setminus (\gamma^1 \cup \Gamma)$. Since C_1 is any arc of C_x such that $\overline{C}_1 \subset C_x$, one has $S = \omega \cap C_x \subset \partial W \setminus (\gamma^1 \cup \Gamma)$. This establishes (25) and thus completes the proof of Step 2 ∎

Step 3. *There is an open subset U_0 of Ω^1 such that $\partial U_0 \subset \gamma^1 \cup (\partial W \setminus \Gamma)$ and that ∂U_0 is piecewise of C^1 type.*

Proof. We have the following two cases.

Case (i): There is an $x_0 \in \partial W \setminus (\Gamma \cup \gamma^1)$ such that $\partial\omega^2_{m_{x_0}} \cap \gamma^1 = \emptyset$ (see Figure 3), i.e. $\partial\omega^2_{m_{x_0}} \cap \partial\omega^1_i = \emptyset \quad (i = 1, ..., m_1)$.

Figures 3 and 4

Then put $U_0 = \omega^2_{m_{x_0}} \cap \Omega^1$. We have $\partial U_0 = \partial\omega^2_{m_{x_0}} \cup (\cup_{i \in I} \partial\omega^1_i)$ where I is the set of the i's in $\{1, ..., m_1\}$ satisfying $\omega^1_i \subset \omega^2_{m_{x_0}}$. Hence U_0 is the required set.

Case (ii): $\partial\omega^2_{m_x} \cap \gamma^1 \neq \emptyset$ for all $x \in \partial W \setminus (\Gamma \cup \gamma^1)$.

Then for each $x \in \partial W \setminus (\Gamma \cup \gamma^1)$, the Jordan arc C_x (cf. Step 2) has its edge points (i.e. the points in $\bar{C}_x \setminus C_x$) in $\partial\omega^1_{m^1_x}$ and $\partial\omega^1_{m^2_x}$ where $\omega^1_{m^1_x}$ and $\omega^1_{m^2_x}$ are holes in Ω^1. To fix ideas, we assume that $m^1_x \leq m^2_x$. The following three situations are to be considered:

- (ii)$_1$ $m^1_{x_0} = m^2_{x_0}$ for some $x_0 \in \partial W \setminus (\Gamma \cup \gamma^1)$.
- (ii)$_2$ $m^1_x \neq m^2_x$ for all $x \in \partial W \setminus (\Gamma \cup \gamma^1)$ and there exist $y_0, z_0 \in \partial W \setminus (\Gamma \cup \gamma^1)$ such that $(m^1_{y_0}, m^2_{y_0}) = (m^1_{z_0}, m^2_{z_0})$ and $C_{y_0} \neq C_{z_0}$.
- (ii)$_3$ $m^1_x \neq m^2_x$ and $C_y = C_z$ or $(m^1_y, m^2_y) \neq (m^1_z, m^2_z)$ for all $x, y, z \in \partial W \setminus (\Gamma \cup \gamma^1)$.

If (ii)$_1$ holds, the set of end points of C_{x_0} has at most two points and, moreover, by the maximality of C_{x_0}, these points are in $\partial\omega^1_{m^1_{x_0}} \subset \gamma_1$. If the end points of C_{x_0} are distinct (see Figure 4), then $\partial\omega^1_{m^1_{x_0}}$ is the union of two Jordan arcs l_α and l_β such that $l_\alpha \cap l_\beta = \{\alpha, \beta\}$. Note that both sets $C_{x_0} \cup l_\alpha$ and $C_{x_0} \cup l_\beta$ are Jordan curves and that one of them, say $\gamma_0 = C_{x_0} \cup l_\alpha$ does not contain the other in its interior. Call U'_0 the domain interior to γ_0 and put $U_0 = U'_0 \cap \Omega^1$. Then clearly U_0 is the desired open set. This proves Step 3 in the case $\alpha \neq \beta$. If $\alpha = \beta$, i.e. if $\bar{C}_{x_0}$ is a Jordan curve, the proof is similar as (in fact easier than) for the case of distinct end points.

If (ii)$_2$ holds (see Figure 5), we denote the end points of C_{y_0} and C_{z_0} by α, β and α', β', respectively. As in the proof for the case (ii)$_1$, we choose U'_0 as the domain interior to the Jordan curve $C_{y_0} \cup l_1 \cup C_{z_0} \cup l_2$ (cf. Figure 5). Put $U_0 = U'_0 \cap \Omega^1$. Just as for the case (ii)$_1$ we get that ∂U_0 is piecewise of C^1 type. This proves Step 3 for the case (ii)$_2$.

