«ACTA»2002

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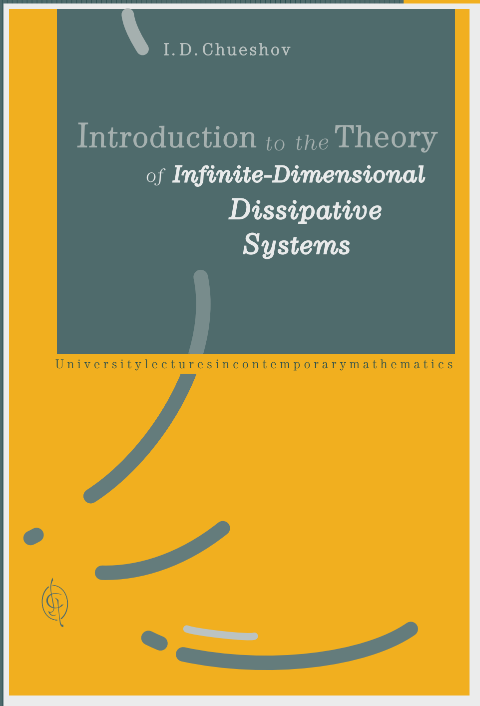
Title: Introduction to the Theory of Infinite-Dimensional Dissipative Systems

ISBN: 966–7021–64–5



This book provides an exhaustive introduction to the scope of main ideas and methods of the theory of infinite-dimensional dissipative dynamical systems which has been rapidly developing in recent years. In the examples systems generated by nonlinear partial differential equations arising in the different problems of modern mechanics of continua are considered. The main goal of the book is to help the reader to master the basic strategies used in the study of infinite-dimensional dissipative systems and to qualify him/her for an independent scientific research in the given branch. Experts in nonlinear dynamics will find many fundamental facts in the convenient and practical form in this book.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during a number of years. This book contains a large number of exercises which make the main text more complete. It is sufficient to know the fundamentals of functional analysis and ordinary differential equations to read the book.

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Chapter 5

Theory of Functionals that Uniquely Determine Long-Time Dynamics

Contents

The results presented in previous chapters show that in many cases the asymptotic behaviour of infinite-dimensional dissipative systems can be described by a finite-dimensional global attractor. However, a detailed study of the structure of attractor has been carried out only for a very limited number of problems. In this regard it is of importance to search for minimal (or close to minimal) sets of natural parameters of the problem that uniquely determine the long-time behaviour of a system. This problem was first discussed by Foias and Prodi [1] and by Ladyzhenskaya [2] for the 2D Navier-Stokes equations. They have proved that the long-time behaviour of solutions is completely determined by the dynamics of the first N Fourier modes if N is sufficiently large. Later on, similar results have been obtained for other parameters and equations. The concepts of determining nodes and determining local volume averages have been introduced. A general approach to the problem on the existence of a finite number of determining parameters has been discussed (see survey [3]).

In this chapter we develop a general theory of determining functionals. This theory enables us, first, to cover all the results mentioned above from a unified point of view and, second, to suggest rather simple conditions under which a set of functionals on the phase space uniquely determines the asymptotic behaviour of the system by its values on the trajectories. The approach presented here relies on the concept of completeness defect of a set of functionals and involves some ideas and results from the approximation theory of infinite-dimensional spaces.

§ 1 Concept of a Set of Determining Functionals

Let us consider a nonautonomous differential equation in a real reflexive Banach space H of the type

$$\frac{du}{dt} = F(u, t), \qquad t > 0, \qquad u\big|_{t=0} = u_0 \; . \tag{1.1}$$

Let $\mathscr{W}$ be a class of solutions to (1.1) defined on the semiaxis $\mathbb{R}_+ \equiv \{t : t \ge 0\}$ such that for any $u(t) \in \mathscr{W}$ there exists a point of time $t_0 > 0$ such that

$$u(t) \in C(t_0, +\infty; H) \cap L^2_{\mathrm{loc}}(t_0, +\infty, V), \tag{1.2}$$

where V is a reflexive Banach space which is continuously embedded into H. Hereinafter $C(a, b; X)$ is the space of strongly continuous functions on $[a, b]$ with the values in X and $L^2_{\mathrm{loc}}(a, b; X)$ has a similar meaning. The symbols $\|\cdot\|_H$ and $\|\cdot\|_V$ stand for the norms in the spaces H and V, $\|\cdot\|_H \le \|\cdot\|_V$.

The following definition is based on the property established in [1] for the Fourier modes of solutions to the 2D Navier-Stokes system with periodic boundary conditions.

Let $\mathcal{L} = \{l_j:\ j = 1, \ldots, N\}$ be a set of continuous linear functionals on V. Then $\mathcal{L}$ is said to be a ***set of asymptotically*** $(V, H, \mathcal{W})$***-determining functionals*** (or elements) for problem (1.1) if for any two solutions $u, v \in \mathcal{W}$ the condition

$$\lim_{t \to \infty} \int_t^{t+1} |l_j(u(\tau)) - l_j(v(\tau))|^2 \, d\tau = 0 \quad \text{for} \quad j = 1, \ldots, N \tag{1.3}$$

implies that

$$\lim_{t \to \infty} \|u(t) - v(t)\|_H = 0 . \tag{1.4}$$

Thus, if $\mathcal{L}$ is a set of asymptotically determining functionals for problem (1.1), the asymptotic behaviour of a solution $u(t)$ is completely determined by the behaviour of a finite number of scalar values $\{l_j(u(t)):\ j = 1, 2, \ldots, N\}$. Further, if no ambiguity results, we will sometimes omit the spaces V, H, and $\mathcal{W}$ in the description of determining functionals.

— Exercise *1.1* Show that condition (1.3) is equivalent to

$$\lim_{t \to \infty} \int_t^{t+1} [\mathcal{N}_{\mathcal{L}}(u(\tau) - v(\tau))]^2 \, d\tau = 0 ,$$

where $\mathcal{N}_{\mathcal{L}}(u)$ is a seminorm in H defined by the equation

$$\mathcal{N}_{\mathcal{L}}(u) = \max_{l \in \mathcal{L}} |l(u)| .$$

— Exercise *1.2* Let u_1 and u_2 be stationary (time-independent) solutions to problem (1.1) lying in the class $\mathcal{W}$. Let $\mathcal{L} = \{l_j:\ j = 1, \ldots, N\}$ be a set of asymptotically determining functionals. Show that condition $l_j(u_1) = l_j(u_2)$ for all $j = 1, 2, \ldots, N$ implies that $u_1 = u_2$.

The following theorem forms the basis for all assertions known to date on the existence of finite sets of asymptotically determining functionals.

Theorem 1.1.

Let $\mathcal{L} = \{l_j:\ j = 1, \ldots, N\}$ ***be a family of continuous linear functionals on*** V. ***Suppose that there exists a continuous function*** $\mathcal{V}(u, t)$ ***on*** $H \times \mathbb{R}_+$ ***with the values in*** $\mathbb{R}_+$ ***which possesses the following properties:***

a) ***there exist positive numbers*** α ***and*** σ ***such that***

$$\mathcal{V}(u, t) \ge \alpha \cdot \|u\|^\sigma \quad \textit{for all} \quad u \in H, \quad t \in \mathbb{R}_+ ; \tag{1.5}$$

b) ***for any two solutions*** $u(t),\ v(t) \in \mathscr{W}$ ***to problem*** (1.1) ***there exist*** (i) ***a point of time*** $t_0 > 0$, (ii) ***a function*** $\psi(t)$ ***that is locally integrable over the half-interval*** $[t_0, \infty)$ ***and such that***

$$\gamma_\psi^+ = \underline{\lim}_{t \to \infty} \int_t^{t+a} \psi(\tau)\, d\tau \ > \ 0 \tag{1.6}$$

and

$$\Gamma_\psi^+ = \overline{\lim}_{t \to \infty} \int_t^{t+a} \max\{0,\ -\psi(\tau)\}\, d\tau \ < \ \infty \tag{1.7}$$

for some $a > 0$, ***and*** (iii) ***a positive constant*** C ***such that for all*** $t \geq s \geq t_0$ ***we have***

$$\mathscr{V}(u(t) - v(t),\ t) + \int_s^t \psi(\tau) \cdot \mathscr{V}(u(\tau) - v((\tau),\ \tau))\, d\tau \ \leq$$

$$\leq \ \mathscr{V}(u(s) - v(s),\ s) + C \cdot \int_s^t \max_{j = 1,\ \ldots,\ N} |l_j(u(\tau)) - l_j(v(\tau))|^2\, d\tau\ . \tag{1.8}$$

Then $\mathscr{L}$ ***is a set of asymptotically*** $(V,\ H,\ \mathscr{W})$***-determining functionals for problem*** (1.1).

It is evident that the proof of this theorem follows from a version of Gronwall's lemma stated below.

Lemma 1.1.

Let $\psi(t)$ *and* $g(t)$ *be two functions that are locally integrable over some half-interval* $[t_0, \infty)$. *Assume that* (1.6) *and* (1.7) *hold and* $g(t)$ *is nonnegative and possesses the property*

$$\lim_{t \to \infty} \int_t^{t+a} g(\tau)\, d\tau \ = \ 0, \qquad a > 0. \tag{1.9}$$

Suppose that $w(t)$ *is a nonnegative continuous function satisfying the inequality*

$$w(t) + \int_s^t \psi(\tau) \cdot w(\tau)\, d\tau \ \leq \ w(s) + \int_s^t g(\tau)\, d\tau \tag{1.10}$$

for all $t \geq s \geq t_0$. *Then* $w(t) \to 0$ *as* $t \to \infty$.

It should be noted that this version of Gronwall's lemma has been used by many authors (see the references in the survey [3]).

Proof.

Let us first show that equation (1.10) implies the inequality

$$w(t) \le w(s)\exp\left\{-\int_s^t \psi(\sigma)\,\mathrm{d}\sigma\right\} + \int_s^t g(\tau)\exp\left\{-\int_\tau^t \psi(\sigma)\,\mathrm{d}\sigma\right\}\mathrm{d}\tau \quad (1.11)$$

for all $t \ge s \ge t_0$. It follows from (1.10) that the function $w(t)$ is absolutely continuous on any finite interval and therefore possesses a derivative $\dot{w}(t)$ almost everywhere. Therewith, equation (1.10) gives us

$$\dot{w}(t) + \psi(t)w(t) \le g(t) \quad (1.12)$$

for almost all t. Multiplying this inequality by

$$e(t) = \exp\left\{\int_s^t \psi(\sigma)\,\mathrm{d}\sigma\right\},$$

we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(t)\,e(t)) \le g(t)e(t)$$

almost everywhere. Integration gives us equation (1.11).

Let us choose the value s such that

$$\int_\tau^{\tau+a} \max\{-\psi(\sigma),\ 0\}\,\mathrm{d}\sigma \le \Gamma+1\ , \qquad \Gamma \equiv \Gamma_\psi^+ \quad (1.13)$$

and

$$\int_\tau^{\tau+a} \psi(\sigma)\,\mathrm{d}\sigma \ge \frac{\gamma}{2}\,, \qquad \gamma \equiv \gamma_\psi^+ \quad (1.14)$$

for all $\tau \ge s$. It is evident that if $t \ge \tau \ge s$ and $k = \left[\frac{t-\tau}{a}\right]$, where $[\,\cdot\,]$ is the integer part of a number, then

$$\int_\tau^t \psi(\sigma)\,\mathrm{d}\sigma = \int_\tau^{ka+\tau} \psi(\sigma)\,\mathrm{d}\sigma + \int_{ka+\tau}^t \psi(\sigma)\,\mathrm{d}\sigma \ge$$

$$\ge \frac{\gamma}{2}k - \int_{ka+\tau}^{(k+1)a+\tau} \max\{-\psi(\sigma),\ 0\}\,\mathrm{d}\sigma \ge \frac{\gamma}{2}k - (\Gamma+1)\ .$$

Thus, for all $t \ge \tau \ge s$

$$\int_\tau^t \psi(\sigma)\,d\sigma \;\geq\; \frac{\gamma}{2a}(t-\tau)-\Big(1+\Gamma+\frac{\gamma}{2}\Big).$$

Consequently, equation (1.11) gives us that

$$w(t) \;\leq\; C(\Gamma,\,\gamma)\left[w(s)\exp\left\{-\frac{\gamma}{2a}(t-s)\right\}+\int_s^t g(r)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\}d\tau\right],$$

where $C(\Gamma,\,\gamma)=\exp\left\{1+\Gamma+\frac{\gamma}{2}\right\}$. Therefore,

$$\overline{\lim_{t\to\infty}}\, w(t) \;\leq\; C(\Gamma,\,\gamma)\cdot\overline{\lim_{t\to\infty}}\int_s^t g(\tau)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\}d\tau\,. \tag{1.15}$$

It is evident that

$$G(t,\,s) \;\equiv\; \int_s^t g(\tau)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\}d\tau \;\leq$$

$$\leq\; \sum_{k=0}^{N}\int_{s+ka}^{s+(k+1)a} g(\tau)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\}d\tau\,,$$

where $N=\left[\frac{t-s}{a}\right]$ is the integer part of the number $\frac{t-s}{a}$. Therefore,

$$G(t,\,s) \;\leq\; \sup_{\sigma\geq s}\int_\sigma^{\sigma+a} g(\tau)\,d\tau\cdot\sum_{k=0}^{N}\int_{s+ka}^{s+(k+1)a}\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\}d\tau \;=$$

$$=\; \sup_{\sigma\geq s}\int_\sigma^{\sigma+a} g(\tau)\,d\tau\int_s^{s+(N+1)a}\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\}d\tau \;=$$

$$=\; \frac{2a}{\gamma}\cdot\sup_{\sigma\geq s}\int_\sigma^{\sigma+a} g(\tau)\,d\tau\cdot\exp\left\{-\frac{\gamma}{2a}(t-s)\right\}\left[e^{\frac{\gamma}{2}(N+1)}-1\right].$$

Since $N=\left[\frac{t-s}{a}\right]$, this implies that

$$\lim_{t\to\infty} G(t,\,s) \;\leq\; C(a,\,\gamma)\cdot\sup_{\sigma\geq s}\int_\sigma^{\sigma+a} g(\tau)\,d\tau \tag{1.16}$$

for any s such that equations (1.13) and (1.14) hold. Hence, equations (1.15) and (1.16) give us that

$$\overline{\lim_{t\to\infty}}\, w(t) \;\le\; C(\Gamma,\, \gamma,\, a)\cdot \sup_{\sigma\ge s} \int_{\sigma}^{\sigma+a} g(\tau)\,d\tau\,.$$

If we tend $s \to \infty$, then with the help of (1.9) we obtain

$$\overline{\lim_{t\to\infty}}\, w(t) = 0\,.$$

This implies the assertion of Lemma 1.1.

In cases when problem (1.1) is the Cauchy problem for a quasilinear partial differential equation, we usually take some norm of the phase space as the function $\mathcal{V}(u,\, t)$ when we try to prove the existence of a finite set of asymptotically determining functionals. For example, the next assertion which follows from Theorem 1.1 is often used for parabolic problems.

Corollary 1.1.

Let V and H be reflexive Banach spaces such that V is continuously and densely embedded into H. Assume that for any two solutions $u_1(t),\, u_2(t) \in \mathcal{W}$ to problem (1.1) *we have*

$$\|u_1(t)-u_2(t)\|_V^2 + \int_s^t \psi(\tau)\|u_1(\tau)-u_2(\tau)\|_V^2\,d\tau \;\le$$

$$\le\; \|u_1(s)-u_2(s)\|_V^2 + K\int_s^t \|u_1-u_2\|_H^2\,d\tau \tag{1.17}$$

for $t \ge s \ge t_0$, where K is a constant and the function $\psi(t)$ depends on $u_1(t)$ and $u_2(t)$ in general and possesses properties (1.6) *and* (1.7). *Assume that the family $\mathcal{L} = \{l_j\colon\; j = 1,\, \ldots,\, N\}$ on V possesses the property*

$$\|v\|_H \;\le\; C\cdot \max_{j=1\ldots N}|l_j(v)| + \varepsilon_{\mathcal{L}}\|v\|_V \tag{1.18}$$

for any $v \in V$, where C and $\varepsilon_{\mathcal{L}}$ are positive constants depending on $\mathcal{L}$. Then $\mathcal{L}$ is a set of asymptotically determining functionals for problem (1.1), *provided*

$$\varepsilon_{\mathcal{L}}^2 \;<\; \frac{1}{K}\cdot \underline{\lim_{t\to\infty}}\; \frac{1}{a}\cdot \int_t^{t+a} \psi(\tau)\,d\tau \;\equiv\; \gamma_\psi^+ a^{-1}K^{-1}\,. \tag{1.19}$$

Proof.

Using the obvious inequality

$$(a+b)^2 \le (1+\delta)a^2 + \Big(1+\frac{1}{\delta}\Big)b^2, \qquad \delta > 0,$$

we find from equation (1.18) that

$$\|v\|_H^2 \le (1+\delta)\varepsilon_{\mathscr{L}}^2 \|v\|_V^2 + C_\delta \cdot \max_{j=1\dots N} |l_j(v)|^2 \tag{1.20}$$

for any $\delta > 0$. Therefore, equation (1.17) implies that

$$\|u_1(t)-u_2(t)\|_V^2 + \int_s^t \Big(\psi(\tau)-(1+\delta)K\varepsilon_{\mathscr{L}}^2\Big)\|u_1(\tau)-u_2(\tau)\|_V^2\, d\tau \le$$

$$\le \|u_1(s)-u_2(s)\|_V^2 + C\int_s^t [\mathscr{N}_{\mathscr{L}}(u_1(\tau)-u_2(\tau))]^2\, d\tau ,$$

where $\mathscr{N}_{\mathscr{L}}(v) = \max_{j=1\dots N} |l_j(v)|$. Consequently, if for some $\delta > 0$ the function

$$\tilde{\psi}(t) = \psi(t) - (1+\delta)K\varepsilon_{\mathscr{L}}^2$$

possesses properties (1.6) and (1.7) with some constants γ and $\Gamma > 0$, then Theorem 1.1 is applicable. A simple verification shows that it is sufficient to require that equation (1.19) be fulfilled. Thus, Corollary 1.1 is proved.

Another variant of Corollary 1.1 useful for applications can be formulated as follows.

Corollary 1.2.

Let V and H be reflexive Banach spaces such that V is continuously embedded into H. Assume that for any two solutions $u(t), v(t) \in \mathscr{W}$ to problem (1.1) *there exists a moment $t_0 > 0$ such that for all $t \ge s \ge t_0$ the equation*

$$\|u(t)-v(t)\|_H^2 + \nu\int_s^t \|u(\tau)-v(\tau)\|_V^2\, d\tau \le$$

$$\le \|u(s)-v(s)\|_H^2 + \int_s^t \phi(\tau)\cdot\|u(\tau)-v(\tau)\|_H^2\, d\tau \tag{1.21}$$

holds. Here $\nu > 0$ and the positive function $\phi(t)$ is locally integrable over the half-interval $[t_0, \infty)$ and satisfies the relation

$$\overline{\lim_{t\to\infty}} \frac{1}{a}\int_t^{t+a} \phi(\tau)\, d\tau \le R \tag{1.22}$$

for some $a > 0$, *where the constant* $R > 0$ *is independent of* $u(t)$ *and* $v(t)$. *Let* $\mathcal{L} = \{l_j\colon\ j = 1, \ldots, N\}$ *be a family of continuous linear functionals on* V *possessing the property*

$$\|w\|_H \ \le \ C_{\mathcal{L}} \cdot \max_{j=1,\ldots,N} |l_j(w)| + \varepsilon_{\mathcal{L}} \cdot \|w\|_V$$

for any $w \in V$. *Here* $C_{\mathcal{L}}$ *and* $\varepsilon_{\mathcal{L}}$ *are positive constants. Then* $\mathcal{L}$ *is a set of asymptotically* $(V, H, \mathcal{W})$*-determining functionals for problem* (1.1), *provided that* $\varepsilon_{\mathcal{L}} < \sqrt{\nu/R}$.

Proof.

Equation (1.20) implies that

$$\|w\|_V^2 \ \ge \ (1+\delta)^{-1} \cdot \varepsilon_{\mathcal{L}}^{-2} \cdot \|w\|_H^2 \ - \ C_{\mathcal{L},\,\delta} \cdot \max_{j=1,\ldots,N} |l_j(w)|^2$$

for any $\delta > 0$. Therefore, (1.21) implies that

$$\|u(t) - v(t)\|_H^2 + \int_s^t \psi(\tau) \cdot \|u(\tau) - v(\tau)\|_H^2 \, d\tau \ \le$$

$$\le \ \|u(s) - v(s)\|_H^2 \ + \ \nu C_{\mathcal{L},\,\delta} \cdot \int_s^t \max_{j=1,\ldots,N} |l_j(u(\tau) - v(\tau))|^2 \, d\tau \ ,$$

where $\psi(t) = \nu(1+\delta)^{-1} \varepsilon_{\mathcal{L}}^{-2} - \phi(t)$. Using (1.22) and applying Theorem 1.1 with $\mathcal{V}(u, t) = \|u\|^2$, we complete the proof of Corollary 1.2.

Other approaches of introduction of the concept of determining functionals are also possible. The definition below is an extension to a more general situation of the property proved by O.A. Ladyzhenskaya [2] for trajectories lying in the global attractor of the 2D Navier-Stokes equations.

Let $\overline{\mathcal{W}}$ be a class of solutions to problem (1.1) on the real axis $\mathbb{R}$ such that $\overline{\mathcal{W}} \subset L^2_{\rm loc}(-\infty, +\infty;\ V)$. A family $\mathcal{L} = \{l_j\colon\ j = 1, \ldots, N\}$ of continuous linear functionals on V is said to be a ***set of*** $(V, \overline{\mathcal{W}})$***-determining functionals*** (or elements) for problem (1.1) if for any two solutions $u,\ v \in \overline{\mathcal{W}}$ the condition

$$l_j(u(t)) = l_j(v(t)) \ \text{ for } \ j = 1, \ldots, N \ \text{ and almost all } \ t \in R \tag{1.23}$$

implies that $u(t) \equiv v(t)$.

It is easy to establish the following analogue of Theorem 1.1.

Theorem 1.2.

Let $\mathcal{L} = \{l_j\colon\ j = 1, \ldots, N\}$ ***be a family of continuous linear functionals on*** V***. Let*** $\overline{\mathcal{W}}$ ***be a class of solutions to problem*** (1.1) ***on the real axis*** $\mathbb{R}$ ***such that***

$$\overline{\mathscr{W}} \subset C(-\infty, +\infty;\ H) \cap L^2_{\mathrm{loc}}(-\infty, +\infty;\ V)\ . \tag{1.24}$$

Assume that there exists a continuous function $\mathscr{V}(u, t)$ ***on*** $H \times \mathbb{R}$ ***with the values in*** $\mathbb{R}$ ***which possesses the following properties:***

a) ***there exist positive numbers*** α ***and*** σ ***such that***

$$\mathscr{V}(u, t) \ge \alpha \cdot \|u\|^{\sigma} \quad \textit{for all} \quad u \in H, \quad t \in \mathbb{R}\ ; \tag{1.25}$$

b) ***for any*** $u(t), v(t) \in \overline{\mathscr{W}}$

$$\sup_{t\,\in\,\mathbb{R}} \mathscr{V}(u(t)-v(t),\ t) < \infty\ ; \tag{1.26}$$

c) ***for any two solutions*** $u(t), v(t) \in \overline{\mathscr{W}}$ ***to problem*** (1.1) ***there exist*** (i) ***a function*** $\psi(t)$ ***locally integrable over the axis*** $\mathbb{R}$ ***with the properties***

$$\gamma_\psi^- \equiv \varliminf_{t \to -\infty} \int\limits_t^{t+a} \psi(\tau)\, d\tau > 0 \tag{1.27}$$

and

$$\Gamma_\psi^- \equiv \varlimsup_{t \to -\infty} \int\limits_t^{t+a} \max\{0,\ -\psi(\tau)\} d\tau < \infty \tag{1.28}$$

for some $a > 0$, ***and*** (ii) ***a positive constant*** C ***such that equation*** (1.8) ***holds for all*** $t \ge s$. ***Then*** $\mathscr{L}$ ***is a set of*** $(V, \overline{\mathscr{W}})$***-determining functionals for problem*** (1.1).

Proof.

It follows from (1.23), (1.8), and (1.11) that the function $w(t) = \mathscr{V}(u(t)-v(t),\ t)$ satisfies the inequality

$$w(t) \le w(s) \cdot \exp\left\{ -\int\limits_s^t \psi(\tau)\, d\tau \right\} \tag{1.29}$$

for all $t \ge s$. Using properties (1.27) and (1.28) it is easy to find that there exist numbers s^*, $a_0 > 0$, and $b_0 > 0$ such that

$$\int\limits_{s_1}^{s_2} \psi(\tau)\, d\tau \ \ge\ a_0 \cdot (s_2 - s_1) - b_0\,, \qquad s_1 \le s_2 \le s^*\,.$$

This equation and boundedness property (1.26) enable us to pass to the limit in (1.29) for fixed t as $s \to -\infty$ and to obtain the required assertion.

Using Theorem 1.2 with $\mathscr{V}(u,\ t) = \|u\|^2$ as above, we obtain the following assertion.

Corollary 1.3.

Let V *and* H *be reflexive Banach spaces such that* V *is continuously embedded into* H. *Let* $\overline{\mathcal{W}}$ *be a class of solutions to problem* (1.1) *on the real axis* $\mathbb{R}$ *possessing property* (1.24) *and such that*

$$\sup_{t\in R}\|u(t)\|_H<\infty \quad \text{for all} \quad u(t)\in\overline{\mathcal{W}}. \tag{1.30}$$

Assume that for any $u(t),\ v(t)\in\overline{\mathcal{W}}$ *and for all real* $t\ge s$ *equation* (1.21) *holds with* $\nu>0$ *and a positive function* $\phi(t)$ *locally integrable over the axis* $\mathbb{R}$ *and satisfying the condition*

$$\overline{\lim_{s\to-\infty}}\ \frac{1}{a}\int_s^{s+a}\phi(\tau)\,d\tau\ \le\ R \tag{1.31}$$

for some $a>0$. *Here* $R>0$ *is a constant independent of* $u(t)$ *and* $v(t)$. *Let* $\mathcal{L}=\{l_j:\ j=1,\ \dots,\ N\}$ *be a family of continuous linear functionals on* V *possessing property* (1.18) *with* $\varepsilon_{\mathcal{L}}<\sqrt{\nu/R}$. *Then* $\mathcal{L}$ *is a set of asymptotically* $(V,\ \overline{\mathcal{W}})$*-determining functionals for problem* (1.1).

Proof.

As in the proof of Corollary 1.2 equations (1.20) and (1.21) imply that

$$\|u(t)-v(t)\|_H+\int_s^t\psi(\tau)\|u(\tau)-v(\tau)\|_H^2\,d\tau\ \le\ \|u(s)-v(s)\|_H^2$$

for all $t\ge s$, where $\psi(\tau)=\nu(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}-\phi(\tau)$ and δ is an arbitrary positive number. Hence

$$\|u(t)-v(t)\|_H^2\ \le\ \|u(s)-v(s)\|_H^2\exp\left\{-\int_s^t\psi(\tau)\,d\tau\right\} \tag{1.32}$$

for all $t\ge s$. Using (1.31) it is easy to find that for any $\eta>0$ there exists $M_\eta>0$ such that

$$\int_{s_1}^{s_2}\phi(\tau)\,d\tau\ \le\ (R+\eta)(s_2-s_1+a)$$

for all $s_1\le s_2\le -M_\eta$. This equation and boundedness property (1.30) enable us to pass to the limit as $s\to-\infty$ in (1.32), provided $\varepsilon_{\mathcal{L}}<\sqrt{\nu/R}$, and to obtain the required assertion.

We now give one more general result on the finiteness of the number of determining functionals. This result does not use Lemma 1.1 and requires only the convergence of functionals on a certain sequence of moments of time.

Theorem 1.3.

