Bednarz, N.; Kieran, C.; Lee, L. (Eds.):

Approaches to Algebra
Perspectives for Research and Teaching

Dordrecht: Kluwer, 1996. – 360 p.
(Mathematics Education Library; 18)
ISBN 0-7923-4145-7

Kaye Stacey, Mollie MacGregor, Melbourne

What should be the main ideas of school algebra? What approaches might be used in the classroom to develop them? How might computer technology enhance algebra learning or change its focus? If you are concerned with any of these questions, read this book. The sixteen contributing authors present a diversity of views, setting the stage for spirited debate on the goals and content of the school algebra curriculum. As the title of the book indicates, a major concern is which of several approaches is appropriate for beginners.

The book is divided into six sections, most sections comprising several chapters. Four aspects of algebra, each providing a different starting point or initial emphasis for an algebra course, make up the central topic of the book. They are:

- algebra as the expression of generality;
- algebra as a problem-solving tool;
- algebra as modeling, using a variety of representations;
- algebra as the study of functions.

Each of these aspects emphasises a different way of giving meaning to elementary algebra and provides the theme for a section of the book. Within each section, authors of individual chapters describe their own opinions, theories or research findings, or reflect on what others have written in previous chapters. The reflective chapters are useful features of this book.

As a prelude to the sections that look at algebra from the four perspectives, there is a valuable section on the history of algebra, describing its roots in arithmetic, geometry, and problem-solving and commenting on the status of "the unknown" in early mathematics. The three chapters in this section, by Charbonneau, Radford and Rojano respectively, alert us to the very slow development of algebra over centuries and the intellectual effort involved. These considerations provide a backdrop for the design of an algebra curriculum. Knowledge of the history of the growth of mathematical concepts might help us understand how students construct them today. However, as Wheeler warns, history may not be a trustworthy guide to the construction of an algebra curriculum for today's students.

Charbonneau explains through examples how algebra as a "science of relations" arose from geometric reasoning. He also shows the origins of the Cartesian method of analysis – the process of assuming the unknown as known in order to find it. Radford gives historical examples of the origins of algebra in both arithmetic and geometry and traces the development of the concepts of unknown and variable. He believes that history gives a better understanding of the significance of knowledge rather than prescribing any approach to teaching. Rojano contends that the separation that students make between algebraic manipulation and the use of algebra in problem-solving may originate in an oversimplified approach to teaching which hides the significance of its origins and the semantic background of its grammar. She provides historical examples to illustrate her observations.

The book presents a range of views on what algebra is and what it means to "think algebraically". Some points of view and suggested teaching strategies reflect an author's personal beliefs about mathematics in general and algebra in particular, whereas others are based on the results of research studies or on practical experience in the classroom. Mason's view, for example, is that algebraic thinking – noticing sameness and difference, seeing generality in patterns, classifying and labelling – is the essential foundation of algebraic behaviour. Through a series of particular examples, Mason develops the point that expressing generality is the key process underlying deep understanding of algebra. Teaching experiments described by Lee indicate that generalising activities are a sound introduction to algebraic thinking and can generate enthusiasm and excitement even in weak students. Lee presents algebra as a culture; students have to learn to function comfortably in the new algebraic culture whilst maintaining links to the old arithmetic culture. She claims that the modern practice of staying within the old culture (arithmetic) while learning to operate in the new one (algebra) is a cause of the difficulties and obstacles enumerated in the literature.

As Radford explains, algebraic thinking involves not only the generalising that leads to concepts of variable and function but also the analytic reasoning that is involved in problem-solving. Analytic reasoning in algebra requires using unknowns as abstract objects that can be manipulated, in contrast with arithmetic methods where reasoning begins with what is known. Many authors are concerned with how to move students from one way of reasoning to the other. Have we any reason to assume that the transition must evolve smoothly? The path may be discontinuous and non-linear. Most contributors to this book, however, seem to believe that the transition can be smooth, via generalising geometric patterns, constructing spreadsheets, developing formulas for solving problems, or seeing graphical representations of changing values of a variable. In fact we see here a proliferation of alternatives to the currently established method of introducing algebra as primarily a rule-based game of transforming expressions. How was it that this approach, now universally condemned, became established as the norm? A review of the history of the teaching of algebra would make interesting reading.

If school algebra begins as a way to solve certain problems, what should these problems be like? Bell proposes some generic problems as authentic algebraic activities that should support all aspects of algebra learning. His method is to begin with a prototype problem in which students themselves can change the elements and structure, thereby exploring ways of dealing with a general class of problem situations. Bednarz and Janvier compare the complexity of typical word problems. They identify features of the problems that make them amenable to either arithmetic or algebraic thinking. The results of their empirical study reveal the contrast between the representation and management of quantities and relationships in spontaneous arithmetic thinking and the kind of reasoning required for algebra. These authors question the value of the currently popular numeric trials approach, which encourages students to initially solve problems by trial and error. Because this procedure encourages children to think of successive calculations rather than general relationships, it does not lead on to the algebraic solution. On the other hand, Rojano provides evidence that numeric trials on a spreadsheet did help some students solve "real-world" problems and build a bridge from thinking about specific quantities to recognising a general pattern or relationship. She presents some interview evidence that there are substantial differences between the concept of variation of the unknown developed by the spreadsheet method and the concept of specific unknown underlying informal trial and error.