Figure 5

Finally, we consider the case (ii)$_3$. We shall prove that Step 3 holds for $U_0 = \Omega^1 \setminus \overline{W}$. Letting $x \in \partial W \setminus (\Gamma \cup \gamma^1)$, Step 2 gives $C_x \subset \partial W \setminus (\Gamma \cup \gamma^1) = \partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1$. Hence

$$\partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1 = \bigcup_{x \in \partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1} C_x. \tag{30}$$

For each pair (p, q) with $p, q \in \{1, ..., m_1\}$ and $p \leq q$, by assumptions (ii)$_3$ there is at most one C_x such that $p = m_x^1$ and $q = m_x^2$. Hence, there are at most m_0 arcs $C_{z_1}, ..., C_{z_{m_0}}$ in $\partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1$ such that

$$\bigcup_{x \in \partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1} C_x = \bigcup_{i=1}^{m_0} C_{z_i} \tag{31}$$

where $m_0 \leq \frac{1}{2} m_1 (m_1 + 1)$ (= the number of pairs (p, q) above). Combining (30) and (31) gives

$$\partial(\Omega^1 \setminus \overline{W}) \setminus \gamma^1 = \bigcup_{i=1}^{m_0} C_{z_i}. \tag{32}$$

Let z_i^1 and z_i^2 be the end points of C_{z_i}, i.e. $\{z_i^1, z_i^2\} = \bar{C}_{z_i} \setminus C_{z_i}$ $(i = 1, ..., m_0)$, and put $B = \{z_i^1, z_i^2 : 1 \leq i \leq m_0\}$. For $x \in (\partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1) \setminus B$ we denote by D_x the maximal arc of $\gamma^1 \setminus B$ such that $x \in D_x$. By the same arguments as for Step 2, we get

$$D_x \subset \partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \qquad \text{for } x \in \partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \setminus B. \tag{33}$$

Since (33) holds, the set $\partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \setminus B$ can be divided into equivalence classes with respect to the equivalence relation

$$x \sim y \qquad \Longleftrightarrow \qquad D_x = D_y$$

(note that the maximality of D_x gives $D_x = D_y$ or $D_x \cap D_y = \emptyset$ for all $x, y \in \partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \setminus B$). We claim that the number of $\sim$-equivalence classes is finite. We first note that, for a couple x, y in a Jordan curve C, there are only two Jordan arcs of C having x, y as its edges points. From the maximality of D_x $(x \in \partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \setminus B)$, its edge points (i.e. the points in $\overline{D}_x \setminus D_x$) are in B. Hence for $z, z' \in B$ there are at most two point $x_1, x_2 \in \partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \setminus B$ such that $D_{x_1} \neq D_{x_2}$ and $\overline{D}_{x_1} \setminus D_{x_1} = \{z, z'\} = \overline{D}_{x_2} \setminus D_{x_2}$. On the other hand, B is finite. Hence the set of $\sim$-equivalence classes is finite.

From the foregoing arguments, we can find $y_1, ..., y_r \in \partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \setminus B$ such that

$$\partial(\Omega^1 \setminus \overline{W}) \cap \gamma^1 \setminus B = \bigcup_{j=1}^{r} D_{y_j}. \tag{34}$$

From (32) and (34) one has

$$\partial(\Omega^1 \setminus \overline{W}) = B \cup (\partial U_0 \setminus \gamma^1) \cup ((\partial U_0 \cap \gamma^1) \setminus B) = B \cup \left(\bigcup_{i=1}^{m_0} C_{z_i} \right) \cup \left(\bigcup_{j=1}^{r} D_{y_j} \right).$$

Here we recall $U_0 = (\Omega^1 \setminus \overline{W})$. This completes the proof of Step 3 ∎

Summarizing, we have shown that in our proof of Theorem 1 by contradiction, the assumption $\Omega^1 \setminus \overline{\Omega}^2 \neq \emptyset$ led to the existence of an open set U_0 as in Step 3.

Step 4. *Using the latter result we will show in this final step that* $u^1 = \text{const}$ *on* Ω^1.

Proof. By Step 3, there is an open set $U_0 \subset \Omega^1$ such that ∂U_0 is piecewise of C^1 type and that $\partial U_0 \subset \gamma^1 \cup (\partial W \setminus \Gamma)$. Hence, there is a finite set $B_1 \subset \gamma^1 \cup \gamma^2$ such that $\partial U_0 \setminus B_1$ is a finite union of C^1-smooth Jordan arcs and that $\{y_i^1, y_j^2 : 1 \leq i \leq k_1, 1 \leq j \leq k_2\} \subset B_1$ where y_i^1 and y_j^2 are as in (17). One has $\partial U_0 = B_1 \cup (\partial U_0 \setminus B_1)$. For $x \in \partial U_0 \setminus B_1$, the following two cases are to be considered:

(a) $x \in \gamma_*^1 = \gamma^1 \setminus \{y_1^1, ..., y_{k_1}^1\}$.

(b) $x \in \partial W \setminus (\gamma^1 \cup \Gamma) \subset \gamma^2$.