Let V and H be reflexive Banach spaces such that V is continuously embedded into H. Assume that $\mathcal{W}$ is a class of solutions to problem (1.1) *possessing property* (1.2). *Assume that there exist constants $C, K \ge 0$, $\beta > \alpha > 0$, and $0 < q < 1$ such that for any pair of solutions $u_1(t)$ and $u_2(t)$ from $\mathcal{W}$ we have*

$$\|u_1(t) - u_2(t)\|_V \le C\|u_1(s) - u_2(s)\|_V , \qquad s \le t \le s + \beta, \tag{1.33}$$

and

$$\|u_1(t) - u_2(t)\|_V \le K\|u_1(t) - u_2(t)\|_H + q\|u_1(s) - u_2(s)\|_V , \quad s+\alpha \le t \le s+\beta \tag{1.34}$$

for s large enough. Let $\mathcal{L}$ be a finite set of continuous linear functionals on V possessing property (1.18) *with $\varepsilon_{\mathcal{L}} < (1-q)K^{-1}$. Assume that $\{t_k\}$ is a sequence of positive numbers such that $t_k \to +\infty$ and $\alpha \le t_{k+1} - t_k \le \beta$. Assume that*

$$\lim_{k \to \infty} l(u_1(t_k) - u_2(t_k)) = 0, \qquad l \in \mathcal{L} . \tag{1.35}$$

Then

$$\|u_1(t) - u_2(t)\|_V \to 0 \quad \textit{as} \quad t \to +\infty . \tag{1.36}$$

It should be noted that relations like (1.33) and (1.34) can be obtained for a wide class of equations (see, e.g., Sections 1.9, 2.2, and 4.6).

Proof.

Let $u(t) = u_1(t) - u_2(t)$. Then equations (1.34) and (1.18) give us

$$\|u(t_k)\|_V \le q_{\mathcal{L}}\|u(t_{k-1})\|_V + C \max_j |l_j(u(t_k))| ,$$

where $q_{\mathcal{L}} = q(1-\varepsilon_{\mathcal{L}}K)^{-1} < 1$. Therefore, after iterations we obtain that

$$\|u(t_n)\|_V \le q_{\mathcal{L}}^n\|u(t_0)\|_V + C \cdot \sum_{k=1}^{n} q_{\mathcal{L}}^{n-k} \max_j |l_j(u(t_k))| .$$

Hence, equation (1.35) implies that $\|u(t_n)\|_V \to 0$ as $n \to \infty$. Therefore, (1.36) follows from equation (1.33). **Theorem 1.3 is proved**.

Application of Corollaries 1.1–1.3 and Theorem 1.3 to the proof of finiteness of a set $\mathcal{L}$ of determining elements requires that the inequalities of the type (1.17) and (1.21), or (1.30) and (1.33), or (1.33) and (1.34), as well as (1.18) with the constant $\varepsilon_{\mathcal{L}}$ small enough be fulfilled. As the analysis of particular examples shows, the fulfilment of estimates (1.17), (1.21), (1.30), (1.31), (1.33), and (1.34) is mainly connected with the dissipativity properties of the system. Methods for obtaining them are rather well-developed (see Chapters 1 and 2 and the references therein) and in many cases the corresponding constants ν, R, K, and q either are close to optimal or can be estimated explicitly in terms of the parameters of equations. Therefore, the problem of description of finite families of functionals that asymptotically determine the dynamics of the process can be reduced to the study of sets of functionals for which estimate (1.18) holds with $\varepsilon_{\mathscr{L}}$ small enough. It is convenient to base this study on the concept of completeness defect of a family of functionals with respect to a pair of spaces.

§ 2 Completeness Defect

Let V and H be reflexive Banach spaces such that V is continuously and densely embedded into H. The ***completeness defect*** of a set $\mathscr{L}$ of linear functionals on V with respect to H is defined as

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\left\{\|w\|_H : \ w \in V, \ l(w)=0, \ l \in \mathscr{L}, \ \|w\|_V \le 1\right\}. \qquad (2.1)$$

It should be noted that the finite dimensionality of Lin $\mathscr{L}$ is not assumed here.

— Exercise *2.1* Prove that the value $\varepsilon_{\mathscr{L}}(V, H)$ can also be defined by one of the following formulae:

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\left\{\|w\|_H : \ w \in V, \ l(w)=0, \ \|w\|_V = 1\right\}, \qquad (2.2)$$

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\left\{\frac{\|w\|_H}{\|w\|_V} : \ w \in V, \ w \ne 0, \ l(w)=0\right\}, \qquad (2.3)$$

$$\varepsilon_{\mathscr{L}}(V, H) = \inf\left\{C : \ \|w\|_H \le C\|w\|_V, \ w \in V, \ l(w)=0\right\}. \qquad (2.4)$$

— Exercise *2.2* Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be two sets in the space V^* of linear functionals on V. Show that $\varepsilon_{\mathscr{L}_1}(V, H) \ge \varepsilon_{\mathscr{L}_2}(V, H)$.

— Exercise *2.3* Let $\mathscr{L} \subset V^*$ and let $\hat{\mathscr{L}}$ be a weakly closed span of the set $\mathscr{L}$ in the space V^*. Show that $\varepsilon_{\mathscr{L}}(V, H) = \varepsilon_{\hat{\mathscr{L}}}(V, H)$.

The following fact explains the name of the value $\varepsilon_{\mathscr{L}}(V, H)$. We remind that a set $\mathscr{L}$ of functionals on V is said to be complete if the condition $l(w) = 0$ for all $l \in \mathscr{L}$ implies that $w = 0$.

— Exercise *2.4* Show that for a set $\mathscr{L}$ of functionals on V to be complete it is necessary and sufficient that $\varepsilon_{\mathscr{L}}(V, H) = 0$.

The following assertion plays an important role in the construction of a set of determining functionals.

Theorem 2.1.

Let $\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}}(V, H)$ ***be the completeness defect of a set*** $\mathscr{L}$ ***of linear functionals on*** V ***with respect to*** H***. Then there exists a constant*** $C_{\mathscr{L}} > 0$ ***such that***

$$\|u\|_H \le \varepsilon_L \|u\|_V + C_L \cdot \sup\left\{|l(u)|:\ l \in \hat{\mathscr{L}},\ \|l\|_* \le 1\right\} \tag{2.5}$$

for any element $u \in V$***, where*** $\hat{\mathscr{L}}$ ***is a weakly closed span of the set*** $\mathscr{L}$ ***in*** V^****.***

Proof.

Let

$$\mathscr{L}^{\perp} = \{v \in V:\ l(v) = 0,\ l \in \mathscr{L}\} \tag{2.6}$$

be the annihilator of $\mathscr{L}$. If $u \in \mathscr{L}^{\perp}$, then it is evident that $l(u) = 0$ for all $l \in \hat{\mathscr{L}}$. Therefore, equation (2.4) implies that

$$\|u\|_H \le \varepsilon_{\mathscr{L}} \|u\|_V \quad \text{for all} \quad u \in \mathscr{L}^{\perp}, \tag{2.7}$$

i.e. for $u \in \mathscr{L}^{\perp}$ equation (2.5) is valid.

Assume that $u \notin \mathscr{L}^{\perp}$. Since $\mathscr{L}^{\perp}$ is a subspace in V, it is easy to verify that there exists an element $w \in \mathscr{L}^{\perp}$ such that

$$\|u - w\|_V = \operatorname{dist}_V(u,\ \mathscr{L}^{\perp}) = \inf\left\{\|u - v\|_V:\ v \in \mathscr{L}^{\perp}\right\}. \tag{2.8}$$

Indeed, let the sequence $\{v_n\} \subset \mathscr{L}^{\perp}$ be such that

$$d \equiv \operatorname{dist}_V(u,\ \mathscr{L}^{\perp}) = \lim_{n \to \infty} \|u - v_n\|_V.$$

It is clear that $\{v_n\}$ is a bounded sequence in V. Therefore, by virtue of the reflexivity of the space V, there exist an element w from $\mathscr{L}^{\perp}$ and a subsequence $\{v_{n_k}\}$ such that v_{n_k} weakly converges to w as $k \to \infty$, i.e. for any functional $f \in V^*$ the equation

$$f(u - w) = \lim_{k \to \infty} f(u - v_{n_k})$$

holds. It follows that

$$|f(u - w)| \le \lim_{k \to \infty} \|u - v_{n_k}\|_V \cdot \|f\|_* \le d\|f\|_*.$$

Therefore, we use the reflexivity of V once again to find that

$$\|u - w\|_V = \sup\left\{|f(u - w)|:\ f \in V^*,\ \|f\|_* = 1\right\} \le d.$$

However, $\|u - w\|_V \ge d$. Hence, $\|u - w\|_V = d$. Thus, equation (2.8) holds.

Equation (2.7) and the continuity of the embedding of V into H imply that

$$\|u\|_H \le \|w\|_H + \|u-w\|_H \le \varepsilon_{\mathscr{L}}\|w\|_V + C\|u-w\|_V .$$

It is clear that

$$\|w\|_V \le \|u\|_V + \|u-w\|_V .$$

Therefore,

$$\|u\|_H \le \varepsilon_{\mathscr{L}}\|u\|_V + (\varepsilon_{\mathscr{L}}+C)\|u-w\|_V . \tag{2.9}$$

Let us now prove that there exists a continuous linear functional l_0 on the space V possessing the properties

$$l_0(u-w) = \|u-w\|_V, \quad \|l_0\|_* = 1, \quad l_0(v) = 0 \quad \text{for} \quad v \in \mathscr{L}^\perp . \tag{2.10}$$

To do that, we define the functional $\tilde{l}_0$ by the formula

$$\tilde{l}_0(m) = a\|u-w\|_V, \qquad m = v + a(u-w),$$

on the subspace

$$M = \left\{ m = v + a(u-w) : \ v \in \mathscr{L}^\perp, \ a \in \mathbb{R} \right\}.$$

It is clear that $\tilde{l}_0$ is a linear functional on M and $\tilde{l}_0(m) = 0$ for $m \in \mathscr{L}^\perp$. Let us calculate its norm. Evidently

$$\|m\|_V \equiv \|v + a(u-w)\|_V = |a| \cdot \left\| u - w + \frac{1}{a}v \right\|_V, \qquad a \ne 0 .$$

Since $w - a^{-1}v \in \mathscr{L}^\perp$, equation (2.8) implies that

$$\|m\|_V \ge |a| \cdot \|u-w\|_V = \left|\tilde{l}_0(m)\right|, \quad m = v + a(u-w), \quad a \ne 0 .$$

Consequently, for any $m \in M$

$$\left|\tilde{l}_0(m)\right| \le \|m\|_V .$$

This implies that $\tilde{l}_0$ has a unit norm as a functional on M. By virtue of the Hahn-Banach theorem the functional $\tilde{l}_0$ can be extended on V without increase of the norm. Therefore, there exists a functional l_0 on V possessing properties (2.10). Therewith l_0 lies in a weakly closed span $\hat{\mathscr{L}}$ of the set $\mathscr{L}$. Indeed, if $l_0 \notin \hat{\mathscr{L}}$, then using the reflexivity of V and reasoning as in the construction of the functional l_0 it is easy to verify that there exists an element $x \in V$ such that $l_0(x) \ne 0$ and $l(x) = 0$ for all $l \in \mathscr{L}$. It is impossible due to (2.10) .

In order to complete the proof of Theorem 2.1 we use equations (2.9) and (2.10). As a result, we obtain that

$$\|u\|_H \le \varepsilon_{\mathscr{L}}\|u\|_V + (\varepsilon_{\mathscr{L}} + C)\, l_0(u-w) .$$

However, $l_0 \in \hat{\mathscr{L}}$, $l_0(u-w) = l_0(u)$, and $\|l_0\|_* = 1$. Therefore, equation (2.5) holds. **Theorem 2.1 is proved**.

Exercise *2.5* Assume that $\mathscr{L} = \{l_j:\ j = 1,\ \dots,\ N\}$ is a finite set in V^*. Show that there exists a constant $C_{\mathscr{L}}$ such that

$$\|u\|_H \le \varepsilon_{\mathscr{L}}(V, H)\|u\|_V + C_{\mathscr{L}} \max_{j=1,\ \dots,\ N} |l_j(u)| \qquad (2.11)$$

for all $u \in V$.

In particular, if the hypotheses of Corollaries 1.1 and 1.3 hold, then Theorem 2.1 and equation (2.11) enable us to get rid of assumption (1.18) by replacing it with the corresponding assumption on the smallness of the completeness defect $\varepsilon_{\mathscr{L}}(V, H)$.

The following assertion provides a way of calculating the completeness defect when we are dealing with Hilbert spaces.

Theorem 2.2.

Let V and H be separable Hilbert spaces such that V is compactly and densely embedded into H. Let K be a selfadjoint positive compact operator in the space V defined by the equality

$$(Ku,\ v)_V = (u,\ v)_H, \qquad u, v \in V.$$

Then the completeness defect of a set $\mathscr{L}$ of functionals on V can be evaluated by the formula

$$\varepsilon_{\mathscr{L}}(V, H) = \sqrt{\mu_{\max}(P_{\mathscr{L}} K P_{\mathscr{L}})}\ , \qquad (2.12)$$

where $P_{\mathscr{L}}$ is the orthoprojector in the space V onto the annihilator

$$\mathscr{L}^{\perp} = \{v \in V:\ l(v) = 0,\ l \in \mathscr{L}\}$$

and $\mu_{\max}(S)$ is the maximal eigenvalue of the operator S.

Proof.

It follows from definition (2.1) that

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\{\|u\|_H:\ u \in B_{\mathscr{L}}\}$$

where $B_{\mathscr{L}} = \mathscr{L}^{\perp} \cap \{v:\ \|v\|_V \le 1\}$ is the unit ball in $\mathscr{L}^{\perp}$. Due to the compactness of the embedding of V into H, the set $B_{\mathscr{L}}$ is compact in H. Therefore, there exists an element $u_0 \in B_{\mathscr{L}}$ such that

$$\varepsilon_{\mathscr{L}}(V, H)^2 = \|u_0\|_H^2 = (Ku_0,\ u_0)_V \equiv \mu\,.$$

Therewith u_0 is the maximum point of the function $(Ku,\ u)_V$ on the set $B_{\mathscr{L}}$. Hence, for any $v \in \mathscr{L}^{\perp}$ and $s \in \mathbb{R}^1$ we have

$$\frac{(K(u_0 + sv),\ u_0 + sv)_V}{\|u_0 + sv\|_V^2} \le \mu \equiv (Ku_0,\ u_0)_V\,.$$

It follows that

$$(K(u_0+sv),\ u_0+sv)_V - \mu\|u_0+sv\|_V^2 \ \le\ 0\,.$$

It is also clear that $\|u_0\|_V = 1$. Therefore,

$$s^2\{(Kv,\ v)_V - \mu\|v\|^2\} + 2s\{(Ku_0,\ v)_V - \mu(u_0,\ v)_V\} \ \le\ 0$$

for all $s \in \mathbb{R}$. This implies that

$$(Ku_0,\ v) - \mu(u_0,\ v)_V \ \le\ 0$$

for any $v \in \mathcal{L}^{\perp}$. If we take $-v$ instead of v in this equation, then we obtain the opposite inequality. Therefore,

$$(Ku_0,\ v) - \mu(u_0,\ v) \ =\ 0\,, \quad v \in \mathcal{L}^{\perp}\,.$$

Consequently,

$$P_{\mathcal{L}}KP_{\mathcal{L}}u_0 \ =\ \mu u_0\,,$$

i.e. $\mu = (Ku_0,\ u_0)_V = \|u_0\|_H^2$ is an eigenvalue of the operator $P_{\mathcal{L}}KP_{\mathcal{L}}$. It is evident that this eigenvalue is maximal. Thus, **Theorem 2.2 is proved**.

Corollary 2.1.

Assume that the hypotheses of Theorem 2.2 hold. Let $\{e_j\}$ be an orthonormal basis in the space V that consists of eigenvectors of the operator K:

$$Ke_j \ =\ \mu_j e_j\,, \quad (e_i,\ e_j)_V = \delta_{ij}\,, \quad \mu_1 \ge \mu_2 \ge \ldots\,, \quad \mu_N \to 0\,.$$

Then the completeness defect of the system of functionals

$$\mathcal{L} = \{l_j \in V^*:\ \ l_j(v) = (v,\ e_j)_V:\ j = 1,\ 2,\ \ldots,\ N\}$$

is given by the formula $\varepsilon_{\mathcal{L}}(V, H) = \sqrt{\mu_{N+1}}$.

To prove this assertion, it is just sufficient to note that $P_{\mathcal{L}}$ is the orthoprojector onto the closure of the span of elements $\{e_j:\ j \ge N+1\}$ and that $P_{\mathcal{L}}$ commutes with K.

— Exercise *2.6* Let A be a positive operator with discrete spectrum in the space H:

$$Ae_k = \lambda_k e_k\,, \quad \lambda_1 \le \lambda_2 \le \ldots\,, \quad \lambda_N \to +\infty\,, \quad (e_k,\ e_j)_H = \delta_{kj}\,,$$

and let $\mathcal{F}_s = D(A^s)$, $s \in \mathbb{R}$, be a scale of spaces generated by the operator A (see Section 2.1). Assume that

$$\mathcal{L} = \{l_j:\ \ l_j(v) = (v,\ e_j)_H:\ j = 1,\ 2,\ \ldots,\ N\}\,. \tag{2.13}$$

Prove that $\varepsilon_{\mathcal{L}}(\mathcal{F}_\sigma,\ \mathcal{F}_s) = \lambda_{N+1}^{-(\sigma-s)}$ for all $\sigma > s$.

It should be noted that the functionals in Exercise 2.6 are often called ***modes***.

Let us give several more facts on general properties of the completeness defect.

Theorem 2.3.

Assume that the hypotheses of Theorem 2.2 ***hold. Assume that*** $\mathcal{L}$ ***is a set of linear functionals on*** V ***and*** $\mathcal{K}_{\mathcal{L}}$ ***is a family of linear bounded operators*** R ***that map*** V ***into*** H ***and are such that*** $Ru = 0$ ***for all*** $u \in \mathcal{L}^{\perp}$***. Let***

$$e_V^H(R) = \sup\left\{\|u - Ru\|_H :\ \|u\|_V \le 1\right\} \tag{2.14}$$

be the global approximation error in H ***arising from the approximation of elements*** $v \in V$ ***by elements*** Rv***. Then***

$$\varepsilon_{\mathcal{L}}(V;\ H) = \min\left\{e_V^H(R) :\ R \in \mathcal{K}_{\mathcal{L}}\right\}. \tag{2.15}$$

Proof.

Let $R \in \mathcal{K}_{\mathcal{L}}$. Equation (2.14) implies that

$$\|u - Ru\|_H \ \le\ e_V^H(R)\|u\|_V, \qquad u \in V.$$

Therefore, for $u \in \mathcal{L}^{\perp}$ we have $\|u\|_H \le e_V^H(R)\|u\|_V$, i.e. $\varepsilon_{\mathcal{L}}(V;\ H) \le e_V^H(R)$ for all $R \in \mathcal{K}_{\mathcal{L}}$. Let us show that there exists an operator $R_0 \in \mathcal{K}_{\mathcal{L}}$ such that $\varepsilon_{\mathcal{L}}(V;\ H) = e_V^H(R_0)$. Equation (2.12) implies that

$$\varepsilon_{\mathcal{L}}(V;\ H) = \left\|K^{1/2}P_{\mathcal{L}}\right\|_{L(V;\ V)} \ =\ \sup\left\{\left\|K^{1/2}P_{\mathcal{L}}\, u\right\|_V :\ \|u\|_V \le 1\right\},$$

where $P_{\mathcal{L}}$ is the orthoprojector in the space V onto $\mathcal{L}^{\perp}$ and $\left\|K^{1/2}P_{\mathcal{L}}\right\|_{L(V,\ V)}$ is the norm of the operator $K^{1/2}P_{\mathcal{L}}$ in the space $L(V,\ V)$ of bounded linear operators in V. Therefore, the definition of the operator K implies that

$$\varepsilon_{\mathcal{L}}(V;\ H) = \sup\left\{\|P_{\mathcal{L}}\, u\|_H :\ \|u\|_V \le 1\right\} = \ e_V^H(I - P_{\mathcal{L}}). \tag{2.16}$$

It is evident that the orthoprojector $Q_{\mathcal{L}} = I - P_{\mathcal{L}}$ belongs to $\mathcal{K}_{\mathcal{L}}$ (it projects onto the subspace that is orthogonal to $\mathcal{L}^{\perp}$ in V). **Theorem 2.3 is proved**.

Exercise *2.7* Assume that $\mathcal{L} = \{l_j :\ \ j = 1,\ \dots,\ N\}$ is a finite set. Show that the family $\mathcal{K}_{\mathcal{L}}$ consists of finite-dimensional operators R of the form

$$Ru = \sum_{j=1}^{N} l_j(u)\,\varphi_j, \qquad u \in V,$$

where $\{\varphi_j :\ j = 1,\ \dots,\ N\}$ is an arbitrary collection of elements of the space V (they do not need to be distinct). How should the choice of elements $\{\varphi_j\}$ be made for the operator $Q_{\mathcal{L}}$ from the proof of Theorem 2.3?

Theorem 2.3 will be used further (see Section 3) to obtain upper estimates of the completeness defect for some specific sets of functionals. The simplest situation is presented in the following example.

Example *2.1*

Let $H = L^2(0,\ l)$ and let $V = (H^2 \cap H_0^1)(0,\ l)$. As usual, here $H^s(0,\ l)$ is the Sobolev space of the order s and $H_0^s(0,\ l)$ is the closure of the set $C_0^\infty(0,\ l)$ in $H^s(0,\ l)$. We define the norms in H and V by the equalities

$$\|u\|_H^2 = \|u\|^2 \equiv \int_0^l (u(x))^2\,dx\,, \qquad \|u\|_V^2 = \|u''\|^2 .$$

Let $h = l/N$, $x_j = jh$, $j = 1,\ \ldots,\ N-1$. Consider a set of functionals

$$\mathcal{L} = \{l(u) = u(x_j):\ \ j = 1,\ \ldots,\ N-1\}$$

on V. Assume that R is a transformation that maps a function $u \in V$ into its linear interpolating spline

$$s(x) = \sum_{j=1}^{N-1} u(x_j)\,\chi\Big(\frac{x}{h} - j\Big).$$

Here $\chi(x) = 1 - |x|$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| > 1$. We apply Theorem 2.3 and obtain

$$\varepsilon_{\mathcal{L}}(V;\ H)\ \le\ \sup\Big\{\|u - s\|:\ u \in V,\ \|u''\| \le 1\Big\}.$$

We use an easy verifiable equation

$$u(x) - s(x) = -\frac{1}{h}\int_{x_j}^{x} d\tau \int_{x_j}^{x_{j+1}} d\xi \int_{\tau}^{\xi} u''(y)\,dy\ , \qquad x \in [x_j,\ x_{j+1}],$$

to obtain the estimate

$$\|u - s\|\ \le\ \frac{h^2}{\sqrt{3}}\|u''\|\ .$$

This implies that $\varepsilon_{\mathcal{L}}(H;\ V)\ \le\ h^2/\sqrt{3}$.

The assertion on the interdependence of the completeness defect $\varepsilon_{\mathcal{L}}$ and the Kolmogorov N-width made below enables us to obtain effective lower estimates for $\varepsilon_{\mathcal{L}}(V;\ H)$.

Let V and H be separable Hilbert spaces such that V is continuously and densely embedded into H. Then the ***Kolmogorov*** N***-width of the embedding of*** V ***into*** H is defined by the relation

$$\varkappa_N = \varkappa_N(V;\ H) = \inf\left\{e_V^H(F):\ F \in \mathcal{F}_N\right\}, \tag{2.17}$$

where $\mathcal{F}_N$ is the family of all N-dimensional subspaces F of the space V and

$$e_V^H(F) = \sup\left\{\mathrm{dist}_H(v,\ F):\ \|v\|_V \le 1\right\}$$

is the global error of approximation of elements $v \in V$ in H by elements of the subspace F. Here

$$\mathrm{dist}_H(v,\ F) = \inf\left\{\|v - f\|_H:\ f \in F\right\}.$$

In other words, the Kolmogorov N-width $\varkappa_N$ of the embedding of V into H is the minimal possible global error of approximation of elements of V in H by elements of some N-dimensional subspace.

Theorem 2.4.

Let V and H be separable Hilbert spaces such that V is continuously and densely embedded into H. Then

$$\varkappa_N(V;\ H) = \min\left\{\varepsilon_{\mathcal{L}}(V;\ H):\ \ \mathcal{L} \subset V^*;\ \dim \mathrm{Lin}\,\mathcal{L} = N\right\} = \sqrt{\mu_{N+1}}\ , \tag{2.18}$$

where $\{\mu_j\}$ is the nonincreasing sequence of eigenvalues of the operator K defined by the equality $(u,\ v)_H = (Ku,\ v)_V$.

The proof of the theorem is based on the lemma given below as well as on the fact that $(u,\ v)_H = (Ku,\ v)_V$, where K is a compact positive operator in the space V (see Theorem 2.2). Further the notation $\{e_j\}_{j=1}^{\infty}$ stands for the proper basis of the operator K in the space V while the notation μ_j stands for the corresponding eigenvalues:

$$Ke_j = \mu_j e_j,\quad \mu_1 \ge \mu_2 \ge \dots,\quad \mu_n \to 0,\quad (e_i,\ e_j)_V = \delta_{ij}\ .$$

It is evident that $\left\{\frac{1}{\sqrt{\mu_i}}\, e_i\right\}$ is an orthonormalized basis in the space H.

Lemma 2.1.

Assume that the hypotheses of Theorem 2.4 *hold. Then*

$$\min\left\{\varepsilon_{\mathcal{L}}(V;\ H):\ \ \mathcal{L} \subset V^*,\ \ \dim \mathrm{Lin}\,\mathcal{L} = N\right\} = \sqrt{\mu_{N+1}}\ . \tag{2.19}$$

Proof.

By virtue of Corollary 2.1 it is sufficient to verify that

$$\varepsilon_{\mathscr{L}}(V;\, H) \geq \sqrt{\mu_{N+1}}$$

for all $\mathscr{L} = \{l_j\colon\ j = 1,\ \ldots,\ N\}$, where l_j are linearly independent functionals on V. Definition (2.1) implies that

$$[\varepsilon_{\mathscr{L}}(V;\, H)]^2 \geq \|u\|_H^2 = \sum_{j=1}^{\infty} \mu_j (u,\, e_j)_V^2 \tag{2.20}$$

for all $u \in V$ such that $\|u\|_V = 1$ and $l_j(u) = 0$, $j = 1,\ \ldots,\ N$. Let us substitute in (2.20) the vector

$$u = \sum_{j=1}^{N+1} c_j\, e_j\ ,$$

where the constants c_j are choosen such that $l_j(u) = 0$ for $j = 1,\ \ldots,\ N$ and $\|u\|_V = 1$. Therewith equation (2.20) implies that

$$[\varepsilon_{\mathscr{L}}(V;\, H)]^2 \geq \sum_{j=1}^{N+1} \mu_j c_j^2 \geq \mu_{N+1} \sum_{j=1}^{N+1} c_j^2 = \mu_{N+1} \|u\|_V^2 = \mu_{N+1}\,.$$

Thus, Lemma 2.1 is proved.

We now prove that $\varkappa_N = \min\{\varepsilon_{\mathscr{L}}\}$. Let us use equation (2.16)

$$\varepsilon_{\mathscr{L}}(V;\, H) = \sup\left\{\|u - Q_{\mathscr{L}} u\|_H\colon\ \|u\|_V \leq 1\right\}. \tag{2.21}$$

Here $Q_{\mathscr{L}}$ is the orthoprojector onto the subspace $Q_{\mathscr{L}}V$ orthogonal to $\mathscr{L}^{\perp}$ in V. It is evident that $Q_{\mathscr{L}}V$ is isomorphic to $\operatorname{Lin}\mathscr{L}$. Therefore, $\dim Q_{\mathscr{L}}V = N$. Hence, equation (2.21) gives us that

$$\varepsilon_{\mathscr{L}}(V;\, H) \geq \sup\left\{\operatorname{dist}_H(u,\, Q_{\mathscr{L}}V)\colon\ \|u\|_V \leq 1\right\} \geq \varkappa_N \tag{2.22}$$

for all $\mathscr{L} \in V^*$ such that $\dim \operatorname{Lin}\mathscr{L} = N$. Conversely, let F be an N-dimensional subspace in V and let $\{f_j\colon\ j = 1,\ \ldots,\ N\}$ be a orthonormalized basis in the space H. Assume that

$$\mathscr{L}_F = \{l_j\colon\ l_j(v) = (f_j,\, v)_H\colon\ j = 1,\ \ldots,\ N\}\,.$$

Let $Q_{H,\,F}$ be the orthoprojector in the space H onto F. It is clear that

$$Q_{H,\,F}\, u = \sum_{j=1}^{N} (u,\, f_j)_H\, f_j\ .$$

Therefore, if $u \in \mathscr{L}_F^{\perp}$, then $Q_{H,\,F}\, u = 0$. It is clear that $Q_{H,\,F}$ is a bounded operator from V into H. Using Theorem 2.3 we find that

$$\varepsilon_{\mathscr{L}_F}(V;\,H) \le e_V^H(Q_{H,\,F}) = \sup\left\{\|u - Q_{H,\,F}\,u\|_H:\ \|u\|_V \le 1\right\}.$$

However, $\|u - Q_{H,\,F}\,u\|_H = \mathrm{dist}_H(u,\,F)$. Hence,

$$\min\{\varepsilon_{\mathscr{L}}\} \le \varepsilon_{\mathscr{L}_F}(V;\,H) \le e_V^H(F) \tag{2.23}$$

for any N-dimensional subspace F in V. Equations (2.22) and (2.23) imply that

$$\varkappa_N(V;\,H) = \min\left\{\varepsilon_{\mathscr{L}}(V;\,H):\ \mathscr{L} \subset V^*,\ \dim \mathrm{Lin}\,\mathscr{L} = N\right\}.$$

This equation together with Lemma 2.1 **completes the proof of Theorem 2.4**.