The possibilities offered by new technology feature strongly in the book, both as an aid to developing traditional algebra and as a doorway to new curriculum goals. Computing technology provides access to representations of variables as quantities with changing values. Nemirovsky describes how students are able to relate the motion of a toy car with the graphs of its speed or distance by using a motion detector linked to a computer. Heid then asks whether the facility to represent motion by graphing will lead to a new kind of mathematical knowledge. If so, how important is pencil-and-paper symbol manipulation in the early phases of algebra learning? Both the modeling approach and the functional approach to algebra are discussed in the context of technology-supported instruction. Nemirovsky is concerned with students interacting with graphical representations for developing concepts of rates of change. Heid, who also sees computer modeling as a gateway to algebraic thinking, describes features of Computer Intensive Algebra which is based on experiences with functions, and she reports the results of some classroom experiments. In their courses, students are provided with function graphers, symbol manipulators and table generators, enabling them to explore families of functions. This is the only part of the book that touches on the looming question of what to do with computer algebra. Kieran, Boileau and Garançon report a series of investigations with the CARAPACE software that caters for beginning students who focus on operations rather than on structures, and builds their algebraic knowledge from natural-language expressions of algorithms. Later there are interesting comments on the differences between a pointwise functional approach (which emphasizes that the function turns each input number into an output number in a regulated way) and a functional approach (which stresses features of the rate of change of functions).

The need for a more thorough examination of the terms used in mathematics research and more agreement about meaning is taken up in Claude Janvier's useful chapter on "Modeling and the Initiation into Algebra". He discusses what is unknown about unknowns, what are the differences between functions, formulas and equations (although they may look identical), and the usual definition of modeling. The term "modeling" had been used quite differently by Nemirovsky in an earlier chapter, to describe the process of interpreting a graph in terms of a situation.

After reading this book the reader is aware of a tension between the different goals for students' learning that the various authors have in mind. At one extreme, a philosophical perspective places emphasis on concepts that inhabit the mathematical world – the Platonic ideals (e.g., functions) whose shadows (e.g., graphs) are seen on the walls of a cave. At the other extreme, a pragmatic approach tries to find out what makes sense to students (e.g., numeric trials to solve a problem), emphasising a familiarity with what is on the floor of the cave first before an investigation of the shadows on the walls and their source. The computer-enriched approach can be seen as moving students to a new cave equipped with the latest technology. With motivation and support from the multiple representation systems that technology provides, it is hoped that students will see beyond the shadows and catch frequent glimpses of the glorious mathematical universe.

Much research in the classroom is needed involving practising teachers and ordinary students before anyone can say where the various approaches would lead if they were widely adopted. There is consensus at present about the goals of algebra learning for bright and highly motivated students. However it is hard to decide on a first priority for weaker students – what we want them to be able to do with algebra and why they are learning it. Is there a minimum level of algebra understanding that all students should reach, and what would it be about: notation, functions, generalisation, or analytic problem solving? In his reflections at the end of the book, Wheeler asks whether mathematics educators distinguish sufficiently between what learning is important for society as a whole and what is important for each individual. All students should have the chance to enter what Lee calls “the algebra culture”, but how far should each be expected to go?

The chapters of the book concerned with current approaches to teaching offer a variety of opinions and some degree of consensus about certain curricular decisions. Wheeler’s commentary in the final chapter does not attempt to draw together the many issues discussed but summarises their differences and the important questions they raise. Like many edited books on aspects of mathematics education, this one does not attempt to compare, contrast and synthesise information provided in the various chapters. Instead, it leaves this task to the reader. As one example, consider the use of numeric trials for solving word problems as opposed to an algebraic method. This is touched on in the chapters by Bednarz and Janvier, in Rojano’s chapter on spreadsheet approaches, and in the chapter by Kieran, Boileau and Garançon. All these authors agree that there is an important difference between arithmetic and algebraic solution patterns. Bednarz and Janvier suggest that the gulf between numeric trials and algebra is greater than the gulf between logical arithmetic reasoning and algebra. They believe this is due to the fact that the use of numeric trials never requires integrating relationships in the problem but always works on one relationship at a time. However Kieran, Boileau and Garançon refer to the effectiveness of the CARAPACE software in building a bridge between numeric trials and algebra. In this approach, numeric trials are used in such a way that they develop understanding of relationships instead of being merely a means for getting answers. Rojano takes a different view, seeing numeric trials as “a real foundation upon which the methods or strategies of algebraic thought are constructed”. In Rojano’s program, students set up a spreadsheet with an initial cell for the unknown, and use this cell as a reference for expressing the rest of the unknowns and data. The value of the unknown is then varied until the problem conditions are met, giving the answer to the problem. Together the three chapters provide insights into the different ways numeric trials can be used and how they relate to algebraic thinking. However the three groups of authors do not refer to each others’ work, and the editors have not attempted to draw it together.

The book is to be read as a whole for the arguments it presents and the questions it provokes. Since it focuses on the work of the contributors, it is clearly not intended to provide an overview of research in the field of initial school algebra instruction. The chapter called “Technology and a Functional Approach to Algebra”, for example, contains only a few references to related studies. Many readers, especially those new to the literature on algebra learning, may be puzzled by some of the terms used. A dictionary will not help you find a meaning for “syncopation of algebra”, “the didactic cut”, “affection of a power” (is this a translation from another language?) or “epistemic norms”. However most of the writing is clear and the important messages it conveys will be understood. The book will be a valuable stimulus for discussion amongst teachers, researchers, and all those interested in the purposes and content of school algebra courses.

Authors

Stacey, Kaye, Prof. Dr., University of Melbourne, Department of Science & Mathematics Education, Parkville, Victoria 3052, Australia

MacGregor, Mollie, Dr., University of Melbourne, Department of Science & Mathematics Education, Parkville, Victoria 3052, Australia. m.macgregor@edfac.unimelb.edu.au