For the case (a), we have

$$\sum_{i,j=1}^{2} a_{ij}(x, u^1(x)) n_j(x) D_i u^1(x) = 0. \tag{35}$$

For the case (b), since $x \in \partial W \setminus (\gamma^1 \cup \Gamma) \subset \gamma^2$, (6) gives

$$\sum_{i,j=1}^{2} a_{ij}(x, u^1(x)) n_j(x) D_i u^1(x) = \sum_{i,j=1}^{2} a_{ij}(x, u^2(x)) n_j(x) D_i u^2(x) = 0. \tag{36}$$

From (35) and (36) we conclude that (35) holds for all $x \in \partial U_0 \setminus B_1$. Multiplying (3) by u^1, integrating on U_0 and using the divergence theorem we get in view of (35) and (36)

$$\int_{U_0} \sum_{i,j=1}^{2} a_{ij}(x, u^1(x)) D_i u^1(x) D_j u^1(x)\, dx + \int_{U_0} c(x) F(u^1(x)) u^1(x)\, dx = 0. \tag{37}$$

Now, if $c(x) > 0$ a.e., then (37) implies $F(u^1(x))u^1(x) = 0$ a.e. on U_0. From (8) it follows that $u^1(x) = 0$ for all $x \in U_0$. By Pederson's theorem on the uniqueness of elliptic continuation [15], the above equality gives $u^1 = 0$ on Ω^1, which implies $f = g \equiv 0$, which is a contradiction.

If $c(x) = 0$ for all $x \in \mathbb{R}^2$, then we get in view of (37) and the ellipticity of the operator $A(x, u)$ (cf. (2)) that $\nabla u^1(x) = 0$ a.e. on U_0. If ω_0 is a component of U_0, then the latter equality gives $u^1 = c_1$ (constant) on ω_0. By uniqueness of elliptic continuation, we have $u^1 = c_1$ on Ω^1. Hence $f = c_1$ and $g = 0$, which is a contradiction. This completes the proof of Theorem 1 ∎

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References

[1] Alessandrini, G.: *Stable determination of a crack from boundary measurements.* Proc. R. Soc. Edinburgh 123A (1993), 497 – 516.

[2] Alessandrini, G.: *Examples of instability in inverse boundary-value problems.* Inverse Problems 13 (1997), 887 – 897.

[3] Alessandrini, G. and E. DiBenedetto: *Determining 2-dimensional cracks in 3-dimensional body: uniqueness & stability.* Indiana Univ. Math. J. 46 (1987), 1 – 83.

[4] Andrieux, S., BenAbda, A. and H. D. Bui: *Sur l'Identification de Fissures Planes via le Concept d'Écart à la Réciprocité en Elasticité.* C.R. Acad. Sci. Paris (Série I) 324 (1997), 1431 – 1438.

[5] Ang, D. D., Mennicken, R. and D. D. Trong: *Crack determination by the electric method, the 3-d case* (in preparation).

[6] Ang, D. D. and D. D. Trong: *Crack detection by the electric method: uniqueness and approximation.* Int. J. Fracture, 1999 (to appear).

[7] Ang, D. D., Trong, D. D. and M. Yamamoto: *Crack identification in plane nonhomogeneous isotropic elasticity.* Preprint. University of Tokyo: Preprint UTMS 98-37, September 8, 1998.

[8] Ang, D. D., Trong, D. D. and M. Yamamoto: *Unique continuation and identification of boundary of an elastic body.* J. Inv. & Ill-posed Prob. 3 (1996), 417 – 428.

[9] Ang, D. D. and L. K. Vy: *Domain identification for harmonic functions.* Acta Appl. Math. 38 (1995), 217 – 238.

[10] Friedman, A. and M. Vogelius: *Determining cracks by boundary measurements.* Indiana Univ. Math. J. 38 (1989), 527 – 556.

[11] McIver, M.: *An inverse problem in electro-magnetic crack detection.* IMA J. Appl. Math. 47 (1991), 127 – 141.

[12] Kim and Seo: *Uniqueness of determination of a collate of finite cracks from two boundary measurements.* SIAM J. Math. Analysis 27 (1996), 1336 – 1340.

[13] Kubo, S.: *Requirements for uniqueness crack identification from electric potential distribution.* In: Inverse Problem in Engineering Sciences (eds.: M. Yamaguchi et al.). Tokyo: Springer-Verlag 1990, pp. 52 – 58.

[14] Bui, H. D.: *Inverse Problems in the Mechanics of Materials: an Introduction.* Boca et al.: CRC Press Inc. 1994.

[15] Pederson, R. N.: *On the unique continuation theorem for certain second and fourth order elliptic equations.* Comm. Pure Appl. Math. 11 (1958), 67 – 80.

[16] Santosa, F. and M. Vogelius: *A complete algorithm to determine cracks from electrical body measurements.* J. Eng. Sci. 29 (1991), 913 – 937.

[17] Trong, D. D.: *Domain identification for a nonlinear elliptic equation.* Z. Anal. Anw. 17 (1998), 1021 – 1024.

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