— Exercise *2.8* Let V_k and H_k be reflexive Banach spaces such that V_k is continuously and densely embedded into H_k, let $\mathscr{L}_k$ be a set of linear functionals on V_k, $k = 1,\,2$. Assume that

$$\mathscr{L} = \overline{\mathscr{L}}_1 \cup \overline{\mathscr{L}}_2 \subset (V_1 \times V_2)^*,$$

where

$$\overline{\mathscr{L}}_k = \left\{ l \in (V_1 \times V_2)^*:\ l(v_1,\,v_2) = l(v_k),\ l \in \mathscr{L}_k \right\}, \qquad k = 1,\,2.$$

Prove that

$$\varepsilon_{\mathscr{L}}(V_1 \times V_2,\,H_1 \times H_2) = \max\left\{\varepsilon_{\mathscr{L}_1}(V_1,\,H_1),\ \varepsilon_{\mathscr{L}_2}(V_2,\,H_2)\right\}.$$

— Exercise *2.9* Use Lemma 2.1 and Corollary 2.1 to calculate the Kolmogorov N-width of the embedding of the space $\mathscr{F}_s = D(A^s)$ into $\mathscr{F}_\sigma = D(A^\sigma)$ for $s > \sigma$, where A is a positive operator with discrete spectrum.

— Exercise *2.10* Show that in Example 2.1 $\varkappa_N(V,\,H) = l^2 \cdot [\pi(N+1)]^{-2}$. Prove that $\pi^{-2} h^2 \le \varepsilon_{\mathscr{L}}(V;\,H) \le h^2/\sqrt{3}$.

— Exercise *2.11* Assume that there are three reflexive Banach spaces $V \subset W \subset H$ such that all embeddings are dense and continuous. Let $\mathscr{L}$ be a set of functionals on W. Prove that $\varepsilon_{\mathscr{L}}(V,\,H) \le \varepsilon_{\mathscr{L}}(V,\,W) \cdot \varepsilon_{\mathscr{L}}(W,\,H)$ (*Hint*: see (2.3)).

— Exercise *2.12* In addition to the hypotheses of Exercise 2.11, assume that the inequality

$$\|u\|_W \le a_\theta \|u\|_H^\theta \cdot \|u\|_V^{1-\theta}, \qquad u \in V,$$

holds for some constants $a_\theta > 0$ and $\theta \in (0,\,1)$. Show that

$$\left[a_\theta^{-1}\,\varepsilon_{\mathscr{L}}(V,\,W)\right]^{\frac{1}{\theta}} \le \varepsilon_{\mathscr{L}}(V,\,H) \le \left[a_\theta\,\varepsilon_{\mathscr{L}}(W,\,H)\right]^{\frac{1}{1-\theta}}.$$

§ 3 Estimates of Completeness Defect in Sobolev Spaces

In this section we consider several families of functionals on Sobolev spaces that are important from the point of view of applications. We also give estimates of the corresponding completeness defects. The exposition is quite brief here. We recommend that the reader who does not master the theory of Sobolev spaces just get acquainted with the statements of Theorems 3.1 and 3.2 and the results of Examples 3.1 and 3.2 and Exercises 3.2–3.6.

We remind some definitions (see, e.g., the book by Lions-Magenes [4]). Let Ω be a domain in $\mathbb{R}^\nu$. The Sobolev space $H^m(\Omega)$ of the order m $(m=0,\ 1,\ 2,\ \dots)$ is a set of functions

$$H^m(\Omega) = \left\{ f \in L^2(\Omega):\ \ \mathcal{D}^j f(x) \in L^2(\Omega),\ \ |j| \le m \right\},$$

where $j = (j_1,\ \dots,\ j_\nu)$, $j_k = 0,\ 1,\ 2,\ \dots$, $|j| = j_1 + \dots + j_\nu$ and

$$\mathcal{D}^j f(x) \;=\; \frac{\partial^{|j|} f}{\partial x_1^{j_1} \cdot \partial x_2^{j_2} \dots \partial x_\nu^{j_\nu}} \;. \tag{3.1}$$

The space $H^m(\Omega)$ is a separable Hilbert space with the inner product

$$(u,\ v)_m = \sum_{|j| \le m} \int_\Omega \mathcal{D}^j u \cdot \mathcal{D}^j v \,\mathrm{d}x \;.$$

Further we also use the space $H_0^m(\Omega)$ which is the closure (in $H^m(\Omega)$) of the set $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω and the space $H^s(\mathbb{R}^\nu)$ which is defined as follows:

$$H^s(\mathbb{R}^\nu) \;=\; \left\{ u(x) \in L^2(\mathbb{R}^\nu):\ \int_{\mathbb{R}^\nu} (1+|y|^2)^s\, |\hat{u}(y)|^2 \,\mathrm{d}y \;\equiv\; \|u\|_s^2 < \infty \right\},$$

where $s \ge 0$, $\hat{u}(y)$ is the Fourier transform of the function $u(x)$,

$$\hat{u}(y) = \int_{\mathbb{R}^\nu} e^{ixy}\, u(x)\,\mathrm{d}x\,,$$

$|y|^2 = y_1^2 + \dots + y_2^2$, and $xy = x_1 y_1 + \dots + x_\nu y_\nu$. Evidently this definition coincides with the previous one for natural s and $\Omega = \mathbb{R}^\nu$.

Exercise *3.1* Show that the norms in the spaces $H^s(\mathbb{R}^\nu)$ possess the property

$$\|u\|_{s(\theta)} \le \|u\|_{s_1}^{\theta} \cdot \|u\|_{s_2}^{1-\theta}, \quad s(\theta) = \theta s_1 + (1-\theta) s_2\,,$$
$$0 \le \theta \le 1\,, \quad s_1,\, s_2 \ge 0\,.$$

We can also define the space $H^s(\Omega)$ as restriction (to Ω) of functions from $H^s(\mathbb{R}^\nu)$ with the norm

$$\|u\|_{s,\,\Omega} \;=\; \inf\left\{\|v\|_{s,\,\mathbb{R}^\nu}:\; v(x)=u(x) \;\text{ in }\; \Omega,\;\; v\in H^s(\mathbb{R}^\nu)\right\}$$

and the space $H_0^s(\Omega)$ as the closure of the set $C_0^\infty(\Omega)$ in $H^s(\Omega)$. The spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ are separable Hilbert spaces. More detailed information on the Sobolev spaces can be found in textbooks on the theory of such spaces (see, e.g., [4], [5]).

The following version of the Sobolev integral representation will be used further.

Lemma 3.1.

Let Ω be a domain in $\mathbb{R}^\nu$ and let $\lambda(x)$ be a function from $L^\infty(\mathbb{R}^\nu)$ such that

$$\operatorname{supp}\lambda \subset\subset \Omega, \qquad \int_{\mathbb{R}^\nu} \lambda(x)\,\mathrm{d}x = 1\,. \tag{3.2}$$

Assume that Ω is a star-like domain with respect to the support $\operatorname{supp}\lambda$ *of the function λ. This means that for any point $x\in\Omega$ the cone*

$$V_x \;=\; \{z=\tau x+(1-\tau)y;\quad 0\le\tau\le 1,\;\; y\in\operatorname{supp}\lambda\} \tag{3.3}$$

belongs to the domain Ω. Then for any function $u(x)\in H^m(\Omega)$ the representation

$$u(x) = P_{m-1}(x;\; u) + \sum_{|\alpha|=m}\frac{m}{\alpha!}\int_{V_x}(x-y)^\alpha K(x,\;y)\,\mathcal{D}^\alpha u(y)\,\mathrm{d}y \tag{3.4}$$

is valid, where

$$P_{m-1}(x,\;u) = \sum_{|\alpha|<m}\frac{1}{\alpha!}\int_{\Omega}\lambda(y)(x-y)^\alpha\,\mathcal{D}^\alpha u(y)\,\mathrm{d}y\;, \tag{3.5}$$

$$\alpha=(\alpha_1,\,\dots,\,\alpha_\nu),\qquad \alpha! = \alpha_1!\dots\alpha_\nu!\,,\qquad z^\alpha = z_1^{\alpha_1}\dots z_\nu^{\alpha_\nu},$$

$$K(x,\;y) = \int_0^1 s^{-\nu-1}\,\lambda\!\left(x+\frac{y-x}{s}\right)\mathrm{d}s\,. \tag{3.6}$$

Proof.

If we multiply Taylor's formula

$$u(x) = \sum_{|\alpha|<m}\frac{(x-y)^\alpha}{\alpha!}\,\mathcal{D}^\alpha u(y)\;+$$

$$+\;m\sum_{|\alpha|=m}\frac{(x-y)^\alpha}{\alpha!}\int_0^1 s^{m-1}\,\mathcal{D}^\alpha u(x+s\,(y-x))\,\mathrm{d}s$$

by $\lambda(y)$ and integrate it over y, then after introducing a new variable $z = x + s(y-x)$ we obtain the assertion of the lemma.

Integral representation (3.4) enables us to obtain the following generalization of the Poincaré inequality.

Lemma 3.2.

Let the hypotheses of Lemma 3.1 *be valid for a bounded domain* $\Omega \subset \mathbb{R}^\nu$ *and for a function* $\lambda(x)$. *Then for any function* $u(x) \in H^1(\Omega)$ *the inequality*

$$\|u - \langle u\rangle_\lambda\|_{L^2(\Omega)} \le \frac{\sigma_\nu}{\nu} d^{\nu+1} \|\lambda\|_{L^\infty(\Omega)} \cdot \|\nabla u\|_{L^2(\Omega)} \tag{3.7}$$

is valid, where $\langle u\rangle_\lambda = \int_\Omega \lambda(x) u(x)\,\mathrm{d}x$, σ_ν *is the surface measure of the unit sphere in* $\mathbb{R}^\nu$ *and* $d = \operatorname{diam}\Omega \equiv \sup\{|x-y|:\ x, y \in \Omega\}$.

Proof.

We use formula (3.4) for $m = 1$:

$$u(x) = \langle u\rangle_\lambda + \sum_{j=1}^{\nu} \int_{V_x} K(x, y)(x_j - y_j) \cdot \frac{\partial u}{\partial y_j}(y)\,\mathrm{d}y\ . \tag{3.8}$$

It is clear that $\lambda(x + (1/s)(y-x)) = 0$ when $s^{-1}|x-y| \ge d \equiv \operatorname{diam}\Omega$. Therefore,

$$K(x, y) = \int_{\mathrm{d}^{-1}|x-y|}^{1} s^{-\nu-1} \lambda\Big(x + \frac{1}{s}(y-x)\Big)\,\mathrm{d}s\,.$$

Consequently,

$$|K(x, y)| \le \frac{\delta}{|x-y|^\nu}\ , \quad \delta \equiv \delta(\lambda, \nu) = \frac{d^\nu}{\nu}\|\lambda\|_{L^\infty(\Omega)}\,. \tag{3.9}$$

Thus, it follows from (3.8) that

$$|u(x) - \langle u\rangle_\lambda| \le \delta \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{\nu-1}}\,\mathrm{d}y \le$$

$$\le \delta\left(\int_\Omega \frac{|\nabla u(y)|^2}{|x-y|^{\nu-1}}\,\mathrm{d}y\right)^{1/2}\left(\int_\Omega \frac{\mathrm{d}y}{|x-y|^{\nu-1}}\right)^{1/2}. \tag{3.10}$$

Let $B_r(x) = \{y:\ |x-y| \le r\}$. Then it is evident that

$$\int_\Omega \frac{\mathrm{d}y}{|x-y|^{\nu-1}} \le \int_{B_{\mathrm{d}}(x)} \frac{\mathrm{d}y}{|x-y|^{\nu-1}} \le \sigma_\nu \cdot d\,, \tag{3.11}$$

where σ_ν is the surface measure of the unit sphere in $\mathbb{R}^\nu$. Therefore, equation (3.10) implies that

$$\left|u(x) - \langle u\rangle_\lambda\right|^2 \le \delta^2 \cdot \sigma_\nu \cdot d \int_\Omega \frac{|\nabla u(y)|^2}{|x-y|^{\nu-1}}\, \mathrm{d}y .$$

After integration with respect to x and using (3.11) we obtain (3.7). Lemma 3.2 is proved.

Lemma 3.3.

Assume that the hypotheses of Lemma 3.1 are valid for a bounded domain Ω from $\mathbb{R}^\nu$ and for a function $\lambda(x)$. Let $m_\nu = [\nu/2]+1$, where $[\cdot]$ is a sign of the integer part of a number. Then for any function $u(x) \in H^{m_\nu}(\Omega)$ we have the inequality

$$\max_{x\in\Omega}\left|u(x) - P_{m_\nu-1}(x;\ u)\right| \le$$

$$\le \frac{c_\nu}{\nu} d^{m_\nu+\frac{\nu}{2}} \|\lambda\|_{L^\infty(\Omega)} \sum_{|\alpha|=m_\nu} \frac{m_\nu}{\alpha!} \|\mathcal{D}^\alpha u\|_{L^2(\Omega)} , \tag{3.12}$$

where $P_{m-1}(x;\ u)$ is defined by formula (3.5), $c_\nu = [\sigma_\nu/(2m_\nu-\nu)]^{1/2}$, σ_ν is the surface measure of the unit sphere in $\mathbb{R}^\nu$, and $d = \operatorname{diam}\Omega$.

Proof.

Using (3.4) and (3.9) we find that

$$\left|u(x) - P_{m_\nu-1}(x;\ u)\right| \le \sum_{|\alpha|=m_\nu} \frac{m_\nu}{\alpha!} \int_{V_x} |x-y|^{m_\nu} |K(x,\ y)| \cdot \left|\mathcal{D}^\alpha u(y)\right| \mathrm{d}y \le$$

$$\le \sum_{|\alpha|=m_\nu} \frac{\delta m_\nu}{\alpha!} \int_\Omega |x-y|^{m_\nu-\nu} \cdot \left|\mathcal{D}^\alpha u(y)\right| \mathrm{d}y \le$$

$$\le \sum_{|\alpha|=m_\nu} \frac{\delta m_\nu}{\alpha!} \|\mathcal{D}^\alpha u\|_{L^2(\Omega)} \cdot \left(\int_\Omega |x-y|^{2(m_\nu-\nu)}\, \mathrm{d}y\right)^{1/2} .$$

As above, we obtain that

$$\int_\Omega |x-y|^{2(m_\nu-\nu)}\, \mathrm{d}y \le \sigma_\nu \int_0^d r^{2m_\nu-\nu-1}\, \mathrm{d}r \le \sigma_\nu d^{2m_\nu-\nu} \cdot (2m_\nu-\nu)^{-1} .$$

Thus, equation (3.12) is valid. Lemma 3.3 is proved.

Lemma 3.4.

Assume that the hypotheses of Lemma 3.3 hold. Then for any function $u(x) \in H^{m_\nu}(\Omega)$ *and for any* $x_* \in \Omega$ *the inequality*

$$\|u - u(x_*)\|_{L^2(\Omega)} \equiv \left(\int_\Omega |u(x) - u(x_*)|^2 \, dx \right)^{1/2} \le$$

$$\le C_\nu \cdot (d^\nu \|\lambda\|_{L^\infty(\Omega)}) \cdot \sum_{j=1}^{m_\nu} d^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \|\mathcal{D}^\alpha u\|_{L^2(\Omega)} \tag{3.13}$$

is valid, where C_ν *is a constant that depends on* ν *only and* d *is the diameter of the domain* Ω.

Proof.

It is evident that

$$\|u - u(x_*)\|_{L^2(\Omega)} \le \|u - \langle u\rangle_\lambda\|_{L^2(\Omega)} + d^{\nu/2} |\langle u\rangle_\lambda - u(x_*)|, \tag{3.14}$$

where

$$\langle u\rangle_\lambda = \int_\Omega \lambda(y)\, u(y)\, dy .$$

The structure of the polynomial $P_{m-1}(x, u)$ implies that

$$|\langle u\rangle_\lambda - u(x)| \le$$

$$\le |u(x) - P_{m_\nu - 1}(x, u)| + \sum_{1 \le |\alpha| \le m_\nu - 1} \frac{1}{\alpha!} \int_\Omega |\lambda(y)(x-y)^\alpha \mathcal{D}^\alpha u(y)| \, dy \le$$

$$\le |u(x) - P_{m_\nu - 1}(x, u)| + \sum_{1 \le |\alpha| \le m_\nu - 1} \frac{d^{|\alpha|}}{\alpha!} \|\lambda\|_{L^2(\Omega)} \cdot \|\mathcal{D}^\alpha u\|_{L^2(\Omega)}$$

for all $x \in \Omega$. Therefore, estimate (3.13) follows from (3.14) and Lemmata 3.2 and 3.3. Lemma 3.4 is proved.

These lemmata enable us to estimate the completeness defect of two families of functionals that are important from the point of view of applications. We consider these families of functionals on the Sobolev spaces in the case when the domain is strongly Lipschitzian, i.e. the domain $\Omega \subset \mathbb{R}^\nu$ possesses the property: for every $x \in \partial\Omega$ there exists a vicinity U such that

$$U \cap \Omega = \{x = (x_1, \ldots, x_\nu): \ x_\nu < f(x_1, \ldots, x_{\nu-1})\}$$

in some system of Cartesian coordinates, where $f(x)$ is a Lipschitzian function. For strongly Lipschitzian domains the space $H^s(\Omega)$ consists of restrictions to Ω of functions from $H^s(\mathbb{R}^\nu)$, $s > 0$ (see [5] or [6]).

Theorem 3.1.

Assume that a bounded strongly Lipschitzian domain Ω ***in*** $\mathbb{R}^\nu$ ***can be divided into subdomains*** $\{\Omega_j\colon\ j = 1, 2, \ldots, N\}$ ***such that***

$$\overline{\Omega} = \bigcup_{j=1}^{N} \overline{\Omega}_j\,, \quad \Omega_i \cap \Omega_j = \varnothing \quad \textit{for} \quad i \neq j. \tag{3.15}$$

Here the bar stands for the closure of a set. Assume that $\lambda_j(x)$ ***is a function in*** $L^\infty(\Omega_j)$ ***such that***

$$\operatorname{supp} \lambda_j \subset\subset \Omega_j\,, \qquad \int_{\Omega_j} \lambda_j(x)\,\mathrm{d}x = 1 \tag{3.16}$$

and Ω_j ***is a star-like domain with respect to*** $\operatorname{supp}\lambda_j$. ***We define the set*** $\mathcal{L}$ ***of generalized local volume averages corresponding to the collection***

$$\mathcal{T} = \{(\Omega_j;\ \lambda_j)\colon\ j = 1, 2, \ldots, N\}$$

as the family of functionals

$$\mathcal{L} = \left\{ l_j(u) = \int_{\Omega_j} \lambda_j(x)\,u(x)\,\mathrm{d}x\,,\ j = 1, 2, \ldots, N \right\}. \tag{3.17}$$

Then the estimate

$$\varepsilon_{\mathcal{L}}(H^s(\Omega),\ H^\sigma(\Omega)) \le \begin{cases} C(\nu, s)(\Lambda d)^{s-\sigma}, & s > 1,\ 0 \le \sigma \le s\,, \\ C_\nu \Lambda^{1-\frac{\sigma}{s}} \cdot d^{s-\sigma}, & 0 \le \sigma \le s \le 1,\ s \neq 0\,, \end{cases} \tag{3.18}$$

holds, where $\Lambda = \max\left\{ d_j^\nu \|\lambda_j\|_{L^\infty(\Omega)}\colon\ j = 1, 2, \ldots, N \right\}$, $\quad d = \max_j d_j$,

$$d_j \equiv \operatorname{diam}\Omega = \sup\{|x - y|\colon\ x, y \in \Omega_j\}\,,$$

$C(\nu, s)$ ***and*** C_ν ***are constants.***

Proof.

Let us define the interpolation operator $\mathcal{R}_{\mathcal{T}}$ for the collection $\mathcal{T}$ by the formula

$$(\mathcal{R}_{\mathcal{T}} u)(x) = l_j(u) = \int_{\Omega_j} \lambda_j(x)\,u(x)\,\mathrm{d}x\,, \qquad x \in \Omega_j\,, \quad j = 1, 2, \ldots, N.$$

It is easy to check that

$$\|u - \mathcal{R}_{\mathcal{T}} u\|_{L^2(\Omega)} \le C_\nu \Lambda \|u\|_{L^2(\Omega)}, \quad u \in L^2(\Omega).$$

It further follows from Lemma 3.2 that

$$\|u - l_j(u)\|_{L^2(\Omega_j)} \le \frac{\sigma_\nu}{\nu} d_j^{\nu+1} \|\lambda_j\|_{L^\infty(\Omega_j)} \|\nabla u\|_{L^2(\Omega_j)}.$$

This implies the estimate

$$\|u - \mathcal{R}_{\mathcal{T}} u\|_{L^2(\Omega)} \le \frac{\sigma_\nu}{\nu} d \Lambda \|u\|_{H^1(\Omega)}.$$

Using the fact (see [4, 6]) that

$$\|u\|_{H^s(\Omega)} \le C \|u\|_{L^2(\Omega)}^{1-s} \cdot \|u\|_{H^1(\Omega)}^{s}, \qquad 0 \le s \le 1,$$

and the interpolation theorem for operators [4] we find that

$$\|u - R_T u\|_{L^2(\Omega)} \le (C_\nu \Lambda)^{1-s} \left(\frac{\sigma_\nu}{\nu} d\Lambda\right)^s \|u\|_{H^s(\Omega)}$$

for all $0 \le s \le 1$. Consequently, Theorem 2.3 gives the equation

$$\varepsilon_{\mathcal{L}}(H^s(\Omega);\ L^2(\Omega)) \le C_\nu \Lambda d^s, \qquad 0 \le s \le 1. \tag{3.19}$$

Using the result of Exercise 2.12 and the interpolation inequalities (see, e.g., [4,6])

$$\|u\|_{H^{s(\theta)}(\Omega)} \le C \|u\|^{\theta}_{H^{s_1}(\Omega)} \cdot \|u\|^{1-\theta}_{H^{s_2}(\Omega)}, \quad s(\theta) = s_1\theta + s_2(1-\theta), \quad 0 \le \theta \le 1, \tag{3.20}$$

it is easy to obtain equation (3.18) from (3.19).

Let us illustrate this theorem by the following example.

Example *3.1*

Let $\Omega = (0,\ l)^\nu$ be a cube in $\mathbb{R}^\nu$ with the edge of the length l. We construct a collection $\mathcal{T}$ which defines local volume averages in the following way. Let $K = (0,\ 1)^\nu$ be the standard unit cube in $\mathbb{R}^\nu$ and let ω be a measurable set in K with the positive Lebesgue measure, $\operatorname{mes}\omega > 0$. We define the function $\lambda(\omega,\ x)$ on K by the formula

$$\lambda(\omega, x) = \begin{cases} [\operatorname{mes}\omega]^{-1}, & x \in \omega, \\ 0, & x \in K \setminus \omega. \end{cases}$$

Assume that

$$\Omega_j = h \cdot (j + K) \equiv \left\{ x = (x_1, \ldots, x_\nu) :\ j_i < \frac{x_i}{h} < j_i + 1, \quad i = 1, 2, \ldots, \nu \right\},$$

$$\lambda_j(x) = \frac{1}{h^\nu} \lambda\left(\omega,\ \frac{x}{h} - j\right), \qquad x \in \Omega_j,$$

for any multi-index $j = (j_1, \ldots, j_\nu)$, where $j_\nu = 0, 1, \ldots, N-1$, $h = l/N$. It is clear that the hypotheses of Theorem 3.1 are valid for the collection $\mathcal{T} = \{\Omega_j, \lambda_j\}$. Moreover,

$$d_j = \operatorname{diam} \Omega_j = \sqrt{\nu} \cdot h, \qquad \Lambda = \frac{\nu^{\nu/2}}{\operatorname{mes} \omega}$$

and hence in this case we have

$$\varepsilon_{\mathcal{L}}(H^s(\Omega); H^\sigma(\Omega)) \leq C_{\nu, s}\left(\frac{h}{\operatorname{mes} \omega}\right)^{s-\sigma} \tag{3.21}$$

for the set $\mathcal{L}$ of functionals of the form (3.17) when $s \geq 1$ and $0 \leq \sigma \leq s$. It should be noted that in this case the number of functionals in the set $\mathcal{L}$ is equal to $\mathcal{N}_{\mathcal{L}} = N^\nu$. Thus, estimate (3.21) can be rewritten as

$$\varepsilon_{\mathcal{L}}(H^s(\Omega), H^\sigma(\Omega)) \leq C_{\nu, s}\left(\frac{l}{\operatorname{mes} \omega}\right)^{s-\sigma} \cdot \left(\frac{1}{\mathcal{N}_{\mathcal{L}}}\right)^{\frac{s-\sigma}{\nu}}.$$

However, one can show (see, e.g., [6]) that the Kolmogorov $N_{\mathcal{L}}$-width of the embedding of $H^s(\Omega)$ into $H^\sigma(\Omega)$ has the same order in $\mathcal{N}_{\mathcal{L}}$, i.e.

$$\varkappa_{\mathcal{N}_{\mathcal{L}}}(H^s(\Omega), H^\sigma(\Omega)) = c_0\left(\frac{1}{N_{\mathcal{L}}}\right)^{\frac{s-\sigma}{\nu}}.$$

Thus, it follows from Theorem 2.4 that local volume averages have a completeness defect that is close (when the number of functionals is fixed) to the minimal. In the example under consideration this fact yields a double inequality

$$c_1 h^{s-\sigma} \leq \varepsilon_L(H^s(\Omega), H^\sigma(\Omega)) \leq c_2 h^{s-\sigma}, \quad \sigma < s, \tag{3.22}$$

where c_1 and c_2 are positive constants that may depend on s, ν, ω, and Ω. Similar relations are valid for domains of a more general type.

Another important example of functionals is given in the following assertion.

Theorem 3.2.

Assume the hypotheses of Theorem** 3.1. **Let us choose a point** x_j **(called a node) in every set** Ω_j **and define a set of functionals on** $H^m(\Omega)$, $m = [\nu/2] + 1$, **by

$$\mathcal{L} = \left\{ l_j(u) = u(x_j) : \ x_j \in \Omega_j, \ j = 1, \ldots, N \right\}. \tag{3.23}$$

Then for all** $s \geq m$ **and** $0 \leq \sigma \leq s$ **the estimate

$$\varepsilon_{\mathcal{L}}(H^s(\Omega); H^\sigma(\Omega)) \leq C(\nu, s)(d\Lambda)^{s-\sigma} \tag{3.24}$$

is valid for the completeness defect of this set of functionals, where

$$d = \max\left\{d_j : \ j = 1,\ 2,\ \dots,\ N\right\}, \qquad d_j = \operatorname{diam}\Omega\,,$$

$$\Lambda = \max\left\{d_j^{\nu}\|\lambda_j\|_{L^\infty(\Omega)} : \ j = 1,\ 2,\ \dots,\ N\right\}.$$

Proof.

Let $u \in H^m(\Omega)$ and let $l_j(u) = u(x_j) = 0$, $j = 1,\ 2,\ \dots,\ N$. Then using (3.13) for $\Omega = \Omega_j$ and $x_* = x_j$ we obtain that

$$\|u\|_{L^2(\Omega_j)} \ \le\ C_\nu\left(d_j^{\nu}\|\lambda_j\|_{L^\infty(\Omega_j)}\right)\sum_{l=1}^{m} d_j^{\,l}\|u\|_{l,\,\Omega_j}\,,$$

where

$$\|u\|_{l,\,\Omega_j} = \sum_{|\alpha| = l}\frac{1}{\alpha!}\|\mathcal{D}^\alpha u\|_{L^2(\Omega_j)}\,.$$

It follows that

$$\|u\|_{L^2(\Omega)} \ \le\ C(\nu)\cdot\Lambda\cdot\sum_{l=1}^{m} d^{\,l}\|u\|_{H^l(\Omega)}$$

for all $u \in H^m(\Omega)$ such that $l_j(u) = u(x_j) = 0$, $j = 1,\ 2,\ \dots,\ N$. Using interpolation inequality (3.20) we find that

$$\|u\|_{L^2(\Omega)} \ \le\ C_\nu\cdot\Lambda\cdot\sum_{l=1}^{m} d^{\,l}\,\|u\|_{L^2(\Omega)}^{1-\frac{l}{m}}\cdot\|u\|_{H^m(\Omega)}^{l/m}\,. \tag{3.25}$$

By virtue of the inequality

$$xy \ \le\ \frac{x^p}{p} + \frac{y^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1\,, \qquad x,\ y \ge 0\,,$$

we get

$$\Lambda\cdot d^{\,l}\cdot\|u\|_{L^2(\Omega)}^{1-\frac{l}{m}}\cdot\|u\|_{H^m(\Omega)}^{l/m} = \|u\|_{L^2}^{1-\frac{l}{m}}\cdot\left(d^{\,m}\Lambda^{\frac{m}{l}}\|u\|_{H^m}\right)^{\frac{l}{m}} \le$$

$$\le (1 - \frac{l}{m})\,\delta^{\frac{m}{m-l}}\,\|u\|_{L^2(\Omega)} + \frac{l}{m}\cdot\delta^{-\frac{m}{l}}\,d^{\,m}\Lambda^{\frac{m}{l}}\,\|u\|_{H^m(\Omega)}$$

for $l = 1,\ 2,\ \dots,\ m-1$ and for all $\delta > 0$. We substitute these inequalities in (3.25) to obtain that

$$\|u\|_{L^2(\Omega)} \ \le\ C_\nu\sum_{l=1}^{m-1}(1 - \frac{l}{m})\,\delta^{\frac{m}{m-l}}\cdot\|u\|_{L^2(\Omega)} +$$

$$+\, C_\nu\left(\sum_{l=1}^{m-1}\frac{l}{m}\,\delta^{-\frac{m}{l}}\,\Lambda^{\frac{m}{l}} + \Lambda\right)d^{\,m}\|u\|_{H^m(\Omega)}\,. \tag{3.26}$$

We choose $\delta = \delta(\nu, m)$ such that

$$C_\nu \sum_{l=1}^{m-1} \left(1 - \frac{l}{m}\right) \delta^{\frac{m}{m-l}} \le \frac{1}{2}.$$

Then equation (3.26) gives us that

$$\|u\|_{L^2(\Omega)} \le C(\nu) \cdot \sum_{l=1}^{m} \frac{l}{m} \Lambda^{\frac{m}{l}} \cdot d^m \|u\|_{H^m(\Omega)}$$

for all $u \in H^m(\Omega)$ such that $u(x_j) = 0$, $j = 1, 2, \ldots, N$. Hence, the estimate

$$\varepsilon_{\mathcal{L}}(H^m(\Omega);\ L^2(\Omega)) \le C(\nu)\, d^m \cdot \sum_{l=1}^{m} \frac{l}{m} \Lambda^{\frac{m}{l}}$$

is valid. Since $\Lambda \ge 1$, this implies inequality (3.24) for $\sigma = 0$ and $s = m = [\nu/2] + 1$. As in Theorem 3.1 further arguments rely on Lemma 2.1 and interpolation inequalities (3.20). **Theorem 3.2 is proved**.

Example *3.2*

We return to the case described in Example 3.1. Let us choose nodes $x_j \in \Omega_j$ and assume that $\omega = K$. Then for a set $\mathcal{L}$ of functionals of the form (3.23) we have

$$\varepsilon_{\mathcal{L}}(H^s(\Omega);\ H^\sigma(\Omega)) \le C_{\nu, s}\, h^{s-\sigma}$$

for all $s \ge [\nu/2] + 1$, $0 \le \sigma \le s$ and for any location of the nodes x_j inside the Ω_j. In the case under consideration double estimate (3.22) is preserved.

In the exercises below several one-dimensional situations are given.

Exercise *3.2* Prove that

$$\varepsilon_{\mathcal{L}}(H^1(0, l);\ L^2(0, l)) = \varkappa_N(H^1(0, l);\ L^2(0, l)) = \left[1 + \left(\frac{\pi N}{l}\right)^2\right]^{-\frac{1}{2}}$$

for

$$\mathcal{L} = \left\{ l_j^c(u) = \int_0^l u(x) \cos\frac{j\pi}{l} x \, \mathrm{d}x,\ \ j = 0, 1, \ldots, N-1 \right\}.$$

Exercise *3.3* Verify that

$$\varepsilon_{\mathcal{L}}(H_0^1(0, l);\ L^2(0, l)) = \varkappa_{N-1}(H_0^1(0, l);\ L^2(0, l)) = \left[1 + \left(\frac{\pi N}{l}\right)^2\right]^{-\frac{1}{2}}$$

for

$$\mathcal{L} = \left\{ l_j^s(u) = \int_0^l u(x) \sin\frac{j\pi x}{l} \, \mathrm{d}x;\ \ j = 1, 2, \ldots, N-1 \right\}.$$

Chapter 5

Exercise 3.4 Let

$$H^1_{\text{per}}(0,\ l) = \{u \in H^1(0,\ l):\ u(0) = u(l)\}.$$

Show that

$$\varepsilon_{\mathscr{L}}(H^1_{\text{per}}(0,\ l);\ L^2(0,\ l)) =$$

$$= \varkappa_{2N-1}(H^1_{\text{per}}(0,\ l);\ L^2(0,\ l)) = \left[1 + \left(\frac{2\pi N}{l}\right)^2\right]^{-\frac{1}{2}} ,$$

where $\mathscr{L}$ consists of functionals l^c_{2k} and l^s_{2k} for $k = 0, 1, \ldots, N-1$ (the functionals l^c_j and l^s_j are defined in Exercises 3.2 and 3.3).

Exercise 3.5 Consider the functionals

$$l_j(u) = \frac{1}{h}\int_0^h u(x_j + \tau)\,d\tau ,$$

$$x_j = jh, \quad h = \frac{l}{N}, \quad j = 0,\ 1,\ \ldots,\ N-1 ,$$

on the space $L^2(0,\ l)$. Assume that an interpolation operator R_h maps an element $u \in L^2(0,\ l)$ into a step-function equal to $l_j(u)$ on the segment $[x_j,\ x_{j+1}]$. Show that

$$\|u - R_h u\|_{L^2(0,\ l)} \le h\|u'\|_{L^2(0,\ l)} .$$

Prove the estimate

$$\frac{h}{\sqrt{\pi^2 + h^2}} \le \varepsilon_{\mathscr{L}}(H^1(0,\ l);\ L^2(0,\ l)) \le h .$$

Exercise 3.6 Consider a set $\mathscr{L}$ of functionals

$$l_j(u) = u(x_j), \quad x_j = jh, \quad h = \frac{l}{N}, \quad j = 0,\ 1,\ \ldots,\ N-1 ,$$

on the space $H^1(0,\ l)$. Assume that an interpolation operator R_h maps an element $u \in H^1(0,\ l)$ into a step-function equal to $l_j(u)$ on the segment $[x_j,\ x_{j+1}]$. Show that

$$\|u - R_h u\|_{L^2(0,\ l)} \le \frac{h}{\sqrt{2}}\|u'\|_{L^2(0,\ l)} .$$

Prove the estimate

$$\frac{h}{\sqrt{\pi^2 + h^2}} \le \varepsilon_{\mathscr{L}}(H^1(0,\ l);\ L^2(0,\ l)) \le \frac{h}{\sqrt{2}} .$$

§ 4 Determining Functionals for Abstract Semilinear Parabolic Equations

In this section we prove a number of assertions on the existence and properties of determining functionals for processes generated in some separable Hilbert space H by an equation of the form

$$\frac{du}{dt} + Au = B(u, t), \qquad t > 0, \quad u\big|_{t=0} = u_0. \tag{4.1}$$

Here A is a positive operator with discrete spectrum (for definition see Section 2.1) and $B(u, t)$ is a continuous mapping from $D(A^{1/2}) \times \mathbb{R}$ into H possessing the properties

$$\|B(0, t)\| \le M_0, \qquad \|B(u_1, t) - B(u_2, t)\| \le M(\rho)\|A^{1/2}(u_1 - u_2)\| \tag{4.2}$$

for all t and for all $u_j \in D(A^{1/2})$ such that $\|A^{1/2}u_j\| \le \rho$, where ρ is an arbitrary positive number, M_0 and $M(\rho)$ are positive numbers.

Assume that problem (4.1) is uniquely solvable in the class of functions

$$\mathcal{W} = C([0, +\infty); H) \cap C([0, +\infty); D(A^{1/2}))$$

and is pointwise dissipative, i.e. there exists $R > 0$ such that

$$\|A^{1/2}u(t)\| \le R \quad \text{when} \quad t \ge t_0(u) \tag{4.3}$$

for all $u(t) \in \mathcal{W}$. Examples of problems of the type (4.1) with the properties listed above can be found in Chapter 2, for example.

The results obtained in Sections 1 and 2 enable us to establish the following assertion.

Theorem 4.1.

For the set of linear functionals $\mathcal{L} = \{l_j\colon j = 1, 2, \dots, N\}$ ***on the space*** $V = D(A^{1/2})$ ***with the norm*** $\|\cdot\|_V = \|A^{1/2}\cdot\|_H$ ***to be*** $(V, V; \mathcal{W})$***-asymptotically determining for problem*** (4.1) ***under conditions*** (4.2) ***and*** (4.3), ***it is sufficient that the completeness defect*** $\varepsilon_{\mathcal{L}}(V, H)$ ***satisfies the inequality***

$$\varepsilon_{\mathcal{L}} \equiv \varepsilon_{\mathcal{L}}(V, H) < M(R)^{-1}, \tag{4.4}$$

where $M(\rho)$ ***and*** R ***are the same as in*** (4.2) ***and*** (4.3).

Proof.

We consider two solutions $u_1(t)$ and $u_2(t)$ to problem (4.1) that lie in $\mathcal{W}$. By virtue of dissipativity property (4.3) we can suppose that

$$\|A^{1/2}u_j(t)\| \le R, \qquad t \ge 0, \quad j = 1, 2. \tag{4.5}$$

Let $u(t) = u_1(t) - u_2(t)$. If we consider $u(t)$ as a solution to the linear problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f(t) \equiv B(u_1(t),\ t) - B(u_2(t),\ t),$$

then it is easy to find that

$$\frac{1}{2}\|u(t)\|^2 + \int_s^t (Au(\tau),\ u(\tau))\,\mathrm{d}\tau \ \le\ \frac{1}{2}\|u(s)\|^2 + M(R)\int_s^t \|A^{1/2}u(\tau)\|\cdot\|u(\tau)\|\,\mathrm{d}\tau$$

for all $t \ge s \ge 0$. We use (2.11) to obtain that

$$M(R)\cdot\|A^{1/2}u\|\cdot\|u\| \ \le\ \varepsilon_{\mathscr{L}}\cdot M(R)\cdot\|A^{1/2}u\|^2 + \delta\|A^{1/2}u\|^2 + C(R,\ \mathscr{L},\ \delta)[N(u)]^2$$

for any $\delta > 0$, where

$$N(u) = \max\{|l_j(u)| :\ j = 1,\ 2,\ \ldots,\ N\}.$$

Therefore,

$$\|u(t)\|^2 + 2\,(1 - \delta - \varepsilon_{\mathscr{L}} M(R))\int_s^t \|A^{1/2}u(\tau)\|^2\,\mathrm{d}\tau \ \le$$

$$\le\ \|u(s)\|^2 + C(R,\ \mathscr{L},\ \delta)\int_s^t [N(u(\tau))]^2\,\mathrm{d}\tau\ . \tag{4.6}$$

Using (4.4) we can choose the parameter $\delta > 0$ such that $1 - \delta - \varepsilon_{\mathscr{L}} M(R) > 0$. Thus, we can apply Theorem 1.1 and find that under condition (4.4) equation

$$\lim_{t\to\infty}\int_t^{t+1} [N(u_1(\tau) - u_2(\tau))]^2\,\mathrm{d}\tau = 0$$

implies the equality

$$\lim_{t\to\infty}\|u_1(t) - u_2(t)\| = 0\,. \tag{4.7}$$

In order to complete the proof of the theorem we should obtain

$$\lim_{t\to\infty}\|A^{1/2}(u_1(t) - u_2(t))\| = 0 \tag{4.8}$$

from (4.7). To prove (4.8) it should be first noted that

$$\lim_{t\to\infty}\|A^{\theta}(u_1(t) - u_2(t))\| = 0 \tag{4.9}$$

for any $0 \le \theta < 1/2$. Indeed, the interpolation inequality (see Exercise 2.1.12)

$$\|A^{\theta}u\| \ \le\ \|u\|^{1-2\theta}\cdot\|A^{1/2}u\|^{2\theta}\,,\qquad 0 \le \theta < 1/2\,,$$

and dissipativity property (4.5) enable us to obtain (4.9) from equation (4.7). Now we use the integral representation of a weak solution (see (2.2.3)) and the method applied in the proof of Lemma 2.4.1 to show (do it yourself) that

$$\|A^{\beta} u_j(t)\| \le C(R, M_0), \quad \frac{1}{2} < \beta < 1, \quad t \ge t_0.$$

Therefore, using the interpolation inequality

$$\|A^{1/2} u\| \le \|A^{1/2-\delta} u\| \cdot \|A^{1/2+\delta} u\|, \qquad 0 < \delta < \frac{1}{2},$$

we obtain (4.8) from (4.9). **Theorem 4.1 is proved**.

Exercise *4.1* Show that if the hypotheses of Theorem 4.1 hold, then equation (4.9) is valid for all $0 \le \theta < 1$.

The reasonings in the proof of Theorem 4.1 lead us to the following assertion.

Corollary 4.1.

Assume that the hypotheses of Theorem 4.1 *hold. Then for any two weak (in* $D(A^{1/2})$*) solutions* $u_1(t)$ *and* $u_2(t)$ *to problem* (4.1) *that are bounded on the whole axis,*

$$\sup\left\{\|A^{1/2} u_i(t)\| : -\infty < t < \infty\right\} \le R, \qquad i = 1, 2, \tag{4.10}$$

the condition $l_j(u_1(t)) = l_j(u_2(t))$ *for* $l_j \in \mathscr{L}$ *and* $t \in \mathbb{R}$ *implies that* $u_1(t) \equiv u_2(t)$.

Proof.

In the situation considered equation (4.6) implies that

$$\|u(t)\|^2 + \beta_{\mathscr{L}} \int_s^t \|A^{1/2} u(\tau)\|^2 d\tau \le \|u(s)\|^2$$

for all $t > s$ and some $\beta_{\mathscr{L}} > 0$. It follows that

$$\|u(t)\| \le e^{-\beta_{\mathscr{L}}(t-s)} \|u(s)\|, \qquad t \ge s.$$

Therefore, if we tend $s \to -\infty$, then using (4.10) we obtain that $\|u(t)\| = 0$ for all $t \in \mathbb{R}$, i.e. $u_1(t) \equiv u_2(t)$.

It should be noted that Corollary 4.1 means that solutions to problem (4.1) that are bounded on the whole axis are uniquely determined by their values on the functionals l_j. It was this property of the functionals $\{l_j\}$ which was used by Ladyzhenskaya [2] to define the notion of determining modes for the two-dimensional Navier-Stokes system. We also note that a more general variant of Theorem 4.1 can be found in [3].

Exercise *4.2* Assume that problem (4.1) is autonomous, i.e. $B(u, t) \equiv B(u)$. Let $\mathscr{A}$ be a global attractor of the dynamical system (V, S_t) generated by weak (in $V = D(A^{1/2})$) solutions to problem (4.1) and assume that a set of functionals $\mathscr{L} = \{l_j : j = 1, \dots, N\}$ possesses property (4.4). Then for any pair of trajectories $u_1(t)$ and $u_2(t)$ lying in the attractor $\mathscr{A}$ the condition $l_j(u_1(t)) = l_j(u_2(t))$ implies that $u_1(t) \equiv u_2(t)$ for all $t \in \mathbb{R}$ and $j = 1, 2, \ldots, N$.

Theorems 4.1 and 2.4 enable us to obtain conditions on the existence of N determining functionals.

Corollary 4.2.

Assume that the Kolmogorov N-width of the embedding of the space $V = D(A^{1/2})$ into H possesses the property $\varkappa_N(V; H) < M(R)^{-1}$. Then there exists a set of asymptotically $(V, V; \mathscr{W})$-determining functionals for problem (4.1) *consisting of N elements.*

Theorem 2.4, Corollary 2.1, and Exercise 2.6 imply that if the hypotheses of Theorem 4.1 hold, then the family of functionals $\mathscr{L}$ given by equation (2.13) is a $(V, V; \mathscr{W})$-determining set for problem (4.1), provided $\lambda_{N+1} > M(R)^2$. Here $M(R)$ and R are the constants from (4.2) and (4.3). It should be noted that the set $\mathscr{L}$ of the form (2.13) for problem (4.1) is often called a set of ***determining modes.*** Thus, Theorem 4.1 and Exercise 2.6 imply that semilinear parabolic equation (4.1) possesses a finite number of determining modes.

When condition (4.2) holds uniformly with respect to ρ, we can omit the requirement of dissipativity (4.3) in Theorem 4.1. Then the following assertion is valid.

Theorem 4.2.

Assume that a continuous mapping $B(u, t)$ from $D(A^{1/2}) \times \mathbb{R}$ into H possesses the properties

$$\|B(0, t)\| \le M_0, \qquad \|B(u_1, t) - B(u_2, t)\| \le M \|A^{1/2}(u_1 - u_2)\| \tag{4.11}$$

for all $u_j \in D(A^{1/2})$. Then a set of linear functionals $\mathscr{L} = \{l_j : j = 1, 2, \ldots, N\}$ on $V = D(A^{1/2})$ is asymptotically $(V, V; \mathscr{W})$-determining for problem (4.1), provided $\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}}(V, H) < M^{-1}$.

Proof.

If we reason as in the proof of Theorem 4.1, we obtain that if $\varepsilon_{\mathscr{L}} < M^{-1}$, then for an arbitrary pair of solutions $u_1(t)$ and $u_2(t)$ emanating from the points u_1 and u_2 at a moment s the equation (see (4.6))

$$\|u(t)\|^2 + \beta \int_s^t \|A^{1/2} u(\tau)\|^2 \, d\tau \le \|u(s)\|^2 + C \int_s^t [N(u(\tau))]^2 \, d\tau \tag{4.12}$$

is valid. Here $u(t) = u_1(t) - u_2(t)$, $\beta = \beta(\varepsilon_{\mathscr{L}}, M)$ and $C(\mathscr{L}, M)$ are positive constants, and

$$N(u) = \max\{|l_j(u)| : j = 1, 2, \ldots, N\}.$$

Therefore, using Theorem 1.1 we conclude that the condition

$$\lim_{t \to \infty} \int_t^{t+1} [N(u(\tau))]^2 \, d\tau = 0 \tag{4.13}$$

implies that

$$\lim_{t \to \infty} \left\{ \|u(t)\|^2 + \int_t^{t+1} \|A^{1/2} u(\tau)\|^2 \, d\tau \right\} = 0 . \tag{4.14}$$

Since $u(t) = u_1(t) - u_2(t)$ is a solution to the linear equation

$$\frac{du}{dt} + Au = f(t) \equiv B(u_1(t), t) - B(u_2(t), t), \tag{4.15}$$

it is easy to verify that

$$\|A^{1/2} u(t)\|^2 \leq \|A^{1/2} u(s)\|^2 + \frac{1}{2} \int_s^t \|f(\tau)\|^2 \, d\tau \tag{4.16}$$

for $t \geq s$. It should be noted that equation (4.16) can be obtained with the help of formal multiplication of (4.15) by $\dot{u}(t)$ with subsequent integration. This conversion can be grounded using the Galerkin approximations. If we integrate equation (4.16) with respect to s from $t-1$ to t, then it is easy to see that

$$\|A^{1/2} u(t)\| \leq \int_{t-1}^t \|A^{1/2} u(s)\|^2 \, ds + \frac{1}{2} \int_{t-1}^t \|f(\tau)\|^2 \, d\tau .$$

Using the structure of the function $f(t)$ and inequality (4.11), we obtain that

$$\|A^{1/2} u(t)\| \leq \left(1 + \frac{M^2}{2}\right) \int_{t-1}^t \|A^{1/2} u(\tau)\|^2 \, d\tau .$$

Consequently, (4.14) gives us that

$$\lim_{t \to \infty} \left\| A^{1/2} (u_1(t) - u_2(t)) \right\| = 0 .$$

Therefore, **Theorem 4.2 is proved**.

Further considerations in this section are related to the problems possessing inertial manifolds (see Chapter 3). In order to cover a wider class of problems, it is convenient to introduce the notion of a process.

Let H be a real reflexive Banach space. A two-parameter family $\{S(t, \tau);\ t \geq \tau;\ \tau, t \in \mathbb{R}\}$ of continuous mappings acting in H is said to be ***evolutionary***, if the following conditions hold:

(a) $S(t, s) \cdot S(s, \tau) = S(t, \tau)$, $t \geq s \geq \tau$, $S(t, t) = I$.

(b) $S(t, s) u_0$ is a strongly continuous function of the variable t.

A pair $(H,\ S(t,\ \tau))$ with $S(t,\ \tau)$ being an evolutionary family in H is often called a ***process***. Therewith the space H is said to be a phase space and the family of mappings $S(t,\ \tau)$ is called an evolutionary operator. A curve

$$\gamma_s[u_0] = \{u(t) \equiv S(t+s,\ s)u_0 :\ \ t \ge 0\}$$

is said to be a trajectory of the process emanating from the point u_0 at the moment s.

It is evident that every dynamical system $(H,\ S_t)$ is a process. However, main examples of processes are given by evolutionary equations of the form (1.1). Therewith the evolutionary operator is defined by the obvious formula

$$S(t,\ s)u_0 = u(t,\ s;\ u_0),$$

where $u(t) = u(t,\ s;\ u_0)$ is the solution to problem (1.1) with the initial condition u_0 at the moment s.

— Exercise *4.3* Assume that the conditions of Section 2.2 and the hypotheses of Theorem 2.2.3 hold for problem (4.1). Show that weak solutions to problem (4.1) generate a process in H.

Similar to the definitions of Chapter 3 we will say that a process $(V;\ S(t,\ \tau))$ acting in a separable Hilbert space V possesses an asymptotically complete finite-dimensional inertial manifold $\{M_t\}$ if there exist a finite-dimensional orthoprojector P in the space V and a continuous function $\Phi(p,\ t):\ PV \times \mathbb{R} \to (1-P)V$ such that

(a) $$\|\Phi(p_1,\ t) - \Phi(p_2,\ t)\|_V \ \le\ L\|p_1 - p_2\|_V \qquad (4.17)$$

for all $p_j \in PV$, $t \in \mathbb{R}$, where L is a positive constant;

(b) the surface

$$M_t = \{p + \Phi(p,\ t):\ \ p \in PV\} \subset V \qquad (4.18)$$

is invariant: $S(t,\ \tau)M_\tau \subset M_t$;

(c) the condition of asymptotical completeness holds: for any $s \in \mathbb{R}$ and $u_0 \in V$ there exists $u_0^* \in M_s$ such that

$$\|S(t,\ s)u_0 - S(t,\ s)u_0^*\|_V \ \le\ Ce^{-\gamma(t-s)}, \qquad t > s, \qquad (4.19)$$

where C and γ are positive constants which may depend on u_0 and $s \in \mathbb{R}$.

— Exercise *4.4* Show that for any two elements $u_1,\ u_2 \in M_s$, $s \in \mathbb{R}$ the following inequality holds

$$(1+L^2)^{-1/2}\|u_1 - u_2\|_V \ \le\ \|P(u_1 - u_2)\|_V \ \le\ \|u_1 - u_2\|_V.$$

— Exercise *4.5* Using equation (4.19) prove that

$$\|(1-P)S(t,\ s)u_0 - \Phi(PS(t,\ s)u_0,\ t)\|_V \ \le\ (1+L)\cdot Ce^{-\gamma(t-s)}$$

for $t \ge s$.

Exercise *4.6* Show that for any two trajectories $u_j(t) = S(t, s)u_j$, $j = 1, 2$, of the process $(V;\ S(t, \tau))$ the condition

$$\lim_{t \to \infty} \|P(u_1(t) - u_2(t))\|_V = 0$$

implies that

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\|_V = 0 .$$

In particular, the results of these exercises mean that M_s is homeomorphic to a subset in $\mathbb{R}^N$, $N = \dim P$, for every $s \in \mathbb{R}$. The corresponding homeomorphism $r\colon\ V \to \mathbb{R}^N$ can be defined by the equality $ru = \{(u, \phi_j)_V\}_{j=1}^N$, where $\{\phi_j\}$ is a basis in PV. Therewith the set of functionals $\{l_j(u) = (u, \varphi_j)_V\colon\ j = 1, \ldots, N\}$ appears to be asymptotically determining for the process. The following theorem contains a sufficient condition of the fact that a set of functionals $\{l_j\}$ possesses the properties mentioned above.

Theorem 4.3.

Assume that** V **and** H **are separable Hilbert spaces such that** V **is continuously and densely embedded into** H. **Let a process** $(V;\ S(t, \tau))$ **possess an asymptotically complete finite-dimensional inertial manifold** $\{M_t\}$. **Assume that the orthoprojector** P **from the definition of** $\{M_t\}$ **can be continuously extended to the mapping from** H **into** V**, i.e. there exists a constant** $\Lambda = \Lambda(P) > 0$ **such that

$$\|Pv\|_V \le \Lambda \cdot \|v\|_H , \qquad v \in V. \tag{4.20}$$

If** $\mathscr{L} = \{l_j\colon\ j = 1, \ldots, N\}$ **is a set of linear functionals on** V **such that

$$\varepsilon_{\mathscr{L}}(V; H) < (1 + L^2)^{-1/2} \Lambda^{-1} , \tag{4.21}$$

then the following conditions are valid:

1) ***there exist positive constants** c_1 **and** c_2 **depending on** $\mathscr{L}$ **such that***

$$c_1 \|u_1 - u_2\|_H \le \max\{|l_j(u_1 - u_2)|\colon\ j = 1, \ldots, N\} \le c_2 \|u_1 - u_2\|_H \tag{4.22}$$

for all** $u_1, u_2 \in M_t$, $t \in \mathbb{R}$**; i.e. the mapping** $r_{\mathscr{L}}$ **acting from** V **into** $\mathbb{R}^N$ **according to the formula** $r_{\mathscr{L}} u = \{l_j(u)\}_{j=1}^N$ **is a Lipschitzian homeomorphism from** M_t **into** $\mathbb{R}^N$ **for every** $t \in \mathbb{R}$**;

2) ***the set of functionals** $\mathscr{L}$ **is determining for the process** $(V;\ S(t, \tau))$ **in the sense that for any two trajectories** $u_j(t) = S(t, s)u_j$ **the condition***

$$\lim_{t \to \infty} (l_j(u_1(t)) - l_j(u_2(t))) = 0$$

implies that

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\|_V = 0 .$$

Proof.

Let $u_1,\ u_2 \in M_t$. Then

$$u_j = Pu_j + \Phi(Pu_j,\ t), \qquad j = 1,\ 2.$$

Therewith equation (4.17) gives us that

$$\|u_1 - u_2\|_V \ \le\ (1+L^2)^{1/2}\|Pu_1 - Pu_2\|_V, \qquad u_j \in M_t. \tag{4.23}$$

Consequently, using Theorem 2.1 and inequality (4.20) we obtain that

$$\|u_1 - u_2\|_H \ \le\ C_{\mathscr{L}} N_{\mathscr{L}}(u_1 - u_2) + \varepsilon_{\mathscr{L}}(1+L^2)^{1/2}\cdot \Lambda\|u_1 - u_2\|_H,$$

where $N_{\mathscr{L}}(u) = \max\{|l_j(u)|:\ j = 1,\ \ldots,\ N\}$ and $\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}}(V;\ H)$. Therefore, equation (4.21) implies that

$$\|u_1 - u_2\|_H \ \le\ C_1(\mathscr{L})\cdot N_{\mathscr{L}}(u_1 - u_2), \quad \text{where} \quad C_1(\mathscr{L}) = C_{\mathscr{L}}\cdot(1 - \varepsilon_{\mathscr{L}}(1+L^2)^{1/2}\Lambda)^{-1}. \tag{4.24}$$

On the other hand, (4.20) and (4.23) give us that

$$|l_j(u_1 - u_2)| \ \le\ C_{\mathscr{L}}\|u_1 - u_2\|_V \ \le\ C_{\mathscr{L}}(1+L^2)^{1/2}\Lambda\|u_1 - u_2\|_H. \tag{4.25}$$

Equations (4.24) and (4.25) imply estimate (4.22). Hence, assertion 1 of the theorem is proved.

Let us prove the second assertion of the theorem. Let $u_j(t) = S(t,\ s)u_j$, $t \ge s$, be trajectories of the process. Since

$$u_j(t) = (Pu_j(t) + \Phi(Pu_j(t),\ t)) + ((1-P)u_j(t) - \Phi(Pu_j(t),\ t)),$$

using (4.17) it is easy to find that

$$\|u_1(t) - u_2(t)\|_V \ \le\ (1+L^2)^{1/2}\|P(u_1(t) - u_2(t))\|_V + \\ + \sum_{j=1,\ 2}\|(1-P)u_j(t) - \Phi(Pu_j(t),\ t)\|_V.$$

The property of asymptotical completeness (4.19) implies (see Exercise 4.5) that

$$\|(1-P)u_j(t) - \Phi(Pu_j(t),\ t)\| \ \le\ Ce^{-\gamma(t-s)}, \qquad t \ge s.$$

Therefore, equation (4.20) gives us the estimate

$$\|u_1(t) - u_2(t)\|_V \ \le\ (1+L^2)^{1/2}\Lambda\|u_1(t) - u_2(t)\|_H + Ce^{-\gamma(t-s)}. \tag{4.26}$$

It follows from Theorem 2.1 that

$$\|u_1(t) - u_2(t)\|_H \ \le\ C_{\mathscr{L}} N_{\mathscr{L}}(u_1(t) - u_2(t)) + \varepsilon_{\mathscr{L}}\|u_1(t) - u_2(t)\|_V.$$

Therefore, provided (4.21) holds, equation (4.26) implies that

$$\|u_1(t) - u_2(t)\|_V \ \le\ A_{\mathscr{L}}\cdot N_{\mathscr{L}}(u_1(t) - u_2(t)) + B_{\mathscr{L}}e^{-\gamma(t-s)}, \qquad t \ge s,$$

where $A_{\mathscr{L}}$ and $B_{\mathscr{L}}$ are positive numbers. Hence, the condition

$$\lim_{t \to \infty} N_{\mathscr{L}}(u_1(t) - u_2(t)) = 0$$

implies that

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\|_V = 0 .$$

Thus, **Theorem 4.3 is proved**.

— Exercise *4.7* Assume that the hypotheses of Theorem 4.3 hold. Let $u_1, u_2 \in M_s$ be such that $l_j(u_1) = l_j(u_2)$ for all $l_j \in \mathscr{L}$. Show that

$$S(t, s)u_1 \equiv S(t, s)u_2 \quad \text{for} \quad t \geq s .$$

— Exercise *4.8* Prove that if the hypotheses of Theorem 4.3 hold, then inequality (4.22) as well as the equation

$$c_1\|u_1 - u_2\|_V \;\leq\; \max\{|l_j(u_1 - u_2)| :\; j = 1, 2, \ldots, N\} \;\leq\; c_2\|u_1 - u_2\|_V$$

is valid for any $u_1, u_2 \in M_t$ and $t \in \mathbb{R}$, where $c_1, c_2 > 0$ are constants depending on $\mathscr{L}$.

Let us return to problem (4.1). Assume that $B(u, t)$ is a continuous mapping from $D(A^\theta) \times \mathbb{R}$ into H, $0 \leq \theta < 1$, possessing the properties

$$\|B(u_0, t)\| \;\leq\; M\Big(1 + \|A^\theta u_0\|\Big), \qquad \|B(u_1, t) - B(u_2, t)\| \;\leq\; M\|A^\theta(u_1 - u_2)\|$$

for all $u_j \in D(A^\theta)$, $0 \leq \theta < 1$. Assume that the spectral gap condition

$$\lambda_{n+1} - \lambda_n \;\geq\; \frac{2M}{q}((1+k)\,\lambda_{n+1}^\theta + \lambda_n^\theta)$$

holds for some n and $0 < q < 2 - \sqrt{2}$. Here $\{\lambda_n\}$ are the eigenvalues of the operator A indexed in the increasing order and k is a constant defined by (3.1.7). Under these conditions there exists (see Chapter 2) a process $(D(A^\theta);\; S(t, s))$ generated by problem (4.1). By virtue of Theorems 3.2.1 and 3.3.1 this process possesses an asymptotically complete finite-dimensional inertial manifold $\{M_t\}$ and the corresponding orthoprojector P is a projector onto the span of the first n eigenvectors of the operator A. Therefore,

$$\|A^\theta Pu\| \leq \lambda_n^{\theta - s}\|A^s u\| , \qquad -\infty < s < \theta .$$

Therewith the Lipschitz constant L for $\Phi(p, t)$ can be estimated by the value $q/(1-q)$. Thus, if $\mathscr{L}$ is a set of functionals on $V = D(A^\theta)$, then in order to apply Theorem 4.3 with $H = D(A^s)$, $-\infty < s < \theta$, it is sufficient to require that

$$\varepsilon_{\mathscr{L}}(D(A^\theta);\; D(A^s)) \;\leq\; \frac{1-q}{\sqrt{2 - 2q + 2q^2}} \cdot \frac{1}{\lambda_n^{\theta - s}} .$$

Due to Theorem 2.4 this estimate can be rewritten as follows:

$$\varepsilon_{\mathscr{L}}(D(A^{\theta});\ D(A^{s})) \ \le\ \frac{1-q}{\sqrt{2-2q+2q^{2}}}\cdot\left(\frac{\lambda_{n+1}}{\lambda_{n}}\right)^{\theta-s}\cdot \varkappa_{n}(D(A^{\theta});\ D(A^{s}))\,,$$

where $\varkappa_n(D(A^{\theta});\ D(A^{s}))$ is the Kolmogorov n-width of the embedding of $D(A^{\theta})$ into $D(A^{s})$, $-\infty < s < \theta$, $0 \le \theta < 1$.

It should be noted that the assertion similar to Theorem 4.3 was first established for the Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + u_x u = 0\,, \qquad x \in (0,\ L)\,, \qquad t > 0\,,$$

with the periodic boundary conditions on $[0,\ L]$ in the case when $\{l_j\}$ is a set of uniformly distributed nodes on $[0,\ L]$, i.e.

$$l_j(u) = u(jh)\,, \quad \text{where} \quad h = \frac{L}{N}\,, \quad j = 0,\ 1,\ \ldots,\ N-1\,.$$

For the references and discussion of general case see survey [3].

In conclusion of this section we give one more theorem on the existence of determining functionals for problem (4.1). The theorem shows that in some cases we can require that the values of functionals on the difference of two solutions tend to zero only on a sequence of moments of time (cf. Theorem 1.3).

Theorem 4.4.

As before, assume that A ***is a positive operator with discrete spectrum:***

$$Ae_k = \lambda_k e_k\,, \quad 0 < \lambda_1 \le \lambda_2 \le \ldots\,, \quad \lim_{k\to\infty} \lambda_k \to \infty\,.$$

Assume that $B(u,\ t)$ ***is a continuous mapping from*** $D(A^{\theta}) \times \mathbb{R}$ ***into*** H ***for some*** $0 < \theta < 1$ ***and the estimate***

$$\|B(u_1,\ t) - B(u_2,\ t)\| \ \le\ M\|A^{\theta}(u_1 - u_2)\|\,, \qquad u_1,\ u_2 \in D(A^{\theta})\,,$$

holds. Let $\mathscr{L} = \{l_j\}$ ***be a finite set of linear functionals on*** $D(A^{\theta})$***. Then for any*** $0 < \alpha < \beta$ ***there exists*** $n = n(\alpha,\ \beta,\ \theta,\ \lambda_1,\ M)$ ***such that the condition***

$$\varepsilon_{\mathscr{L}} \equiv \varepsilon_{\mathscr{L}}(D(A^{\theta}),\ H) < \frac{1}{2}\,\lambda_n^{-\theta}$$

implies that the set of functionals $\mathscr{L}$ ***is determining for problem*** (4.1) ***in the sense that if for some pair of solutions*** $\{u_1(t);\ u_2(t)\}$ ***and for some sequence*** $\{t_k\}$ ***such that***

$$\lim_{k\to\infty} t_k = \infty\,, \qquad \alpha \le t_{k+1} - t_k \le \beta\,, \qquad k \ge 1\,,$$

the condition

$$\lim_{k\to\infty} l_j(u_1(t_k) - u_2(t_k)) = 0\,, \qquad l_j \in \mathscr{L}\,,$$

holds, then

$$\lim_{t\to\infty} \|A^{\theta}(u_1(t) - u_2(t))\| = 0\,. \tag{4.27}$$

Proof.

Let $u(t) = u_1(t) - u_2(t)$. Then the results of Chapter 2 (see Theorem 2.2.3 and Exercise 2.2.7) imply that

$$\|A^\theta u(t)\| \le a_1 e^{a_2(t-s)} \|A^\theta u(s)\| , \qquad (4.28)$$

$$\|(1-P_n)A^\theta u(t)\| \le \left\{ e^{-\lambda_{n+1}(t-s)} + \frac{a_3}{\lambda_{n+1}^{1-\theta}} e^{a_2(t-s)} \right\} \|A^\theta u(s)\| \qquad (4.29)$$

for $t \ge s \ge 0$, where a_1, a_2, and a_3 are positive numbers depending on θ, λ_1, and M and P_n is the orthoprojector onto $\mathrm{Lin}\{e_1, \ldots, e_n\}$. It follows form (4.29) that

$$\|(1-P_n)A^\theta u(t)\| \le q_{\alpha,\beta} \|A^\theta u(s)\| , \qquad s+\alpha \le t \le s+\beta , \qquad (4.30)$$

where

$$q_{\alpha,\beta} = e^{-\lambda_{n+1}\alpha} + \frac{a_3}{\lambda_{n+1}^{1-\theta}} e^{a_2\beta} .$$

Let us choose $n = n(\alpha, \beta, \theta, \lambda_1, M)$ such that $q_{\alpha,\beta} \le 1/2$. Then equation (4.30) gives us that

$$\|A^\theta u(t)\| \le \lambda_n^\theta \|u(t)\| + \frac{1}{2} \|A^\theta u(s)\| , \qquad s+\alpha \le t \le s+\beta .$$

This inequality as well as estimate (4.28) enables us to use Theorem 1.3 with $V = D(A^\theta)$ and **to complete the proof of Theorem 4.4**.

— Exercise *4.9* Assume that n is chosen such that $q_{\alpha,\beta} < 1$ in the proof of Theorem 4.4. Show that the condition $\|P_n(u_1(t_k) - u_2(t_k))\| \to 0$ as $k \to \infty$ implies (4.27).

The results presented in this section can also be proved for semilinear retarded equations. For example, we can consider a retarded perturbation of problem (4.1) of the following form

$$\begin{cases} \dfrac{du}{dt} + Au = B(u, t) + Q(u_t, t) , \\ u\big|_{t \in [-r, 0]} = \varphi(t) \in C_r \equiv C([-r, 0], D(A^{1/2})) , \end{cases}$$

where, as usual (see Section 2.8), u_t is an element of C_r defined with the help of $u(t)$ by the equality $u_t(\theta) = u(t+\theta)$, $\theta \in [-r, 0]$, and Q is a continuous mapping from $C_r \times \mathbb{R}$ into H possessing the property

$$\|Q(v_1, t) - Q(v_2, t)\|^2 \le M_1 \cdot \int_{-r}^{0} \|A^{1/2}(v_1(\theta) - v_2(\theta))\|^2 \, d\theta$$

for any $v_1, v_2 \in C_r$. The corresponding scheme of reasoning is similar to the method used in [3], where the second order in time retarded equations are considered.

§ 5 Determining Functionals for Reaction-Diffusion Systems

In this section we consider systems of parabolic equations of the reaction-diffusion type and find conditions under which a finite set of linear functionals given on the phase space uniquely determines the asymptotic behaviour of solutions. In particular, the results obtained enable us to prove the existence of finite collections of determining modes, nodes, and local volume averages for the class of systems under consideration. It also appears that in some cases determining functionals can be given only on a part of components of the state vector. As an example, we consider a system of equations which describes the Belousov-Zhabotinsky reaction and the Navier-Stokes equations.

Assume that Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \ge 1$, $H^s(\Omega)$ is the Sobolev space of the order s on Ω, and $H_0^s(\Omega)$ is the closure (in $H^s(\Omega)$) of the set of infinitely differentiable functions with compact support in Ω. Let $\|\cdot\|_s$ be a norm in $H^s(\Omega)$ and let $\|\cdot\|$ and $(\cdot, \cdot)$ be a norm and an inner product in $L^2(\Omega)$, respectively. Further we also use the spaces

$$\boldsymbol{H}^s = (H^s(\Omega))^m \equiv H^s(\Omega) \times \ldots \times H^s(\Omega), \qquad m \ge 1 .$$

Notations $\boldsymbol{L}^2$ and $\boldsymbol{H}_0^s$ have a similar meaning. We denote the norms and the inner products in $\boldsymbol{L}^2$ and $\boldsymbol{H}^s$ as in $L^2(\Omega)$ and $H^s(\Omega)$.

We consider the following system of equations

$$\partial_t u = a(x, t)\Delta u - f(x, u, \nabla u;\ t), \quad x \in \Omega, \quad t > 0, \tag{5.1}$$

$$u\big|_{\partial\Omega} = 0, \qquad u(x,\ 0) = u_0(x),$$

as the main model. Here $u(x,\ t) = (u_1(x,\ t);\ \ldots;\ u_m(x,\ t))$, Δ is the Laplace operator, $\nabla u_k = (\partial_{x_1} u_k,\ \ldots,\ \partial_{x_n} u_k)$, and $a(x,\ t)$ is an m-by-m matrix with the elements from $L^\infty(\overline{\Omega} \times \mathbb{R}_+)$ such that for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}_+$

$$a_+(x,\ t) \ \equiv\ \frac{1}{2}\cdot(a + a^*) \ \ge\ \mu_0 \cdot I, \qquad \mu_0 > 0 . \tag{5.2}$$

We also assume that the continuous function

$$f = (f_1;\ \ldots;\ f_m):\quad \overline{\Omega} \times \mathbb{R}^{(n+1)m} \times \mathbb{R}_+ \to \mathbb{R}^m$$

is such that problem (5.1) has solutions which belong to a class $\mathscr{W}$ of functions on $\mathbb{R}_+$ with the following properties:

a) for any $u \in \mathscr{W}$ there exists $t_0 > 0$ such that

$$u(t) \in C(t_0,\ +\infty;\ \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1), \quad \partial_t u(t) \in C(t_0,\ +\infty;\ \boldsymbol{L}^2), \tag{5.3}$$

where $C(a, b;\ X)$ is the space of strongly continuous functions on $[a,\ b]$ with the values in X;

b) there exists a constant $k > 0$ such that for any $u, v \in \mathcal{W}$ there exists $t_1 > 0$ such that for $t > t_1$ we have

$$\|f(u, \nabla u;\ t) - f(v, \nabla v;\ t)\| \ \le\ k \cdot \Big(\|u - v\| + \|\nabla u - \nabla v\| \Big). \tag{5.4}$$

It should be noted that if $a(x, t)$ is a diagonal matrix with the elements from $C^2(\overline{\Omega} \times \mathbb{R}_+)$ and f is a continuously differentiable mapping such that

$$|f(x, u, p;\ t) - f(x, v, q;\ t)| \ \le\ k \cdot (|u - v| + |p - q|) \tag{5.5}$$

for all $u, v \in \mathbb{R}^m$, $p, q \in \mathbb{R}^{nm}$, $x \in \overline{\Omega}$, and $t \in \mathbb{R}_+$, then under natural compatibility conditions problem (5.1) has a unique classical solution [7] which evidently possesses properties (5.3) and (5.4). In cases when the dynamical system generated by equations (5.1) is dissipative, the global Lipschitz condition (5.5) can be weakened. For example (see [8]), if a is a constant matrix and

$$f(x, u, \nabla u;\ t) = \bar{f}(x, u) + \sum_{j=1}^{n} b_j(x) \partial_{x_j} u + g(x)\,,$$

where $b_j = (b_j^1, \ldots, b_j^m) \in \boldsymbol{L}^\infty$, $g = (g_1, \ldots, g_m) \in \boldsymbol{L}^2$, and $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_m)$ is continuously differentiable and satisfies the conditions

$$\bar{f}(x, u)u \ \ge\ \mu_1 |u|^{p_0}, \qquad |\bar{f}(x, u)| \ \le\ \mu_2 |u|^{p_0 - 1} + C\,, \qquad p_0 > 2\,,$$

$$\left|\frac{\partial \bar{f}_k}{\partial u_j}\right| \ \le\ C \cdot (1 + |u|^{p_1})\,, \qquad 1 \le k \le m\,, \qquad 1 \le j \le n\,,$$

where $\mu_1, \mu_2 > 0$ and $p_1 < \min(4/n,\ 2/(n-2))$ for $n > 2$, then any solution to problem (5.1) with the initial condition from $\boldsymbol{L}^2$ is unique and possesses properties (5.3) and (5.4).

Let us formulate our main assertion.

Theorem 5.1.

A set $\mathcal{L} = \{l_j\colon\ j = 1, \ldots, N\}$ ***of linearly independent continuous linear functionals on*** $\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ ***is an asymptotically determining set with respect to the space*** $\boldsymbol{H}_0^1$ ***for problem*** (5.1) ***in the class*** $\mathcal{W}$ ***if***

$$\varepsilon_{\mathcal{L}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1,\ L^2) \ <\ \frac{c_0 \mu_0}{k\sqrt{2}} \cdot \left(1 + \frac{27}{4} \cdot \left(\frac{k}{\mu_0}\right)^2\right)^{-1/2} \equiv p(k,\ \mu_0)\,, \tag{5.6}$$

where $c_0^{-1} = \sup\{\|w\|_2\colon\ w \in (\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1)(\Omega),\ \|\Delta w\| \le 1\}$***, and*** μ_0 ***and*** k ***are constants from*** (5.2) ***and*** (5.4)***. This means that if inequality*** (5.6) ***holds, then for some pair of solutions*** $u, v \in \mathcal{W}$ ***the equation***

$$\lim_{t \to \infty} \int_t^{t+1} \big|l_j(u(\tau)) - l_j(v(\tau))\big|^2 \, d\tau = 0, \qquad j = 1, \ldots, N, \tag{5.7}$$

implies their asymptotic closeness in the space H^1:

$$\lim_{t \to \infty} \|u(t) - v(t)\|_1 = 0. \tag{5.8}$$

Proof.

Let $u, v \in \mathscr{W}$. Then equation (5.1) for $w = u - v$ gives us that

$$\partial_t w = a(x, t)\Delta w - (f(x, u, \nabla u;\ t) - f(x, v, \nabla v;\ t)).$$

If we multiply this by Δw in L^2 scalarwise and use equation (5.4), then we find that

$$\frac{1}{2} \cdot \frac{d}{dt}\|\Delta w(t)\|^2 + \mu_0 \|\Delta w(t)\|^2 \le k\Big(\|w(t)\| + \|(\nabla w)(t)\|\Big) \cdot \|\Delta w(t)\|$$

for $t > 0$ large enough. Therefore, the inequality $\|\nabla w\|^2 = |(w, \Delta w)| \le \|w\| \cdot \|\Delta w\|$ enables us to obtain the estimate

$$\frac{d}{dt}\|\nabla w\|^2 + \mu_0 \|\Delta w\|^2 \le \frac{2k^2}{\mu_0}\left(1 + \frac{27}{4}\left(\frac{k}{\mu_0}\right)^2\right)\|w\|^2. \tag{5.9}$$

Theorem 2.1 implies that

$$\|w\|^2 \le C(N, \delta) \max_j |l_j(w)|^2 + (1 + \delta) \cdot \varepsilon_{\mathscr{L}}^2 \cdot \|w\|_2^2 \tag{5.10}$$

for all $w \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ and for any $\delta > 0$, where $C(N, \delta) > 0$ is a constant and $\varepsilon_{\mathscr{L}} \equiv \equiv \varepsilon_{\mathscr{L}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1, L^2)$. Consequently, estimate (5.9) gives us that

$$\frac{d}{dt}\|\nabla w(t)\|^2 + \mu_0 \cdot \left(1 - \frac{(1+\delta)\varepsilon_{\mathscr{L}}^2}{p(k, \mu_0)^2}\right) \cdot \|\Delta w(t)\|^2 \le C \cdot \max_j |l_j(w)|^2,$$

where $p(k, \mu_0)$ is defined by equation (5.6). It follows that if estimate (5.6) is valid, then there exists $\beta > 0$ such that

$$\|\nabla w(t)\|^2 \le e^{-\beta(t - t_0)}\|\nabla w(t_0)\|^2 + C \cdot \int_{t_0}^{t} e^{-\beta(t - \tau)} \max_j |l_j(w(\tau))|^2 \, d\tau$$

for all $t \ge t_0$, where t_0 is large enough. Therefore, equation (5.7) implies (5.8). Thus, **Theorem 5.1 is proved**.

Exercise *5.1* Assume that $u(t)$ and $v(t)$ are two solutions to equation (5.1) defined for all $t \in \mathbb{R}$. Let (5.3) and (5.4) hold for every $t_0 \in \mathbb{R}$ and let

$$\sup_{t < 0}\Big(\|\nabla u(t)\| + \|\nabla v(t)\|\Big) < \infty.$$

Prove that if the hypotheses of Theorem 5.1 hold, then equalities $l_j(u(t)) = l_j(v(t))$ for $j = 1, \ldots, N$ and $t < s$ for some $s \le \infty$ imply that $u(t) \equiv v(t)$ for all $t < s$.

Let us give several examples of sets of determining functionals for problem (5.1).

E x a m p l e *5.1* (determining modes, $m \geq 1$)

Let $\{e_k\}$ be eigenelements of the operator $-\Delta$ in $\boldsymbol{L}^2$ with the Dirichlet boundary conditions on $\partial\Omega$ and let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the corresponding eigenvalues. Then the completeness defect of the set

$$\mathscr{L} = \left\{ l_j : \quad l_j(w) = \int_\Omega w(x) \cdot e_j(x)\,\mathrm{d}x\,,\ j = 1, \ldots, N \right\}$$

can be easily estimated as follows: $\varepsilon_{\mathscr{L}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1,\ \boldsymbol{L}^2) \leq \sqrt{n} \cdot \lambda_{N+1}^{-1}$ (see Exercise 2.6). Thus, if N possesses the property $\lambda_{N+1} > \sqrt{n} \cdot p(k,\ \mu_0)^{-1}$, then $\mathscr{L}$ is a set of asymptotically $(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1,\ \boldsymbol{H}_0^1,\ \mathscr{W})$-determining functionals for problem (5.1).

Considerations of Section 5.3 also enable us to give the following examples.

E x a m p l e *5.2* (determining generalized local volume averages)

Assume that the domain Ω is divided into local Lipschitzian subdomains $\{\Omega_j :\ j = 1, \ldots, N\}$, with diameters not exceeding some given number $h > 0$. Assume that on every domain Ω_j a function $\lambda_j(x) \in L^\infty(\Omega_j)$ is given such that the domain Ω_j is star-like with respect to $\operatorname{supp} \lambda_j$ and the conditions

$$\int_{\Omega_j} \lambda_j(x)\,\mathrm{d}x = 1\,, \qquad \operatorname*{ess\,sup}_{x \in \Omega_j} |\lambda_j(x)| \leq \frac{\Lambda}{h^n}\,,$$

hold, where the constant $\Lambda > 0$ does not depend on h and j. Theorem 3.1 implies that

$$\varepsilon_{\mathscr{L}_h}(H^2(\Omega) \cap H_0^1(\Omega);\ L^2(\Omega)) \leq c_n h^2 \cdot \Lambda^2$$

for the set of functionals

$$\mathscr{L}_h = \left\{ l_j(u) = \int_{\Omega_j} \lambda_j(x)\,u(x)\,\mathrm{d}x\,,\ j = 1, 2, \ldots, N \right\}.$$

Therewith the reasonings in the proof of Theorem 3.1 imply that $c_n = \sigma_n^2\, n^{-2}$, where σ_n is the area of the unit sphere in $\mathbb{R}^n$. For every $l_j \in \mathscr{L}_h$ we define the functionals $l_j^{(k)}$ on $\boldsymbol{H}^2$ by the formula

$$l_j^{(k)}(w) = l_j(w_k)\,, \qquad w = (w_1, \ldots, w_m) \in \boldsymbol{H}^2\,, \qquad k = 1, 2, \ldots, m\,. \tag{5.11}$$

Let

$$\mathscr{L}_h^{(m)} = \{ l_j^{(k)}(u) :\ k = 1, 2, \ldots, m\,,\ \ l_j \in \mathscr{L}_h \}\,. \tag{5.12}$$

One can check (see Exercise 2.8) that

$$\varepsilon_{\mathscr{L}_h^{(m)}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1;\ \boldsymbol{L}^2) \le c_m \cdot h^2 \cdot \Lambda^2 .$$

Therefore, if h is small enough, then $\mathscr{L}_h^{(m)}$ is a set of asymptotically determining functionals for problem (5.1).

Example 5.3 (determining nodes, $n \le 3$)

Let Ω be a convex smooth domain in $\mathbb{R}^n$, $n \le 3$. Let $h > 0$ and let

$$\Omega_j = \Omega \cap \{x = (x_1, \dots, x_n):\ j_k h < x_k < (j_k + 1)h,\ k = 1, 2, 3\},$$

where $j = (j_1, \dots, j_n) \in \mathbb{Z}^n \cap \Omega$. Let us choose a point x_j in every subdomain Ω_j and define the set of nodes

$$\mathscr{L}_h = \{l_j(u) = u(x_j):\ j \in \mathbb{Z}^n \cap \Omega\}, \qquad n \le 3. \tag{5.13}$$

Theorem 3.2 enables us to state that

$$\varepsilon_{\mathscr{L}_h}(H^2(\Omega) \cap H_0^1(\Omega);\ L^2(\Omega)) \le c \cdot h^2 ,$$

where c is an absolute constant. Therefore, the set of functionals $\mathscr{L}_h^{(m)}$ defined by formulae (5.11) and (5.12) with $\mathscr{L}_h$ given by equality (5.13) possesses the property

$$\varepsilon_{\mathscr{L}_h^{(m)}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1;\ \boldsymbol{L}^2) \le c \cdot h^2 .$$

Consequently, $\mathscr{L}_h^{(m)}$ is a set of asymptotically determining functionals for problem (5.1) in the class $\mathscr{W}$, provided that h is small enough.

It is also clear that the result of Exercise 2.8 enables us to construct mixed determining functionals: they are determining nodes or local volume averages depending on the components of a state vector. Other variants are also possible.

However, it is possible that not all the components of the solution vector $u(x, t)$ appear to be essential for the asymptotic behaviour to be uniquely determined. A theorem below shows when this situation can occur.

Let I be a subset of $\{1, \dots, m\}$. Let us introduce the spaces

$$\boldsymbol{H}_I^s = \{w = (w_1, \dots, w_m) \in \boldsymbol{H}^s:\ w_k \equiv 0, k \notin I\}, \qquad s \in \mathbb{R}.$$

We identify these spaces with $(H^s(\Omega))^{|I|}$, where $|I|$ is the number of elements of the set I. Notations $\boldsymbol{L}_I^2$ and $\boldsymbol{H}_{0,I}^s$ have the similar meaning. The set $\mathscr{L}$ of linear functionals on $\boldsymbol{H}_I^2$ is said to be determining if $p_I^* \mathscr{L}$ is determining, where p_I is the natural projection of $\boldsymbol{H}^s$ onto $\boldsymbol{H}_I^s$.

Theorem 5.2.

Let $a = \operatorname{diag}(d_1, \ldots, d_m)$ *be a diagonal matrix with constant elements and let* $\{I, I'\}$ *be a partition of the set* $\{1, \ldots, m\}$ *into two disjoint subsets. Assume that there exist positive constants* ω, k^*, θ_i, *where* $i = 1, \ldots, m$, *such that for any pair of solutions* $u, v \in \mathscr{W}$ *the following inequality holds (hereinafter* $w = u - v$ *):*

$$\sum_{i \in I} \theta_i \left\{ -\frac{d_i}{2} \|\Delta w_i\|^2 + (f_i(u, \nabla u;\ t) - f_i(v, \nabla v;\ t),\ \Delta w_i) \right\} +$$

$$+ \sum_{i \in I'} \theta_i \left\{ -d_i \|\nabla w_i\|^2 - (f_i(u, \nabla u;\ t) - f_i(v, \nabla v;\ t),\ w_i) \right\} \le$$

$$\le -\omega \sum_{i \in I'} \|w_i\|^2 + k^* \sum_{i \in I} \|w_i\|^2 . \tag{5.14}$$

Then a set $= \{l_j\colon\ j = 340(1), \ldots,$ *of linearly independent continuous linear functionals on* $H_I^2 \cap H_{0,I}^1$ *is an asymptotically determining set with respect to the space* H_0^1 *for problem* (5.1) *in the class* $\mathscr{W}$ *if*

$$\varepsilon_{\mathscr{L}} \equiv \varepsilon_{\mathscr{L}}(H_I^2 \cap H_{0,I}^1,\ L_I^2) < c_0 \cdot \min_{i \in I} \sqrt{\frac{d_i \theta_i}{2k^*}} , \tag{5.15}$$

where $c_0 > 0$ *is defined as in* (5.6). *This means that if two solutions* $u, v \in \mathscr{W}$ *possess the property*

$$\lim_{t \to \infty} \int_t^{t+1} \left| l_j(p_I u(\tau)) - l_j(p_I v(\tau)) \right|^2 \mathrm{d}\tau = 0 \quad \textit{for} \quad j = 1, \ldots, N, \tag{5.16}$$

where p_I *is the natural projection of* H^s *onto* H_I^s, *then equation* (5.8) *holds.*

Proof.

Let $u, v \in \mathscr{W}$ and $w = u - v$. Then

$$\partial_t w_i = d_i \Delta w_i - (f_i(x, u, \nabla u;\ t) - f_i(x, v, \nabla v;\ t)) \tag{5.17}$$

for $i = 1, \ldots, m$. In $L^2(\Omega)$ we scalarwise multiply equations (5.17) by $-\theta_i \Delta w_i$ for $i \in I$ and by $\theta_i w_i$ for $i \in I'$ and summarize the results. Using inequality (5.14) we find that

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \Phi(w(t)) + \frac{1}{2} \sum_{i \in I} d_i \theta_i \|\Delta w_i\|^2 + \omega \sum_{i \in I'} \|w_i\|^2 \le k^* \sum_{i \in I} \|w_i\|^2 ,$$

where

$$\Phi(w) = \sum_{i \in I} \theta_i \|\nabla w_i\|^2 + \sum_{i \in I'} \theta_i \|w_i\|^2 .$$

As in the proof of Theorem 5.1 (see (5.10)) we have

$$\sum_{i \in I} \|w_i\|^2 \leq C(N, \delta) \max_j |l_j(p_I w)|^2 + \frac{1+\delta}{c_0^2} \varepsilon_{\mathcal{L}}^2 \cdot \sum_{i \in I} \|\Delta w_i\|^2$$

for every $\delta > 0$. Therefore, provided (5.15) holds, we obtain that

$$\frac{d}{dt} \Phi(w(t)) + \beta \Phi(w(t)) \leq C \cdot \max_j |l_j(p_I w(t))|^2$$

with some constant $\beta > 0$. Equation (5.16) implies that $\Phi(w(t)) \to 0$ as $t \to 0$. Hence,

$$\lim_{t \to \infty} \|w(t)\| = 0 . \tag{5.18}$$

However, equation (5.9) and the inequality

$$\|\nabla w\| \leq C \cdot \|\Delta w\|, \qquad w \in H^2 \cap H_0^1,$$

imply that

$$\|\nabla w(t)\|^2 \leq e^{-\beta(t-t_0)} \|\nabla w(t_0)\|^2 + C \int_{t_0}^{t} e^{-\beta(t-t_0)} \|w(\tau)\|^2 \, d\tau$$

for all $t \geq t_0$ with t_0 large enough and for $\beta > 0$. Therefore, equation (5.8) follows from (5.18). **Theorem 5.2 is proved**.

The abstract form of Theorem 5.2 can be found in [3].

As an application of Theorem 5.2 we consider a system of equations which describe the Belousov-Zhabotinsky reaction. This system (see [9], [10], and the references therein) can be obtained from (5.1) if we take $n \leq 3$, $m = 3$, $a(x, t) = \mathrm{diag}(d_1, d_2, d_3)$ and

$$f(x, u, \nabla u; t) \equiv f(u) = (f_1(u); f_2(u); f_3(u)),$$

where

$$f_1(u) = -\alpha(u_2 - u_1 u_2 + u_1 - \beta u_1^2),$$

$$f_2(u) = -\frac{1}{\alpha}(\gamma u_3 - u_2 - u_1 u_2), \qquad f_3(u) = -\delta(u_1 - u_3).$$

Here α, β, γ, and δ are positive numbers. The theorem on the existence of classical solutions can be proved without any difficulty (see, e.g., [7]). It is well-known [10] that if $a_3 > a_1 > \max(1, \beta^{-1})$ and $a_2 > \gamma a_3$, then the domain

$$D = \{u \equiv (u_1, u_2, u_3): \ 0 \leq u_j \leq a_j, \ j = 1, 2, 3\} \subset \mathbb{R}^3$$

is invariant (if the initial condition vector lies in D for all $x \in \Omega$, then $u(x, t) \in D$ for $x \in \Omega$ and $t > 0$). Let $\mathcal{W} \equiv \mathcal{W}_D$ be a set of classical solutions the initial conditions of which have the values in D. It is clear that assumptions (5.3) and (5.4) are valid for $\mathcal{W}$. Simple calculations show that the numbers ω, k^*, θ_2, and θ_3 can be chosen such that equation (5.14) holds for $I = \{1\}$, $I' = \{2, 3\}$, and $\theta_1 = 1$. Indeed, let smooth functions $u(x)$ and $v(x)$ be such that $u(x), v(x) \in D$ for all $x \in \Omega$ and let $w(x) = u(x) - v(x)$. Then it is evident that there exist constants $C_j > 0$ such that

$$\Phi_1 \equiv (f_1(u) - f_1(v),\ \Delta w_1) \le \frac{d_1}{2}\|\Delta w_1\|^2 + C_1 \cdot \left(\|w_1\|^2 + \|w_2\|^2\right),$$

$$\Phi_2 \equiv -(f_2(u) - f_2(v),\ w_2) \le -\frac{1}{2\alpha}\|w_2\|^2 + C_2 \cdot \left(\|w_1\|^2 + \|w_3\|^2\right),$$

$$\Phi_3 \equiv -(f_3(u) - f_3(v),\ w_3) \le -\frac{\delta}{2}\|w_3\|^2 + C_3 \cdot \|w_1\|^2 .$$

Consequently, for any $\theta_2, \theta_3 > 0$ we have

$$\Phi_1 + \theta_2 \Phi_2 + \theta_3 \Phi_3 \le \frac{d_1}{2}\|\Delta w_1\|^2 + (C_1 + \theta_2 C_2 + \theta_3 C_3)\|w_1\|^2 +$$
$$+\left(C_1 - \frac{\theta_2}{2\alpha}\right)\|w_2\|^2 + \left(\theta_2 C_2 - \frac{\theta_3 \delta}{2}\right)\|w_3\|^2 .$$

It follows that there is a possibility to choose the parameters θ_2 and θ_3 such that

$$\Phi_1 + \theta_2 \Phi_2 + \theta_3 \Phi_3 \le \frac{d_1}{2}\|\Delta w_1\|^2 + k^*\|w_1\|^2 - \omega\left(\|w_2\|^2 + \|w_3\|^2\right)$$

with positive constants k^* and ω. This enables us to prove (5.14) and, hence, the validity of the assertions of Theorem 5.2 for the system of Belousov-Zhabotinsky equations. Therefore, if $\mathscr{L} = \{l_j :\ j = 1, \ldots, N\}$ is a set of linear functionals on $H^2(\Omega) \cap H_0^1(\Omega)$ such that $\varepsilon_{\mathscr{L}}((H^2 \cap H_0^1)(\Omega),\ L^2(\Omega))$ is small enough, then the condition

$$\lim_{t \to \infty} \int_t^{t+1} \left| l_j(u_1(\tau)) - l_j(v_1(\tau)) \right|^2 d\tau = 0 , \qquad j = 1, \ldots, N ,$$

for some pair of solutions $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ and $v(t) = (v_1(t),\ v_2(t),\ v_3(t))$ which lie in $\mathscr{W}$ implies that

$$\|u(t) - v(t)\|_1 \to 0 \quad \text{as} \quad t \to \infty .$$

In particular, this means that the asymptotic behaviour of solutions to the Belousov-Zhabotinsky system is uniquely determined by the behaviour of one of the components of the state vector. A similar effect for the other equations is discussed in the sections to follow.

The approach presented above can also be used in the study of the Navier-Stokes system. As an example, let us consider equations that describe the dynamics of a viscous incompressible fluid in the domain $\Omega \equiv T^2 = (0, L) \times (0, L)$ with periodic boundary conditions:

$$\partial_t u - \nu\Delta u + (u, \nabla)u + \nabla p = F(x, t), \quad x \in T^2, \quad t > 0,$$
$$\nabla u = 0, \quad x \in T^2, \quad u(x, 0) = u_0(x), \tag{5.19}$$

where the unknown velocity vector $u(x, t) = (u_1(x, t);\ u_2(x, t))$ and pressure $p(x, t)$ are L-periodic functions with respect to spatial variables, $\nu > 0$, and $F(x, t)$ is the external force.

Let us introduce some definitions. Let $\mathcal{V}$ be a space of trigonometric polynomials $v(x)$ of the period L with the values in $\mathbb{R}^2$ such that $\operatorname{div} v = 0$ and $\int_{T^2} v(x)\,\mathrm{d}x = 0$. Let H be the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$, let Π be the orthoprojector onto H in $\boldsymbol{L}^2$, let $A = -\Pi\Delta u = -\Delta u$ and $B(u, v) = \Pi(u, \nabla)v$ for all u and v from $D(A) = H \cap \boldsymbol{H}^2$. We remind (see, e.g., [11]) that A is a positive operator with discrete spectrum and the bilinear operator $B(u, v)$ is a continuous mapping from $D(A) \times D(A)$ into H. In this case problem (5.19) can be rewritten in the form

$$\partial_t u + \nu A u + B(u, u) = \Pi F(t), \quad u\big|_{t=0} = u_0 \in H. \tag{5.20}$$

It is well-known (see, e.g., [11]) that if $u_0 \in H$ and $\Pi F(t) \in L^\infty(\mathbb{R}_+;\ H)$, then problem (5.20) has a unique solution $u(t)$ such that

$$u(t) \in C(\mathbb{R}_+;\ H) \cap C(t_0, +\infty;\ D(A)), \quad t_0 > 0. \tag{5.21}$$

One can prove (see [9] and [12]) that it possesses the property

$$\overline{\lim_{t\to\infty}}\ \frac{1}{a}\int_t^{t+a} \|Au(\tau)\|^2\,\mathrm{d}\tau \ \le\ \frac{F^2}{\nu^2}\left(1 + \frac{1}{a\nu\lambda_1}\right) \tag{5.22}$$

for any $a > 0$. Here $\lambda_1 = (2\pi/L)^2$ is the first eigenvalue of the operator A in H and

$$F = \overline{\lim_{t\to\infty}}\ \|\Pi F(t)\|.$$

Lemma 5.1.

Let $u, v \in D(A)$ and let $w = u - v$. Then

$$|(B(u, u) - B(v, v),\ Aw)| \ \le\ \sqrt{2}\,\|w\|_\infty \|\nabla w\|\,\|Au\|, \tag{5.23}$$

$$|(B(u, u) - B(v, v),\ Aw)| \ \le\ c\sqrt{L}\,\big\|\nabla w_k\big\| \cdot \|\nabla w\|^{1/2} \cdot \|\Delta w\|^{1/2} \cdot \|Au\|, \tag{5.24}$$

where $\|\cdot\|_\infty$ is the L^∞-norm, w_k is the k-th component of the vector w, $k = 1, 2$, and c is an absolute constant.

Proof.

Using the identity (see [12])

$$(B(u, w),\ Aw) + (B(w, v),\ Aw) + (B(w, w),\ Au) = 0$$

for $u, v \in D(A)$ and $w = u - v$, it is easy to find that

$$(B(u, u) - B(v, v),\ Aw) \ =\ -(B(w, w),\ Au).$$

Therefore, it is sufficient to estimate the norm $\|B(w, w)\|$. The incompressibility condition $\nabla w = 0$ gives us that

$$(w, \nabla)w = (-w_1\partial_2 w_2 + w_2\partial_2 w_1, \; w_1\partial_1 w_2 - w_2\partial_1 w_1),$$

where ∂_i is the derivative with respect to the variable x_i. Consequently,

$$\|(w, \nabla)w\|^2 \le 2\left(\|(w_1, \nabla)w_2\|^2 + \|(w_2, \nabla)w_1\|^2\right). \tag{5.25}$$

This implies (5.23). Let us prove (5.24). For the sake of definiteness we let $k = 1$. We can also assume that $w \in \mathcal{V}$. Then (5.25) gives us that

$$\|(w, \nabla)w\| \le \sqrt{2}\left(\|w_1\|_{L^4}\|\nabla w_2\|_{L^4} + \|w_2\|_\infty\|\nabla w_1\|\right).$$

We use the inequalities (see, e.g., [11], [12])

$$\|v\|_{L^\infty} \le a_0\|v\|^{1/2}\cdot\|\Delta v\|^{1/2}$$

and

$$\|v\|_{L^4} \le a_1\|v\|^{1/2}\cdot\|\nabla v\|^{1/2},$$

where a_0 and a_1 are absolute constants (their explicit equations can be found in [12]). These inequalities as well as a simply verifiable estimate $\|v\| \le$ $\le (L/2\pi)\cdot\|\nabla v\|$ imply that

$$\|(w, \nabla)w\| \le \sqrt{\frac{L}{\pi}}(a_1^2 + a_0)\|\nabla w_1\|\cdot\|\nabla w\|^{1/2}\cdot\|\Delta w\|^{1/2}.$$

This proves (5.24) for $k = 1$.

Theorem 5.3.

1. ***A set*** $\mathcal{L} = \{l_j : j = 1, 2, \dots, N\}$ ***of linearly independent continuous linear functionals on*** $D(A) = \boldsymbol{H}^2 \cap \boldsymbol{H}$ ***is an asymptotically determining set with respect to*** $\boldsymbol{H}^1$ ***for problem*** (5.20) ***in the class of solutions with property*** (5.21) ***if***

$$\varepsilon_{\mathcal{L}} \equiv \varepsilon_{\mathcal{L}}(D(A), H) < c_1 G \equiv c_1 \nu^2\left(\overline{\lim_{t\to\infty}}\|\Pi F(t)\|\right)^{-1}, \tag{5.26}$$

where c_1 ***is an absolute constant.***

2. ***Let*** $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ ***be a set of linear functionals on*** $H^2(\Omega)$ ***and let***

$$\varepsilon'_{\mathcal{L}} \equiv \varepsilon_{\mathcal{L}}(H^2(\Omega), L^2(\Omega)) < c_2 \nu^4 L^{-3}\left(\overline{\lim_{t\to\infty}}\|\Pi F(t)\|\right)^{-2},$$

where c_2 ***is an absolute constant. Then every set*** $p_k^*\mathcal{L}$ ***is an asymptotically determining set with respect to*** $\boldsymbol{H}^1$ ***for problem*** (5.20) ***in the class of solutions with property*** (5.21)***. Here*** p_k ***is the natural projection onto the*** k***-th component of the velocity vector,*** $p_k(u_1; u_2) = u_k$, $k = 1, 2$.

Proof.

Let $u(t)$ and $v(t)$ be solutions to problem (5.20) possessing property (5.21). Then equations (5.20) and (5.23) imply that

$$\frac{1}{2}\cdot\frac{d}{dt}\|\nabla w\|^2+\nu\|\Delta w\|^2 \;\le\; \sqrt{2}\,\|w\|_\infty\cdot\|\nabla w\|\cdot\|Au\|$$

for $w = u-v$. As above, Theorem 2.1 gives us the estimate

$$\|w\| \;\le\; C\eta(w)+\varepsilon_{\mathscr{L}}\|\Delta w\|\,,\qquad \eta(w)=\max_j\left|l_j(w)\right|.$$

Therefore,

$$\|w\|_\infty \;\le\; a_0\cdot\|w\|^{1/2}\cdot\|\Delta w\|^{1/2} \;\le\; c_1\sqrt{\eta(w)}\cdot\|\Delta w\|^{1/2}+a_0\sqrt{\varepsilon_{\mathscr{L}}}\cdot\|\Delta w\|\,.$$

The inequality

$$\|\nabla w\|^2 \;\le\; \|w\|\cdot\|\Delta w\|$$

enables us to obtain the estimate (see Exercise 2.12) $\varepsilon_{\mathscr{L}}(D(A),\ D(A^{1/2})) \le \varepsilon_{\mathscr{L}}^{1/2}$ and hence

$$\|\Delta w\|^2 \;\ge\; -C_\delta\,\eta(w)^2+(1-\delta)\,\varepsilon_{\mathscr{L}}^{-1}\cdot\|\nabla w\|^2$$

for every $\delta > 0$. Therefore, we use dissipativity properties of solutions (see [8] and [9] as well as Chapter 2) to obtain that

$$\frac{d}{dt}\|\nabla w(t)\|^2+\alpha(t)\cdot\|\nabla w\|^2 \;\le\; C\cdot\left[\eta(w)^2+\sqrt{\eta(w)}\right],$$

where

$$\alpha(t)=\nu(1-\delta)\,\varepsilon_{\mathscr{L}}^{-1}-2a_0^2\,\varepsilon_{\mathscr{L}}\,\nu^{-1}\|Au(t)\|^2\,.$$

Equation (5.22) for $a = (\nu\lambda_1)^{-1}$ implies that the function $\alpha(t)$ possesses properties (1.6) and (1.7), provided (5.26) holds. Therefore, we apply Lemma 1.1 to obtain that $\|\nabla w(t)\| \to 0$ as $t\to\infty$, provided

$$\lim_{t\to\infty}\int_t^{t+1}\left[\eta(w(\tau))\right]^2 d\tau = 0\,.$$

In order to prove the second part of the theorem, we use similar arguments. For the sake of definiteness let us consider the case $k = 1$ only. It follows from (5.20) and (5.24) that

$$\frac{1}{2}\cdot\frac{d}{dt}\|\nabla w\|^2+\nu\|\Delta w\|^2 \;\le\; c\sqrt{L}\left\|\nabla w_1\right\|^{1/2}\cdot\|\nabla w\|\cdot\|\Delta w\|^{1/2}\cdot\|Au\|\,. \tag{5.27}$$

The definition of completeness defect implies that

$$\left\|\nabla w_1\right\|^{1/2} \;\le\; C\sqrt{\eta(w_1)}+\sqrt{\varepsilon'_{\mathscr{L}}}\left\|\Delta w_1\right\|^{1/2} \;\le\; C\sqrt{\eta(w_1)}+\sqrt{\varepsilon'_{\mathscr{L}}}\,\|\Delta w\|^{1/2},$$

where $\eta(w_1)=\max_j\left|l_j(w_1)\right|$. Therefore,

$$c\sqrt{L}\|\nabla w_1\|^{1/2}\|\nabla w\|\cdot\|\Delta w\|^{1/2}\|Au\| \le$$

$$\le c\sqrt{L}\cdot\sqrt{\varepsilon'_{\mathcal{L}}}\cdot\|\nabla w\|\cdot\|\Delta w\|\cdot\|Au\| + C\sqrt{\eta(w_1)}\|\nabla w\|\cdot\|\Delta w\|^{1/2}\cdot\|Au\| \le$$

$$\le C[\eta(w_1)]^2 + \left(\varkappa + \frac{Lc^2}{\nu}\varepsilon'_{\mathcal{L}}\right)\cdot\|\nabla w\|^2\cdot\|Au\|^2 + \frac{\nu}{2}\|\Delta w\|^2$$

for any $\varkappa > 0$, where the constant C depends on $\varkappa$, ν, $\mathcal{L}$, and L. Consequently, from (5.27) we obtain that

$$\frac{d}{dt}\|\nabla w\|^2 + \nu\|\Delta w\|^2 \le C[\eta(w_1)]^2 + 2\left(\varkappa + \frac{Lc^2}{\nu}\varepsilon'_{\mathcal{L}}\right)\cdot\|\nabla w\|^2\|Au\|^2 .$$

Therefore, we can choose $\varkappa = \delta\cdot L\cdot c^2\cdot\nu^{-1}\cdot\varepsilon'_{\mathcal{L}}$ and find that

$$\frac{d}{dt}\|\nabla w\|^2 + \left[\nu\left(\frac{2\pi}{L}\right)^2 - 2(1+\delta)\frac{Lc^2}{\nu}\varepsilon'_{\mathcal{L}}\cdot\|Au\|^2\right]\|\nabla w\|^2 \le C[\eta(w_1)]^2 ,$$

where $\delta > 0$ is an arbitrary number. Further arguments repeat those in the proof of the first assertion. **Theorem 5.3 is proved**.

It should be noted that assertion 1 of Theorem 5.3 and the results of Section 3 enable us to obtain estimates for the number of determining nodes and local volume averages that are close to optimal (see the references in the survey [3]). At the same time, although assertion 2 uses only one component of the velocity vector, in general it makes it necessary to consider a much greater number of determining functionals in comparison with assertion 1. It should also be noted that assertion 2 remains true if instead of w_k we consider the projections of the velocity vector onto an arbitrary a priori chosen direction [3]. Furthermore, analogues of Theorems 1.3 and 4.4 can be proved for the Navier-Stokes system (5.19) (the corresponding variants of estimates (4.28) and (4.29) can be derived from the arguments in [2], [8], and [9]).

§ 6 Determining Functionals in the Problem of Nerve Impulse Transmission

We consider the following system of partial differential equations suggested by Hodgkin and Huxley for the description of the mechanism of nerve impulse transmission:

$$\partial_t u - d_0\partial_x^2 u + g(u, v_1, v_2, v_3) = 0 , \quad x \in (0, L) , \quad t > 0 , \tag{6.1}$$

$$\partial_t v_j - d_j\partial_x^2 v_j + k_j(u)\cdot(v_j - h_j(u)) = 0 , \quad x \in (0, L) , \quad t > 0 , \quad j = 1, 2, 3 . \tag{6.2}$$

Here $d_0 > 0$, $d_j \geq 0$, and

$$g(u, v_1, v_2, v_3) = -\gamma_1 v_1^3 v_2(\delta_1 - u) - \gamma_2 v_3^4(\delta_2 - u) - \gamma_3(\delta_3 - u) , \qquad (6.3)$$

where $\gamma_j > 0$ and $\delta_1 > \delta_3 > 0 > \delta_2$. We also assume that $k_j(u)$ and $h_j(u)$ are the given continuously differentiable functions such that $k_j(u) > 0$ and $0 < h_j(u) < 1$, $j = 1, 2, 3$. In this model u describes the electric potential in the nerve and v_j is the density of a chemical matter and can vary between 0 and 1. Problem (6.1) and (6.2) has been studied by many authors (see, e.g., [9], [10], [13] and the references therein) for different boundary conditions. The results of numerical simulation given in [13] show that the asymptotic behaviour of solutions to this problem can be quite complicated. In this chapter we focus on the existence and the structure of determining functionals for problem (6.1) and (6.2). In particular, we prove that the asymptotic behaviour of densities v_j is uniquely determined by sets of functionals defined on the electric potential u only. Thus, the component u of the state vector (u, v_1, v_2, v_3) is leading in some sense.

We equip equations (6.1) and (6.2) with the initial data

$$u\big|_{t=0} = u_0(x) , \qquad v_j\big|_{t=0} = v_{j0}(x) , \qquad j = 1, 2, 3 , \qquad (6.4)$$

and with one of the following boundary conditions:

$$u\big|_{x=0} = u\big|_{x=L} = d_j v_j\big|_{x=0} = d_j v_j\big|_{x=L} = 0 , \qquad t > 0 , \qquad (6.5a)$$

$$\partial_x u\big|_{x=0} = \partial_x u\big|_{x=L} = d_j \cdot \partial_x v_j\big|_{x=0} = d_j \cdot \partial_x v_j\big|_{x=L} = 0 , \qquad t > 0 , \qquad (6.5b)$$

$$u(x+L, t) - u(x, t) = d_j \cdot (v_j(x+L, t) - v_j(x, t)) = 0 , \qquad x \in \mathbb{R}^1 , \quad t > 0 , (6.5c)$$

where $j = 1, 2, 3$. Thus, we have no boundary conditions for the function $v_j(x, t)$ when the corresponding diffusion coefficient d_j is equal to zero for some $j = 1, 2, 3$.

Let us now describe some properties of solutions to problem (6.1)–(6.5). First of all it should be noted (see, e.g., [10]) that the parallelepiped

$$D = \left\{ U \equiv (u, v_1, v_2, v_3) : \ \delta_2 \leq u \leq \delta_1 , \ 0 \leq v_j \leq 1 , \ j = 1, 2, 3 \right\} \subset \mathbb{R}^4$$

is a positively invariant set for problem (6.1)–(6.5). This means that if the initial data $U_0(x) = (u_0(x), v_{10}(x), v_{20}(x), v_{30}(x))$ belongs to D for almost all $x \in [0, L]$, then

$$U(t) \equiv (u(x, t), v_1(x, t), v_2(x, t), v_3(x, t)) \in D$$

for $x \in [0, L]$ and for all $t > 0$ for which the solution to problem (6.1)–(6.5) exists.

Let $\mathbb{H}_0 \equiv [H]^4 \equiv [L^2(0, L)]^4$ be the space consisting of vector-functions $U(x) \equiv (u, v_1, v_2, v_3)$, where $u \in L^2(0, L)$, $v_j \in L^2(0, L)$, $j = 1, 2, 3$. We equip it with the standard norm. Let

$$\mathbb{H}_0(D) = \left\{ U(x) \in \mathbb{H}_0 : \ U(x) \in D \ \text{ for almost all } \ x \in (0, L) \right\} .$$

Depending upon the boundary conditions (6.5 a, b, or c) we use the following notations $\mathbb{H}_1 = [V_1]^4$ and $\mathbb{H}_2 = [V_2]^4$, where

$$V_1 = H_0^1(0, L), \quad \text{or} \quad H^1(0, L), \quad \text{or} \quad H^1_{\text{per}}(0, L) \tag{6.6}$$

and

$$V_2 = H^2(0, L) \cap H_0^1(0, L), \quad \text{or}$$

$$\left\{ u(x) \in H^2(0, L) : \partial_x u \Big|_{x=0, \ x=L} = 0 \right\}, \quad \text{or} \quad H^2_{\text{per}}(0, L), \tag{6.7}$$

respectively. Hereinafter $H^s(0, L)$ is the Sobolev space of the order s on $(0, L)$, H_0^1 and H^1_{per} are subspaces in $H^1(0, L)$ corresponding to the boundary conditions (6.5a) and (6.5c). The norm in $H^s(0, L)$ is defined by the equality

$$\|u\|_s^2 = \left\|\partial_x^s u\right\|^2 + \|u\|^2 = \int_0^L \left(\left|\partial_x^s u(x)\right|^2 + |u(x)|^2 \right) \mathrm{d}x , \qquad s = 1, \ 2, \ \ldots$$

We use notations $\|\cdot\|$ and $(\cdot, \cdot)$ for the norm and the inner product in $H \equiv$ $\equiv L^2(0, L)$. Further we assume that $C(0, T; X)$ is the space of strongly continuous functions on $[0, T]$ with the values in X. The notation $L^p(0, T; X)$ has a similar meaning.

Let $d_j > 0$ for all $j = 1, 2, 3$. Then for every vector $U_0 \in \mathbb{H}_0(D)$ problem (6.1)–(6.5) has a unique solution $U(t) \in \mathbb{H}_0(D)$ defined for all t (see, e.g., [9], [10]). This solution lies in

$$C(0, T; \ \mathbb{H}_0(D)) \ \cap \ L^2(0, T; \ \mathbb{H}_1)$$

for any segment $[0, T]$ and if $U_0 \in \mathbb{H}_0(D) \cap \mathbb{H}_1$, then

$$U(t) \in C(0, T; \ \mathbb{H}_0(D) \cap \mathbb{H}_1) \cap L^2(0, T; \ \mathbb{H}_2) . \tag{6.8}$$

Therefore, we can define the evolutionary semigroup S_t in the space $\mathbb{H}_0(D) \cap \mathbb{H}_1$ by the formula

$$S_t U_0 = U(t) \equiv (u(x, t), \ v_1(x, t), \ v_2(x, t), \ v_3(x, t)) ,$$

where $U(t)$ is a solution to problem (6.1)–(6.5) with the initial conditions

$$U_0 \equiv (u_0(x), \ v_{10}(x), \ v_{20}(x), \ v_{30}(x)) .$$

The dynamical system $(\mathbb{H}_0(D) \cap \mathbb{H}_1; \ S_t)$ has been studied by many authors. In particular, it has been proved that it possesses a finite-dimensional global attractor [9].

If $d_1 = d_2 = d_3 = 0$, then the corresponding evolutionary semigroup can be defined in the space $\mathbb{V}_1 = (V_1 \times [H^1(0, L)]^3) \cap \mathbb{H}_0(D)$. In this case for any segment $[0, T]$ we have

$$S_t U_0 = U(t) \in C(0, T; \mathbb{V}_1) \cap L^2(0, T; V_2 \times (H^1(0, L))^3), \tag{6.9}$$

if $U_0 \in \mathbb{V}_1$. This assertion can be easily obtained by using the general methods of Chapter 2.

The following assertion is the main result of this section.

Theorem 6.1.

Let $\mathcal{L} = \{l_j : j = 1, \ldots, N\}$ ***be a finite set of continuous linear functionals on the space*** V_s, $s = 1, 2$ ***(see*** (6.6) ***and*** (6.7)***). Assume that***

$$\varepsilon_{\mathcal{L}}^{(1)} \equiv \varepsilon_{\mathcal{L}}(V_1, H) \le \sqrt{\frac{d_0}{d_0 + K_1}} \tag{6.10}$$

or

$$\varepsilon_{\mathcal{L}}^{(2)} \equiv \varepsilon_{\mathcal{L}}(V_2, H) \le \frac{d_0}{\sqrt{d_0^2 + K_2^2}}, \tag{6.11}$$

where

$$K_1 = (\delta_1 - \delta_2) \cdot \sum_{j=1}^{3} \frac{\beta_j (A_j + B_j)}{k_j^*} \tag{6.12}$$

and

$$K_2 = 2 \cdot \left((\gamma_1 + \gamma_2)^2 + 5(\delta_1 - \delta_2)^2 \cdot \sum_{j=1}^{3} \left(\frac{\beta_j (A_j + B_j)}{2 k_j^*} \right)^2 \right)^{1/2} \tag{6.13}$$

with $\beta_1 = 3\gamma_1$, $\beta_2 = \gamma_1$, $\beta_3 = 4\gamma_2$***, and***

$$A_j = \max\left\{ |\partial_u k_j(u)| : \delta_2 \le u \le \delta_1 \right\}, \qquad B_j = \max\left\{ |\partial_u (k_j h_j)(u)| : \delta_2 \le u \le \delta_1 \right\},$$

$$k_j^* = \min\left\{ k_j(u) : \delta_2 \le u \le \delta_1 \right\}.$$

Then $\mathcal{L}$ ***is an asymptotically determining set with respect to the space*** $\mathbb{H}_0$ ***for problem*** (6.1)–(6.5) ***in the sense that for any two solutions***

$$U(t) = (u(x, t), v_1(x, t), v_2(x, t), v_3(x, t))$$

and

$$U^*(t) = (u^*(x, t), v_1^*(x, t), v_2^*(x, t), v_3^*(x, t))$$

satisfying either (6.8) ***with*** $d_j > 0$***, or (6.9) with*** $d_j = 0$, $j = 1, 2, 3$***, the condition***

$$\lim_{t \to \infty} \int_t^{t+1} \left| l_j(u(\tau)) - l_j(u^*(\tau)) \right|^2 \mathrm{d}\tau = 0 \quad \textit{for} \quad j = 1, \dots, N \tag{6.14}$$

implies that

$$\lim_{t \to \infty} \left\{ \|u(t) - u^*(t)\|^2 + \sum_{j=1}^{3} \|v_j(t) - v_j^*(t)\|^2 \right\} = 0 \ . \tag{6.15}$$

Proof.

Assume that (6.10) is valid. Let

$$U(t) = (u(x, t),\ v_1(x, t),\ v_2(x, t),\ v_3(x, t))$$

and

$$U^*(t) = (u^*(x, t),\ v_1^*(x, t),\ v_2^*(x, t),\ v_3^*(x, t))$$

be solutions satisfying either (6.8) with $d_j > 0$, or (6.9) with $d_j = 0$, $j = 1, 2, 3$. It is clear that

$$G(U, U^*) \equiv g(u, v_1, v_2, v_3) - g(u^*, v_1^*, v_2^*, v_3^*) =$$
$$= (\gamma_1 v_1^3 v_2 + \gamma_2 v_3^4 + \gamma_3)(u - u^*) + h(U, U^*) \ ,$$

where

$$h(U, U^*) = -\gamma_1 (v_1^3 v_2 - v_1^{*3} v_2^*)(\delta_1 - u^*) - \gamma_2 (v_3^4 - v_3^{*4})(\delta_2 - u^*) .$$

Since $U, U^* \in D$, it is evident that

$$|h(U, U^*)| \ \le \ \sum_{j=1}^{3} a_j |v_j - v_j^*| \ ,$$

where $a_j = (\delta_1 - \delta_2) \cdot \beta_j$. Let $w = u - u^*$ and $\psi_j = v_j - v_j^*$, $j = 1, 2, 3$. It follows from (6.1) that

$$\partial_t w - d_0 \partial_x^2 w + G(U, U^*) = 0 , \qquad x \in (0, L) , \qquad t > 0 . \tag{6.16}$$

If we multiply (6.16) by w in $L^2(0, L)$, then it is easy to find that

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \|w\|^2 + d_0 \|\partial_x w\|^2 + \gamma_3 \|w\|^2 \ \le \ \sum_{j=1}^{3} a_j \|\psi_j\| \cdot \|w\| \ . \tag{6.17}$$

Equation (6.2) also implies that

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \|\psi_j\|^2 + d_j \|\partial_x \psi_j\|^2 + k_j^* \|\psi_j\|^2 \ \le \ (A_j + B_j) \|\psi_j\| \cdot \|w\| . \tag{6.18}$$

Thus, for any $0 < \varepsilon < 1$ and $\theta_j > 0$, $j = 1, 2, 3$, we obtain that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right)+d_0\|\partial_x w\|^2+\gamma_3\|w\|^2+\varepsilon\sum_{j=1}^{3}k_j^*\theta_j\|\psi_j\|^2 \le$$
$$\le -\sum_{j=1}^{3}(1-\varepsilon)k_j^*\theta_j\|\psi_j\|^2+\sum_{j=1}^{3}[a_j+\theta_j(A_j+B_j)]\|\psi_j\|\cdot\|w\| \le$$
$$\le \sum_{j=1}^{3}\frac{[a_j+\theta_j(A_j+B_j)]^2}{4(1-\varepsilon)\theta_j k_j^*}\cdot\|w\|^2 . \tag{6.19}$$

Theorem 2.1 gives us that

$$\|w\|^2 \le C_{\mathscr{L},\eta}\cdot\max_j|l_j(w)|^2+(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2\cdot\left(\|\partial_x w\|^2+\|w\|^2\right)$$

for any $\eta>0$. Therefore,

$$\|\partial_x w\|^2 \ge \left(\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2}-1\right)\cdot\|w\|^2-C_{\mathscr{L},\eta}\cdot\max_j|l_j(w)|^2 . \tag{6.20}$$

Consequently, it follows from (6.19) that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right)+d_0\cdot\omega(\mathscr{L},\varepsilon,\theta,\eta)\|w\|^2+$$
$$+\varepsilon\sum_{j=1}^{3}k_j^*\theta_j\|\psi_j\|^2 \le C_{\mathscr{L},\eta}\cdot\max_j|l_j(w)|^2 \tag{6.21}$$

for any $0<\varepsilon<1$, $\theta_j>0$, and $\eta>0$, where

$$\omega(\mathscr{L},\varepsilon,\theta,\eta)=\frac{\gamma_3}{d_0}+\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2}-1-\sum_{j=1}^{3}\frac{[a_j+\theta_j(A_j+B_j)]^2}{4(1-\varepsilon)\theta_j k_j^* d_0} .$$

We choose $\theta_j=a_j\cdot(A_j+B_j)^{-1}$ and obtain that

$$\omega(\mathscr{L},\varepsilon,\theta,\eta)=\frac{\gamma_3}{d_0}+\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2}-1-\frac{K_1}{(1-\varepsilon)d_0} .$$

It is easy to see that if (6.10) is valid, then there exist $0<\varepsilon<1$ and $\eta>0$ such that $\omega(\mathscr{L},\varepsilon,\theta,\eta)>0$. Therefore, equation (6.21) gives us that

$$\|w(t)\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j(t)\|^2 \le$$
$$\le \left(\|w_0\|^2+\sum_{j=1}^{3}\theta_j\|\psi_{j0}\|^2\right)\cdot e^{-\omega^* t}+C_{\mathscr{L},\eta}\cdot\int_0^t e^{-\omega^*(t-\tau)}\max_j|l_j(w(\tau))|^2\,\mathrm{d}\tau ,$$

where $\omega^*>0$. Thus, if (6.10) is valid, then equation (6.14) implies (6.15).

Let us now assume that (6.11) is valid. Since

$$-(G(U,\,U^*),\,\partial_x^2 w)\ \le\ -\gamma_3\|\partial_x w\|^2+\Big((\gamma_1+\gamma_2)\cdot\|w\|+\|h(U,\,U^*)\|\Big)\cdot\|\partial_x^2 w\|,$$

then it follows from (6.16) that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x w\|^2+\frac{d_0}{2}\|\partial_x^2 w\|^2+\gamma_3\|\partial_x w\|^2\ \le$$

$$\le\ \frac{2}{d_0}\cdot(\gamma_1+\gamma_2)^2\cdot\|w\|^2+2\cdot\sum_{j=1}^{3}\frac{a_j^2}{d_0}\cdot\|\psi_j\|^2\ . \tag{6.22}$$

Therefore, we can use equation (6.18) and obtain (cf. (6.19)) that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_x w\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right)+\frac{d_0}{2}\|\partial_x^2 w\|^2+\gamma_3\|\partial_x w\|^2+\varepsilon\sum_{j+1}^{3}k_j^*\theta_j\|\psi_j\|^2\ \le$$

$$\le\ \left(\frac{2}{d_0}\cdot(\gamma_1+\gamma_2)^2+\frac{9}{4}\cdot\sum_{j=1}^{3}\frac{a_j^2(A_j+B_j)^2}{(1-\varepsilon)^2\,[k_j^*]^2 d_0}\right)\cdot\|w\|^2$$

for any $0<\varepsilon<1$ and $\theta_j=3a_j^2\cdot[(1-\varepsilon)k_j^*d_0]^{-1}$. As above, we find that

$$\|\partial_x^2 w\|^2\ \ge\ \left(\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(2)}]^2}-1\right)\cdot\|w\|^2-C_{\mathscr{L},\,\eta}\cdot\max_j|l_j(w)|^2$$

for any $\eta>0$. Therefore,

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_x w\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right)+\gamma_3\|\partial_x w\|^2+\varepsilon\sum_{j+1}^{3}k_j^*\theta_j\|\psi_j\|^2\ \le\ C_{\mathscr{L},\,\eta}\cdot\max_j|l_j(w)|^2,$$

provided that

$$\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(2)}]^2}-1-\frac{4}{d_0^2}\cdot(\gamma_1+\gamma_2)^2-\sum_{j=1}^{3}\frac{9a_j^2(A_j+B_j)^2}{2(1-\varepsilon)^2[k_j^*]^2 d_0^2}\ \ge\ 0\ .$$

As in the first part of the proof, we can now conclude that if (6.11) is valid, then (6.14) implies the equations

$$\lim_{t\to\infty}\|\partial_x w(t)\|=0\quad\text{and}\quad\lim_{t\to\infty}\|\psi_j(t)\|=0\,,\qquad j=1,\,2,\,3\,. \tag{6.23}$$

It follows from (6.17) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+\gamma_3\|w\|^2\ \le\ \frac{3}{\gamma_3}\sum_{j=1}^{3}a_j^2\,\|\psi_j(t)\|^2\ .$$

Therefore, as above, we obtain equation (6.15) for the case (6.11). **Theorem 6.1 is proved**.

As in the previous sections, modes, nodes and generalized local volume averages can be choosen as determining functionals in Theorem 6.1.

— Exercise *6.1* Let $\{e_k\}$ be a basis in $L^2(0, L)$ which consists of eigenvectors of the operator ∂_x^2 with one of the boundary conditions (6.5). Show that the set

$$\mathcal{L} = \left\{ l_j(u) = \int_0^L e_j(x) u(x)\, \mathrm{d}x\,, \quad j = 1, \ldots, N \right\}$$

is determining (in the sense of (6.14) and (6.15)) for problem (6.1)–(6.5) for N large enough.

— Exercise *6.2* Show that in the case of the Neumann boundary conditions (6.5b) it is sufficient to choose the number N in Exercise 6.1 such that

$$N > \frac{L}{\pi} \sqrt{\frac{K_j}{d_0}}\,, \qquad j = 1 \text{ or } 2\,. \tag{6.24}$$

Obtain a similar estimate for the other boundary conditions (*Hint*: see Exercises 3.2–3.4).

— Exercise *6.3* Let

$$\mathcal{L} = \left\{ l_j : \; l_j(u) = u(x_j),\; x_j = jh\,,\; h = \frac{L}{N},\; j = 1, \ldots, N \right\}. \tag{6.25}$$

Show that for every $w \in V_1$ the estimate

$$\|\partial_x w\|^2 \geq \frac{2}{(1+\eta)h^2} \|w\|^2 - C_{N,\eta} \max |l_j(w)|^2 \tag{6.26}$$

holds for any $\eta > 0$ (*Hint*: see Exercise 3.6).

— Exercise *6.4* Use estimate (6.26) instead of (6.20) in the proof of Theorem 6.1 to show that the set of functionals (6.25) is determining for problem (6.1)–(6.5), provided that $N > L \cdot \sqrt{(2K_1)/d_0}$.

— Exercise *6.5* Obtain the assertions similar to those given in Exercises 6.3 and 6.4 for the following set of functionals

$$\mathcal{L} = \left\{ l_j(w) = \frac{1}{h} \int_0^h \lambda\left(\frac{\tau}{h}\right) u(x_j + \tau)\, \mathrm{d}\tau\,, \right.$$

$$\left. x_j = jh\,,\; h = \frac{L}{N},\; j = 0, \ldots, N-1 \right\},$$

where the function $\lambda(x) \in L^\infty(\mathbb{R}^1)$ possesses the properties

$$\int_{-\infty}^{\infty} \lambda(x)\,\mathrm{d}x = 1\,, \qquad \operatorname{supp}\lambda(x) \subset [0,\ 1]\,.$$

It should be noted that in their work Hodgkin and Huxley used the following expressions (see [13]) for $k_j(u)$ and $h_j(u)$:

$$k_j(u) = \alpha_j(u) + \beta_j(u)\,, \qquad h_j(u) = \frac{\alpha_j(u)}{\alpha_j(u) + \beta_j(u)}\,,$$

where
$$\alpha_1(u) = e(-0.1\,u + 2.5)\,, \qquad \beta_1(u) = 4\exp(-u/18)\,;$$

$$\alpha_2(u) = \frac{0.07}{e(-0.05\,u)}\,, \qquad \beta_2(u) = \frac{1}{1 + \exp(-0.1\,u + 3)}\,;$$

$$\alpha_3(u) = 0.1\,e(-0.1\,u + 1)\,, \qquad \beta_3(u) = 0.125\exp(-u/80)\,.$$

Here $e(z) = z/(e^z - 1)$. They also supposed that $\delta_1 = 115$, $\delta_2 = -12$, $\gamma_1 = 120$, and $\gamma_2 = 36$. As calculations show, in this case $K_1 \le 5.2 \cdot 10^4$ and $K_2 \le 7.4 \cdot 10^4$. Therefore (see Exercise 6.4), the nodes $\{x_j = jh,\ \ h = l/N,\ \ j = 0,\ 1,\ 2,\ \ldots,\ N\}$ are determining for problem (6.1)–(6.5) when $N \ge 2.3 \cdot 10^2 \cdot L/\sqrt{d_0}$. Of course, similar estimates are valid for modes and generalized volume averages.

Thus, for the asymptotic dynamics of the system to be determined by a small number of functionals, we should require the smallness of the parameter $L/\sqrt{d_0}$. However, using the results available (see [14]) on the analyticity of solutions to problem (6.1)–(6.5) one can show (see Theorem 6.2 below) that the values of all components of the state vector $U = (u,\ v_1,\ v_2,\ v_3)$ in two sufficiently close nodes uniquely determine the asymptotic dynamics of the system considered not depending on the value of the parameter $L/\sqrt{d_0}$. Therewith some regularity conditions for the coefficients of equations (6.1) and (6.2) are necessary.

Let us consider the periodic initial-boundary value problem (6.1)–(6.5c). Assume that $d_j > 0$ for all j and the functions $k_j(u)$ and $h_j(u)$ are polynomials such that $k_j(u) > 0$ and $0 \le h_j(u) \le 1$ for $u \in [\delta_2,\ \delta_1]$. In this case (see [14]) every solution

$$U(t) = (u(x,\ t),\ v_1(x,\ t),\ v_2(x,\ t),\ v_3(x,\ t))$$

possesses the following ***Gevrey regularity*** property: there exists $t_* > 0$ such that

$$\sum_{l=-\infty}^{\infty}\left(\left|F_l(u(t))\right|^2 + \sum_{j=1}^{3}\left|F_l(v_j(t))\right|^2\right)\cdot e^{\tau|l|} \ \le\ C \qquad (6.27)$$

for some $\tau > 0$ and for all $t \ge t_*$. Here $F_l(w)$ are the Fourier coefficients of the function $w(x)$:

$$F_l(w) = \frac{1}{L} \cdot \int_0^L w(x) \cdot \exp\left\{i \frac{2\pi l}{L} x\right\} \mathrm{d}x\,, \qquad l = 0,\ \pm 1,\ \pm 2,\ \ldots$$

In particular, property (6.27) implies that every solution to problem (6.1)–(6.5c) becomes a real analytic function for all t large enough. This property enables us to prove the following assertion.

Theorem 6.2.

Let $d_j > 0$ ***for all*** j ***and let*** $k_j(u)$ ***and*** $h_j(u)$ ***be the polynomials possessing the properties***

$$k_j(u) > 0, \quad 0 \le h_j(u) \le 1 \quad \textit{for} \quad u \in [\delta_2,\ \delta_1].$$

Let x_1 ***and*** x_2 ***be two nodes such that*** $0 \le x_1 < x_2 \le L$ ***and*** $x_2 - x_1 < \sqrt{2 d_0 / K_1}$, ***where*** K_1 ***is defined by formula*** (6.12). ***Then for every two solutions***

$$U(t) = (u(x,\ t),\ v_1(x,\ t),\ v_2(x,\ t),\ v_3(x,\ t))$$

and

$$U^*(t) = (u^*(x,\ t),\ v_1^*(x,\ t),\ v_2^*(x,\ t),\ v_3^*(x,\ t))$$

to problem (6.1)–(6.5 c) ***the condition***

$$\lim_{t \to \infty} \max_{l = 1,\, 2} \left\{ \left| u(x_l,\ t) - u^*(x_l,\ t) \right| + \sum_{j=1}^{3} \left| v_j(x_l,\ t) - v_j^*(x_l,\ t) \right| \right\} = 0$$

implies their asymptotic closeness in the space $\mathbb{H}_0$***:***

$$\lim_{t \to \infty} \left\{ \| u(t) - u^*(t) \|^2 + \sum_{j=1}^{3} \left\| v_j(t) - v_j^*(t) \right\|^2 \right\} = 0\,. \tag{6.28}$$

Proof.

Let $w = u - u^*$ and let $\psi_j = v_j - v_j^*$, $j = 1,\ 2,\ 3$. We introduce the notations:

$$\Delta = \{x\colon\ x_1 \le x \le x_2\}, \qquad |\Delta| = x_2 - x_1\,, \quad \text{and} \quad \|w\|_\Delta^2 = \int_{x_1}^{x_2} |w(x)|^2 \,\mathrm{d}x\,.$$

Let

$$m(t,\ \Delta) = \max_{l = 1,\ 2} \left\{ |w(x_l,\ t)| + \sum_{j=1}^{3} |\psi_j(x_l,\ t)| \right\}.$$

As in the proof of Theorem 6.1, it is easy to find that

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \|w\|_\Delta^2 + d_0 \|\partial_x w\|_\Delta^2 + \gamma_3 \|w\|_\Delta^2 \le$$

$$\leq \sum_{l=1,\ 2} \left|w(x_l,\, t)\cdot \partial_x w\,(x_l,\, t)\right| + \sum_{j=1}^{3} a_j \|\psi_j\|_\Delta \cdot \|w\|_\Delta$$

and

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_j\|_\Delta^2 + k_j^*\|\psi_j\|_\Delta^2 \quad\leq \sum_{l=1,\ 2} \left|\psi_j(x_l,\, t)\cdot \partial_x \psi_j(x_l,\, t)\right| + (A_j + B_j)\|\psi_j\|_\Delta \cdot \|w\|_\Delta\,.$$

It follows from (6.27) that

$$\sup_{t\,>\,0}\ \max_{x\,\in\,[0,\,L]} \left\{\left|\partial_x w(x,\, t)\right| + \sum_{j=1}^{3}\left|\partial_x \psi_j(x,\, t)\right|\right\} < \infty\,.$$

Therefore, for any $0 < \varepsilon < 1$ and $\theta_j = a_j\cdot(A_j + B_j)^{-1}$, $j = 1,\ 2,\ 3$, we have

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w(t)\|_\Delta^2 + \sum_{j=1}^{3}\theta_j\|\psi_j\|_\Delta^2\right) + d_0\|\partial_x w\|_\Delta^2 + \gamma_3\|w\|_\Delta^2 + \varepsilon\sum_{j+1} k_j^*\theta_j\|\psi_j\|_\Delta^2 \quad\leq$$

$$\leq\quad K_1\cdot(1-\varepsilon)^{-1}\cdot\|w\|_\Delta^2 \ +\ m(t,\, \Delta)\ ,$$

where K_1 is defined by formula (6.12). Simple calculations give us that

$$\|w\|_\Delta^2 \ \equiv\ \int_{x_1}^{x_2} |w(x)|^2\,\mathrm{d}x \ \leq\ (1+\eta)\frac{|\Delta|^2}{2}\cdot\int_{x_1}^{x_2}\left|\partial_x w(x)\right|^2\mathrm{d}x \ +\ C_\eta\cdot|\Delta|\cdot\left|w(x_1)\right|^2$$

for any $\eta > 0$. Consequently, if $|\Delta|^2 < 2\,d_0/K_1$, then there exist $0 < \varepsilon < 1$ and $\eta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_\Delta^2 + \sum_{j=1}^{3}\theta_j\|\psi_j\|_\Delta^2\right) + \omega\cdot\left(\|w\|_\Delta^2 + \sum_{j=1}^{3}\theta_j\|\psi_j\|_\Delta^2\right) \ \leq\ C\cdot m(t,\ \Delta)\,,$$

where ω is a positive constant. As in the proof of Theorem 6.1, it follows that condition $m(t,\ \Delta) \to 0$ as $t\to\infty$ implies that

$$\lim_{t\to\infty}\|U(t) - U^*(t)\|_\Delta \equiv \lim_{t\to\infty}\left(\|w\|_\Delta^2 + \sum_{j=1}^{3}\theta_j\|\psi_j\|_\Delta^2\right) = 0\,. \qquad (6.29)$$

Let us now prove (6.28). We do it by reductio ad absurdum. Assume that there exists a sequence $t_n \to +\infty$ such that

$$\overline{\lim_{n\to\infty}}\left\|U(t_n) - U^*(t_n)\right\| \ >\ 0\,. \qquad (6.30)$$

Let $\{V_n\}$ and $\{V_n^*\}$ be sequences lying in the attractor $\mathcal{A}$ of the dynamical system generated by equations (6.1)–(6.5c) and such that

$$\left\|U(t_n) - V_n\right\| \to 0\,, \qquad \left\|U^*(t_n) - V_n^*\right\| \to 0\,. \qquad (6.31)$$

Using the compactness of the attractor we obtain that there exist a sequence $\{n_k\}$ and elements $V,\ V^* \in \mathcal{A}$ such that $V_{n_k} \to V$ and $V^*_{n_k} \to V^*$. Since

$$\|V_n - V^*_n\|_\Delta \le \|U(t_n) - U^*(t_n)\|_\Delta + \|U(t_n) - V_n\| + \|U^*(t_n) - V^*_n\|,$$

it follows from (6.29) and (6.31) that

$$\|V - V^*\|_\Delta = \lim_{k \to \infty} \|V_{n_k} - V^*_{n_k}\|_\Delta = 0 .$$

Therefore, $V(x) = V^*(x)$ for $x \in \Delta$. However, the Gevrey regularity property implies that elements of the attractor are real analytic functions. The theorem on the uniqueness of such functions gives us that $V(x) \equiv V^*(x)$ for $x \in [0, L]$. Hence, $\|V_{n_k} - V^*_{n_k}\| \to 0$ as $k \to \infty$. Therefore, equation (6.31) implies that

$$\lim_{k \to \infty} \|U(t_{n_k}) - U^*(t_{n_k})\| = 0 .$$

This contradicts assumption (6.30). **Theorem 6.2 is proved**.

It should be noted that the connection between the Gevrey regularity and the existence of two determining nodes was established in the paper [15] for the first time. The results similar to Theorem 6.2 can also be obtained for other equations (see the references in [3]). However, the requirements of the spatial unidimensionality of the problem and the Gevrey regularity of its solutions are crucial.

§ 7 Determining Functionals for Second Order in Time Equations

In a separable Hilbert space H we consider the problem

$$\ddot{u} + \gamma \dot{u} + A u = B(u, t), \qquad u|_{t=s} = u_0, \qquad \dot{u}|_{t=s} = u_1, \tag{7.1}$$

where the dot over u stands for the derivative with respect to t, A is a positive operator with discrete spectrum, $\gamma > 0$ is a constant, and $B(u, t)$ is a continuous mapping from $D(A^\theta) \times \mathbb{R}$ into H with the property

$$\|B(u_1, t) - B(u_2, t)\| \le M(\rho) \cdot \|A^\theta(u_1 - u_2)\| \tag{7.2}$$

for some $0 \le \theta < 1/2$ and for all $u_j \in D(A^{1/2})$ such that $\|A^{1/2} u_j\| \le \rho$. Assume that for any $s \in \mathbb{R}$, $u_0 \in D(A^{1/2})$, and $u_1 \in H$ problem (7.1) is uniquely solvable in the class of functions

$$C([s, +\infty); D(A^{1/2})) \cap C^1([s, +\infty), H) \tag{7.3}$$

and defines a process $(\mathcal{H};\ S(t, \tau))$ in the space $\mathcal{H} = D(A^{1/2}) \times H$ with the evolutionary operator given by the formula

$$S(t, s)(u_0;\ u_1) = (u(t);\ \dot{u}(t)), \tag{7.4}$$

where $u(t)$ is a solution to problem (7.1) in the class (7.3). Assume that the process $(\mathcal{H};\ S(t,\ \tau))$ is pointwise dissipative, i.e. there exists $R > 0$ such that

$$\|S(t,\ s)y_0\|_{\mathcal{H}} \ \le\ R\ , \qquad t\ \ge\ s + t_0(y_0) \tag{7.5}$$

for all initial data $y_0 = (u_0;\ u_1) \in \mathcal{H}$. The nonlinear wave equation (see the book by A. V. Babin and M. I. Vishik [8])

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \gamma \dfrac{\partial u}{\partial t} - \Delta u = f(u), \quad x \in \Omega,\ t > 0\ , \\ u\big|_{\partial\Omega} = 0\,,\ u\big|_{t=0} = u_0(x),\ \dfrac{\partial u}{\partial t}\bigg|_{t=0} = u_1(x)\ , \end{cases}$$

is an example of problem (7.1) which possesses all the properties listed above. Here Ω is a bounded domain in $\mathbb{R}^d$ and the function $f(u) \in C^1(\mathbb{R})$ possesses the properties:

$$-(\lambda_1 - \varepsilon)u^2 - C_1 \ \le\ \int_0^u f(v)\,\mathrm{d}v \ \le\ C_2\, u\, f(u) + C_3 + \frac{1}{2}(\lambda_1 - \varepsilon)u^2\ ,$$

$$|f'(u)| \ \le\ C_4(1 + |u|^{\beta}),$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary conditions on $\partial\Omega$, $C_j > 0$ and $\varepsilon > 0$ are constants, $\beta \le 2(d-2)^{-1}$ for $d \ge 3$ and β is arbitrary for $d = 2$.

Theorem 7.1.

Let $\mathcal{L} = \{l_j:\ \ j = 1,\ \ldots,\ N\}$ ***be a set of continuous linear functionals on*** $D(A^{1/2})$. ***Assume that***

$$\varepsilon \ \equiv\ \varepsilon_{\mathcal{L}}(D(A^{1/2});\ H) \ <\ \left[\frac{\gamma}{M(R)(4 + 4\lambda_1^{-1}\gamma^2)^{3/2}}\right]^{\frac{1}{1-2\theta}}, \tag{7.6}$$

where R ***is the radius of dissipativity (see*** (7.5)***),*** $M(\rho)$ ***and*** $\theta \in [0,\ 1/2)$ ***are the constants from*** (7.2)***, and*** λ_1 ***is the first eigenvalue of the operator*** A. ***Then*** $\mathcal{L}$ ***is an asymptotically determining set for problem*** (7.1) ***in the sense that for a pair of solutions*** $u_1(t)$ ***and*** $u_2(t)$ ***from the class*** (7.3) ***the condition***

$$\lim_{t\to\infty} \int_t^{t+1} \big|l_j(u_1(\tau) - u_2(\tau))\big|\,\mathrm{d}\tau = 0 \quad \textit{for} \quad j = 1,\ \ldots,\ N \tag{7.7}$$

implies that

$$\lim_{t\to\infty} \left\{ \big\|A^{1/2}(u_1(t) - u_2(t))\big\|^2 + \big\|\dot{u}_1(t) - \dot{u}_2(t)\big\|^2 \right\} = 0\,. \tag{7.8}$$

Proof.

We rewrite problem (7.1) in the form

$$\frac{dy}{dt} + \mathcal{A}y = \mathcal{B}(y,\, t)\,, \qquad y\big|_{t=s} = y_0\,, \tag{7.9}$$

where

$$y = (u;\ \dot{u})\,, \quad \mathcal{A}y = (-\dot{u}\,;\ \gamma\dot{u} + Au)\,, \quad \mathcal{B}(y,\, t) = (0;\ B(u,\, t))\,.$$

Lemma 7.1.

There exists an exponential operator $\exp\{-t\mathcal{A}\}$ *in the space* $\mathcal{H} = D(A^{1/2}) \times H$ *and*

$$\|\exp\{-t\mathcal{A}\}y\|_{\mathcal{H}} \;\le\; 2\sqrt{1 + \frac{\gamma^2}{\lambda_1}}\cdot \exp\left\{-\frac{\lambda_1\gamma t}{4\lambda_1 + 2\gamma^2}\right\}\|y\|_{\mathcal{H}}\,, \tag{7.10}$$

where λ_1 *is the first eigenvalue of the operator* A.

Proof.

Let $y_0 = (u_0;\ u_1)$. Then it is evident that $y(t) = e^{-t\mathcal{A}}y_0 = (u(t);\ \dot{u}(t))$, where $u(t)$ is a solution to the problem

$$\ddot{u} + \gamma\dot{u} + Au = 0\,, \qquad u\big|_{t=0} = u_0\,, \qquad \dot{u}\big|_{t=0} = u_1\,, \tag{7.11}$$

(see Section 3.7 for the solvability of this problem and the properties of solutions). Let us consider the functional

$$V(u) = \frac{1}{2}\Big(\|u\|^2 + \|A^{1/2}u\|^2\Big) + \nu\Big(u,\ \dot{u} + \frac{\gamma}{2}u\Big), \qquad 0 < \nu < \gamma\,,$$

on the space $\mathcal{H} = D(A^{1/2}) \times H$. It is clear that

$$\frac{1}{2}(1 - \frac{\nu}{\gamma})\|\dot{u}\|^2 + \frac{1}{2}\|A^{1/2}u\|^2 \;\le\; V(u) \;\le$$

$$\le\; \Big(\frac{1}{2} + \frac{\nu}{2\gamma}\Big)\|\dot{u}\|^2 + (\,\frac{1}{2} + \lambda_1^{-1}\nu\gamma)\,\|A^{1/2}u\|^2\;. \tag{7.12}$$

Moreover, for a solution $u(t)$ to problem (7.11) from the class (7.3) with $s = 0$ we have that

$$\frac{d}{dt}V(u(t)) \;=\; -(\gamma - \nu)\|\dot{u}\|^2 - \nu\|A^{1/2}u\|^2\,. \tag{7.13}$$

Therefore, it follows from (7.12) and (7.13) that

$$\frac{d}{dt}V(u(t)) + \beta V(u(t)) \le$$

$$\le\; -\Big(\gamma - \nu - \Big(\frac{1}{2} + \frac{\nu}{2\gamma}\Big)\,\beta\Big)\|\dot{u}\|^2 - \Big(\nu - \Big(\frac{1}{2} + \lambda_1^{-1}\nu\gamma\Big)\,\beta\Big)\|A^{1/2}u\|^2\;.$$

Hence, for $\nu = (1/2)\gamma$ and $\beta = (2 + \lambda_1^{-1}\gamma^2)^{-1}\cdot\gamma$ we obtain that

$$\frac{d}{dt}V(u(t)) + \beta V(u(t)) \le 0\,,$$

$$\frac{1}{4}\Big(\|\dot{u}\|^2 + \|A^{1/2}u\|^2\Big) \le V(u) \le \Big(\frac{3}{4} + \lambda_1^{-1}\gamma^2\Big)\Big(\|\dot{u}\|^2 + \|A^{1/2}u\|^2\Big).$$

This implies estimate (7.10). Lemma 7.1 is proved.

It follows from (7.9) that

$$y(t) = e^{-(t-s)\mathcal{A}}y(s) + \int_s^t e^{-(t-\tau)\mathcal{A}}\,\mathcal{B}(y(\tau),\,\tau)\,d\tau\,.$$

Therefore, with the help of Lemma 7.1 for the difference of two solutions $y_j(t) = (u_j(t);\ \dot{u}_j(t))$, $j = 1,\ 2$, we obtain the estimate

$$\|y(t)\|_{\mathcal{H}} \le De^{-\beta(t-s)}\|y(s)\|_{\mathcal{H}} + D\int_s^t e^{-\beta(t-\tau)}\big\|B(u_1(\tau)) - B(u_2(\tau))\big\|\,d\tau\,, \quad (7.14)$$

where $y(t) = y_1(t) - y_2(t)$ and the constants D and β have the form

$$D = 2\sqrt{1 + \lambda_1^{-1}\gamma^2}\,, \qquad \beta = \frac{\gamma\lambda_1}{4\lambda_1 + 2\gamma^2}\,.$$

By virtue of the dissipativity (7.5) we can assume that $\|y_j(t)\|_{\mathcal{H}} \le R$ for all $t \ge s \ge s_0$. Therefore, equations (7.14) and (7.2) imply that

$$\|y(t)\|_{\mathcal{H}} \le De^{-\beta(t-s)}\|y(s)\|_{\mathcal{H}} + DM(R)\int_s^t e^{-\beta(t-\tau)}\big\|A^{\theta}(u_1(\tau) - u_2(\tau))\big\|\,d\tau\,.$$

The interpolation inequality (see Exercise 2.1.12)

$$\|A^{\theta}u\| \le \|u\|^{1-2\theta}\|A^{1/2}u\|^{2\theta}\,, \qquad 0 \le \theta \le \frac{1}{2}\,,$$

Theorem 2.1, and the result of Exercise 2.12 give us that

$$\|A^{\theta}u\| \le C_{\mathcal{L}}\max_j\big|l_j(u)\big| + \varepsilon_{\mathcal{L}}^{1-2\theta}\|A^{1/2}u\|\,,$$

where $\varepsilon_{\mathcal{L}} = \varepsilon_{\mathcal{L}}(D(A^{1/2}),\ H)$. Therefore,

$$\|y(t)\|_{\mathcal{H}} \le De^{-\beta(t-s)}\|y(s)\|_{\mathcal{H}} + DM(R)\cdot\varepsilon_{\mathcal{L}}^{1-2\theta}\int_s^t e^{-\beta(t-\tau)}\|y(\tau)\|_{\mathcal{H}}\,d\tau +$$

$$+\,C(\mathcal{L},\ D,\ R)\cdot\int_s^t e^{-\beta(t-\tau)}N_{\mathcal{L}}(u(\tau))\,d\tau\,,$$

where $N_{\mathcal{L}}(u) = \max\{|l_j(u)|:\ j = 1,\ 2,\ \dots,\ N\}$. If we introduce a new unknown function $\psi(t) = e^{\beta t}\|y(t)\|_{\mathcal{H}}$ in this inequality, then we obtain the equation

$$\psi(t) \le D\psi(s) + \alpha\int_s^t \psi(\tau)\,d\tau + C\int_s^t e^{\beta\tau} N_{\mathscr{L}}(u(\tau))\,d\tau\,,$$

where $\alpha = DM(R)\varepsilon_{\mathscr{L}}^{1-2\theta}$. We apply Gronwall's lemma to obtain

$$\psi(t) \le D\psi(s)e^{\alpha(t-s)} + Ce^{\alpha t}\int_s^t e^{-\alpha\tau}\left\{\int_s^\tau e^{\beta\xi} N_{\mathscr{L}}(u(\xi))\,d\xi\right\} d\tau\,.$$

After integration by parts we get

$$\|y(t)\|_{\mathscr{H}} \le D\|y(s)\|_{\mathscr{H}} \exp\{-(\beta-\alpha)(t-s)\} + C\int_s^t e^{-(\beta-\alpha)(t-\tau)} N_{\mathscr{L}}(u(\tau))\,d\tau\,.$$

If equation (7.6) holds, then $\omega = \beta - \alpha > 0$. Therewith it is evident that for $0 < a < t-s$ we have the estimate

$$\|y(t)\|_{\mathscr{H}} \le D\|y(s)\|_{\mathscr{H}} e^{-\omega(t-s)} + C\int_{t-a}^t e^{-\omega(t-\tau)} N_{\mathscr{L}}(u(\tau))\,d\tau + $$
$$+ C\int_s^{t-a} e^{-\omega(t-\tau)} N_{\mathscr{L}}(u(\tau))\,d\tau\,.$$

Therefore, using the dissipativity property we obtain that

$$\|y(t)\|_{\mathscr{H}} \le DR\cdot e^{-\omega(t-s)} + C\int_{t-a}^t N_{\mathscr{L}}(u(\tau))\,d\tau + C(R,\,\mathscr{L})e^{-\omega a}$$

for $t \ge s$ and $0 < a < t-s$. If we fix a and tend the parameter t to infinity, then with the help of (7.7) we find that

$$\overline{\lim_{t\to\infty}}\,\|y(t)\|_{\mathscr{H}} \le C(R,\,\mathscr{L})e^{-\omega a}$$

for any $a > 0$. This implies (7.8). **Theorem 7.1 is proved**.

Unfortunately, because of the fact that condition (7.2) is assumed to hold only for $0 \le \theta < 1/2$, Theorem 7.1 cannot be applied to the problem on nonlinear plate oscillations considered in Chapter 4. However, the arguments in the proof of Theorem 7.1 can be slightly modified and the theorem can still be proved for this case using the properties of solutions to linear nonautonomous problems (see Section 4.2). However, instead of a modification we suggest another approach (see also [3]) which helps us to prove the assertions on the existence of sets of determining functionals for second order in time equations. As an example, let us consider a problem of plate oscillations .

Thus, in a separable Hilbert space H we consider the equation

$$\begin{cases} \ddot{u} + \gamma\dot{u} + A^2 u + M\left(\|A^{1/2}u\|^2\right)Au + Lu = p(t)\ , & (7.15) \\ u\big|_{t=0} = u_0\,, \quad \dot{u}\big|_{t=0} = u_1\ . & (7.16) \end{cases}$$

We assume that A is an operator with discrete spectrum and the function $M(z)$ lies in $C^1(\mathbb{R}_+)$ and possesses the properties:

a) $$\mathscr{M}(z) \equiv \int_0^z M(\xi)\,d\xi \ \geq\ -az - b\,, \qquad (7.17)$$

where $0 \leq a < \lambda_1$, $b \in \mathbb{R}$, and λ_1 is the first eigenvalue of the operator A;

b) there exist numbers $a_j > 0$ such that

$$zM(z) - a_1\,\mathscr{M}(z) \ \geq\ a_2\, z^{1+\alpha} - a_3\,, \qquad z \geq 0 \qquad (7.18)$$

with some constant $\alpha > 0$.

We also require the existence of $0 \leq \theta < 1$ and $C > 0$ such that

$$\|Lu\| \leq C\|A^\theta u\|\,, \qquad u \in D(A^\theta)\,. \qquad (7.19)$$

These assumptions enable us to state (see Sections 4.3 and 4.5) that if

$$u_0 \in D(A)\,, \qquad u_1 \in H\,, \qquad p(t) \in L^\infty(\mathbb{R}_+, H)\,, \qquad \gamma > 0\,, \qquad (7.20)$$

then problem (7.15) and (7.16) is uniquely solvable in the class of functions

$$\mathscr{W} = C(\mathbb{R}_+;\ D(A)) \cap C^1(\mathbb{R}_+;\ H)\,. \qquad (7.21)$$

Therewith there exists $R > 0$ such that

$$\|Au(t)\|^2 + \|\dot{u}(t)\|^2 \ \leq\ R^2\,, \qquad t \ \geq\ t_0(u_0,\ u_1) \qquad (7.22)$$

for any solution $u(t) \in \mathscr{W}$ to problem (7.15) and (7.16).

Theorem 7.2.

Assume that conditions (7.17)–(7.20) ***hold. Let*** $\mathscr{L} = \{l_j:\ j = 1,\ 2,\ \ldots,\ N\}$ ***be a set of continuous linear functionals on*** $D(A)$. ***Then there exists*** $\varepsilon_0 > 0$ ***depending both on*** R ***and the parameter of equation*** (7.15) ***such that the condition*** $\varepsilon \equiv \varepsilon_{\mathscr{L}}(D(A);\ H) < \varepsilon_0$ ***implies that*** $\mathscr{L}$ ***is an asymptotically determining set of functionals with respect to*** $D(A) \times H$ ***for problem*** (7.15) ***and*** (7.16) ***in the class of solutions*** $\mathscr{W}$, ***i.e. for two solutions*** $u_1(t)$ ***and*** $u_2(t)$ ***from*** $\mathscr{W}$ ***the condition***

$$\lim_{t\to\infty} \int_t^{t+1} \left|l_j(u_1(\tau) - u_2(\tau))\right|^2 d\tau = 0\,, \qquad l_j \in \mathscr{L} \qquad (7.23)$$

implies that

$$\lim_{t \to \infty} \left\{ \|\dot{u}_1(t) - \dot{u}_2(t)\|^2 + \|A(u_1(t) - u_2(t))\|^2 \right\} = 0 . \tag{7.24}$$

Proof.

Let $u_1(t)$ and $u_2(t)$ be solutions to problem (7.15) and (7.16) lying in $\mathscr{W}$. Due to equation (7.22) we can assume that these solutions possess the property

$$\|Au_j(t)\|^2 + \|\dot{u}_j(t)\|^2 \le R^2 , \qquad t \ge 0 , \quad j = 1,\ 2 . \tag{7.25}$$

Let us consider the function $u(t) = u_1(t) - u_2(t)$ as a solution to equation

$$\ddot{u} + \gamma\dot{u} + A^2 u + M\left(\|A^{1/2} u_1(t)\|^2\right) Au = F(u_1(t),\ u_2(t)) , \tag{7.26}$$

where

$$F(u_1,\ u_2) = \left[M\left(\|A^{1/2} u_2\|^2\right) - M\left(\|A^{1/2} u_1\|^2\right)\right] Au_2 - L(u_1 - u_2) .$$

It follows from (7.19) and (7.25) that

$$\|F(u_1(t),\ u_2(t))\| \le C_R\left(\|A^{1/2} u\| + \|A^{\theta} u\|\right). \tag{7.27}$$

Let us consider the functional

$$V(u,\ \dot{u};\ t) = \frac{1}{2} E(u,\ \dot{u};\ t) + \nu\left\{(u,\ \dot{u}) + \frac{\gamma}{2}\|u\|^2\right\} \tag{7.28}$$

on the space $\mathscr{H} = D(A) \times H$, where

$$E(u,\ \dot{u};\ t) = \|\dot{u}\|^2 + \|Au\|^2 + M\left(\|A^{1/2} u_1(t)\|^2\right) \|A^{1/2} u\|^2 + \mu\|u\|^2$$

and the positive parameters μ and ν will be chosen below. It is clear that for $(u;\ \dot{u}) \in \mathscr{H}$ we have

$$E(u,\ \dot{u};\ t) \ge \|\dot{u}\|^2 + \|Au\|^2 + m_R\|A^{1/2} u\|^2 + \mu\|u\|^2 ,$$

where $m_R = \min\{M(z):\ 0 \le z \le \lambda_1^{-1} R^2\}$. Moreover,

$$-\frac{1}{2\gamma}\|\dot{u}\|^2 \le (u,\ \dot{u}) + \frac{\gamma}{2}\|\dot{u}\|^2 \le \frac{1}{2\gamma}\|\dot{u}\|^2 + \gamma\|u\|^2 .$$

Therefore, the value μ can be chosen such that

$$\alpha_1\left(\|Au\|^2 + \|\dot{u}\|^2\right) \le V(u,\ t) \le \alpha_2\left(\|Au\|^2 + \|\dot{u}\|^2\right) \tag{7.29}$$

for all $0 < \nu < \gamma$, where α_1 and α_2 are positive numbers depending on R . Let us now estimate the value $(d/dt) V(u(t),\ \dot{u}(t);\ t)$. Due to (7.26) we have that

$$\frac{1}{2}\frac{d}{dt} E(u(t),\ \dot{u}(t);\ t) = -\gamma\|\dot{u}(t)\|^2 + \mu(u(t),\ \dot{u}(t)) +$$

$$+ M'\left(\|A^{1/2} u_1(t)\|^2\right)(Au_1(t),\ \dot{u}_1(t))\|A^{1/2} u(t)\|^2 + (F(u_1(t),\ u_2(t)),\ \dot{u}(t)) .$$

With the help of (7.25) and (7.27) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(u,\, \dot{u};\; t) \;\le\; -\frac{\gamma}{2}\|\dot{u}(t)\|^2 + C_R\left(\|A^{1/2}u\|^2 + \|A^{\theta}u\|^2\right).$$

Using (7.26) and (7.27) it is also easy to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{(u,\, \dot{u}) + \frac{\gamma}{2}(u,\, u)\right\} \;=\; \|\dot{u}\|^2 + (u,\; \ddot{u} + \gamma\dot{u}) \;\le$$

$$\le\; \|\dot{u}\|^2 - \|Au\|^2 + M_R\|A^{1/2}u\|^2 + \; C_R\left(\|A^{1/2}u\|^2 + \|A^{\theta}u\|^2\right)\|u\| \;,$$

where

$$M_R = \max\left\{|M(z)|:\;\; 0 \le z \le \lambda_1^{-1}R^2\right\}.$$

We choose $\nu = \gamma/4$ and use the estimate of the form

$$\|A^{\beta}u\| \;\le\; \|Au\|^{\beta}\cdot\|u\|^{1-\beta} \;\le\; \varepsilon\|Au\| + C_{\varepsilon}\|u\|\,, \qquad 0 < \beta < 1\,, \quad \varepsilon > 0\,,$$

to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u,\, \dot{u};\; t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}E(u,\, \dot{u};\; t) + \nu\left(u,\; \dot{u} + \frac{\gamma}{2}u\right)\right\} \;\le$$

$$\le\; -\frac{\gamma}{8}\left(\|Au\|^2 + \|\dot{u}\|^2\right) + C_R\|u(t)\|^2 \;.$$

Therefore, using the estimate

$$\|u(t)\|^2 \;\le\; C_{\mathscr{L}}\max_j\left|l_j(u)\right|^2 + 2\,\varepsilon_{\mathscr{L}}(D(A),\; H)\|Au\|^2$$

and equation (7.29) we obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u(t),\, \dot{u}(t);\; t) + \omega V(u(t),\, \dot{u}(t);\; t) \;\le\; C\,\max_j\left|l_j(u(t))\right|^2 \;,$$

provided $\varepsilon_{\mathscr{L}}(D(A),\, H) < \varepsilon_0 = \dfrac{\gamma}{16}\,C_R^{-1}$. Here ω is a positive constant. As above, this easily implies (7.24), provided (7.23) holds. **Theorem 7.2 is proved**.

— Exercise *7.1* Show that the method used in the proof of Theorem 7.2 also enables us to obtain the assertion of Theorem 7.1 for problem (7.1).

— Exercise *7.2* Using the results of Section 4.2 related to the linear variant of equation (7.15), prove that the method of the proof of Theorem 7.1 can also be applied in the proof of Theorem 7.2.

Thus, the methods presented in the proofs of Theorems 7.1 and 7.2 are close to each other. The same methods with slight modifications can also be used in the study of problems like (7.1) with additional retarded terms (see [3]).

— Exercise *7.3* Using the estimates for the difference of two solutions to equation (7.15) proved in Lemmata 4.6.1 and 4.6.2, find an analogue of Theorems 1.3 and 4.4 for the problem (7.15) and (7.16).

§ 8 On Boundary Determining Functionals

The fact (see Sections 5–7 as well as paper [3]) that in some cases determining functionals can be defined on some auxiliary space admits in our opinion an interesting generalization which leads to the concept of boundary determining functionals. We now clarify this by giving the following simple example.

In a smooth bounded domain $\Omega \subset \mathbb{R}^d$ we consider a parabolic equation with the nonlinear boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \Delta u - f(u), \quad x \in \Omega, \ t > 0 , \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} + h(u)|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) . \end{cases} \tag{8.1}$$

Assume that ν is a positive parameter, $f(z)$ and $h(z)$ are continuously differentiable functions on $\mathbb{R}^1$ such that

$$f'(z) \ge -\alpha, \qquad |h'(z)| \le \beta, \tag{8.2}$$

where $\alpha \ge 0$ and $\beta > 0$ are constants. Let

$$\mathscr{W} = C^{2,1}(\Omega \times \mathbb{R}_+) \cap C^{1,0}(\overline{\Omega \times \mathbb{R}_+}). \tag{8.3}$$

Here $C^{2,1}(\Omega \times \mathbb{R}_+)$ is a set of functions $u(x, t)$ on $\Omega \times \mathbb{R}_+$ that are twice continuously differentiable with respect to x and continuously differentiable with respect to t. The notation $C^{1,0}(\overline{\Omega \times \mathbb{R}_+})$ has a similar meaning, the bar denotes the closure of a set.

Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions to problem (8.1) lying in the class $\mathscr{W}$ (we do not discuss the existence of such solutions here and refer the reader to the book [7]). We consider the difference $u(t) = u_1(t) - u_2(t)$. Then (8.1) evidently implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u(t)\|^2_{L^2(\Omega)} + (f(u_1(t)) - f(u_2(t)), \ u(t))_{L^2(\Omega)} =$$

$$= -\nu \int_{\partial\Omega} u(t)(h(u_1(t)) - h(u_2(t)))\, \mathrm{d}\sigma \ .$$

Using (8.2) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u(t)\|^2_{L^2(\Omega)} - \alpha\|u(t)\|^2_{L^2(\Omega)} \;\le\; \nu\beta\cdot\|u(t)\|^2_{L^2(\partial\Omega)} \,. \quad (8.4)$$

One can show that there exist constants c_1 and c_2 depending on the domain Ω only and such that

$$\|u\|^2_{L^2(\Omega)} \;\le\; c_1\Big(\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big), \quad (8.5)$$

$$\|u\|^2_{H^{1/2}(\partial\Omega)} \;\le\; c_2\Big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big). \quad (8.6)$$

Here $H^s(\partial\Omega)$ is the Sobolev space of the order s on the boundary of the domain Ω. Equations (8.4)–(8.6) enable us to obtain the following assertion.

Theorem 8.1.

Let $\mathscr{L} = \{l_j\colon\ j = 1, \ldots, N\}$ ***be a set of continuous linear functionals on the space*** $H^{1/2}(\partial\Omega)$. ***Assume that*** $\alpha c_1 < \nu$ ***and***

$$\varepsilon_{\mathscr{L}} \;\equiv\; \varepsilon_{\mathscr{L}}(H^{1/2}(\partial\Omega),\, L^2(\partial\Omega)) \;<\; \left[\frac{\nu - \alpha c_1}{\nu(1+c_1)\,c_2(1+\beta)}\right]^{1/2} \;\equiv\; \varepsilon_0 \,, \quad (8.7)$$

where the constants ν, α, β, c_1, ***and*** c_2 ***are defined in equations*** (8.1), (8.2), (8.5), ***and*** (8.6). ***Then*** $\mathscr{L}$ ***is an asymptotically determining set with respect to*** $L^2(\Omega)$ ***for problem*** (8.1) ***in the class of classical solutions*** $\mathscr{W}$.

Proof.

Let $u(t) = u_1(t) - u_2(t)$, where $u_j(t) \in \mathscr{W}$ are solutions to problem (8.1). Theorem 2.1 implies that

$$\|u\|^2_{L^2(\partial\Omega)} \;\le\; C_{\mathscr{L},\,\delta}\,\max_j |l_j(u)|^2 + (1+\delta)\,\varepsilon^2_{\mathscr{L}}\|u\|^2_{H^{1/2}(\Omega)} \quad (8.8)$$

for any $\delta > 0$. Equations (8.5) and (8.6) imply that

$$\|u\|^2_{L^2(\partial\Omega)} \;\le\; C_{\mathscr{L},\,\delta}\,\max_j |l_j(u)|^2 + (1+c_1)c_2(1+\delta)\,\varepsilon^2_{\mathscr{L}}\Big(\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big).$$

Therefore, equation (8.4) gives us that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\Omega)} + \nu\Big[\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big] - \alpha\|u(t)\|^2_{L^2(\Omega)} \;\le$$

$$\le\; \nu(1+\beta)\|u(t)\|_{L^2(\partial\Omega)} \;\le\; \nu(1+c_1)\,c_2(1+\delta)(1+\beta)\,\varepsilon^2_{\mathscr{L}}\Big(\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big) +$$

$$+\, C\max_j |l_j(u)|^2 \,.$$

Using estimate (8.5) once again we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\nu}{c_1}\left\{1 - (1+c_1)c_2(1+\delta)(1+\beta)\varepsilon^2_{\mathscr{L}} - \alpha\frac{c_1}{\nu}\right\}\|u\|^2 \;\le\; C\max_j |l_j(u)|^2, \quad (8.9)$$

provided that

$$1-(1+c_1)c_2(1+\delta)(1+\beta)\varepsilon_{\mathscr{L}}^2 - \alpha\frac{c_1}{\nu} > 0\,. \tag{8.10}$$

It is evident that (8.10) with some $\delta > 0$ follows from (8.7). Therefore, inequality (8.9) enables us to **complete the proof** of the theorem.

Thus, the analogue of Theorem 3.1 for smooth surfaces enables us to state that problem (6.1) has finite determining sets of boundary local surface averages.

An assertion similar to Theorem 8.1 can also be obtained (see [3]) for a nonlinear wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} = \Delta u - f(u)\,, \qquad x \in \Omega \subset \mathbb{R}^d\,, \qquad t > 0\,,$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = -\alpha\left.\frac{\partial u}{\partial t}\right|_{\Gamma} - \varphi\left(u|_{\Gamma}\right), \quad u|_{\partial\Omega\setminus\Gamma} = 0\,, \quad u|_{t=0} = u_0(x)\,, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x)\,.$$

Here Γ is a smooth open subset on the boundary of Ω, $f(u)$ and $\varphi(u)$ are bounded continuously differentiable functions, and α and γ are positive parameters.

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