If $G = MX$ is a subgraph of another graph Y, we call X a *minor* of Y and write $X \preccurlyeq Y$. Note that every subgraph of a graph is also its minor; in particular, every graph is its own minor. By Proposition 1.7.1, any minor of a graph can be obtained from it by first deleting some vertices and edges, and then contracting some further edges. Conversely, any graph obtained from another by repeated deletions and contractions (in any order) is its minor: this is clear for one deletion or contraction, and follows for several from the transitivity of the minor relation (Proposition 1.7.3).

If we replace the edges of X with independent paths between their ends (so that none of these paths has an inner vertex on another path or in X), we call the graph G obtained a *subdivision* of X and write $G = TX$.[8] If $G = TX$ is the subgraph of another graph Y, then X is a *topological minor* of Y (Fig. 1.7.3).

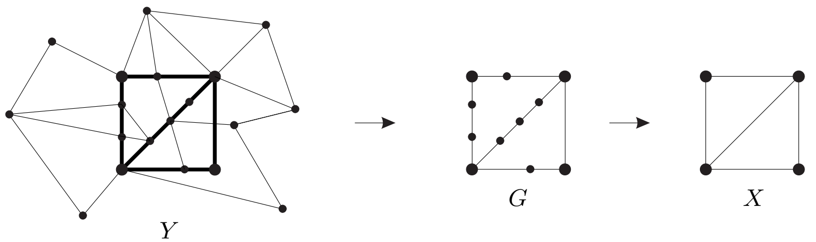

Fig. 1.7.3. $Y \supseteq G = TX$, so X is a topological minor of Y

If $G = TX$, we view $V(X)$ as a subset of $V(G)$ and call these vertices the *branch vertices* of G; the other vertices of G are its *subdividing vertices*. Thus, all subdividing vertices have degree 2, while the branch vertices retain their degree from X.

Proposition 1.7.2.

(i) *Every TX is also an MX (Fig. 1.7.4); thus, every topological minor of a graph is also its (ordinary) minor.*

(ii) *If $\Delta(X) \leqslant 3$, then every MX contains a TX; thus, every minor with maximum degree at most 3 of a graph is also its topological minor.* □

Proposition 1.7.3. *The minor relation $\preccurlyeq$ and the topological-minor relation are partial orderings on the class of finite graphs, i.e. they are reflexive, antisymmetric and transitive.* □

[8] So again TX denotes an entire class of graphs: all those which, viewed as a topological space in the obvious way, are homeomorphic to X. The T in TX stands for 'topological'.

The 'all' is true only if $\delta(X) \geqslant 3$. Graphs obtained from X by suppressing vertices of degree 2 (see p. 29) are not considered as a TX.

Let $u_1, \dots, u_k$ be those k vertices in U that are not an end of a path in $\mathcal{P}$. For each $i = 1, \dots, k$, let L_i be the U-path in K (i.e., the subdivided edge of the K^{3k}) from u_i to the end of P_i in U, and let v_i be the first vertex of L_i on any path $P \in \mathcal{P}$. By definition of $\mathcal{P}$, P has no more edges outside $E(K)$ than $Pv_iL_iu_i$ does, so $v_iP = v_iL_i$ and hence $P = P_i$ (Fig. 3.5.1). Similarly, if M_i denotes the U-path in K from u_i to the end of Q_i in U, and w_i denotes the first vertex of M_i on any path in $\mathcal{P}$, then this path is Q_i. Then the paths $s_iP_iv_iL_iu_iM_iw_iQ_it_i$ are disjoint for different i and show that G is k-linked. □

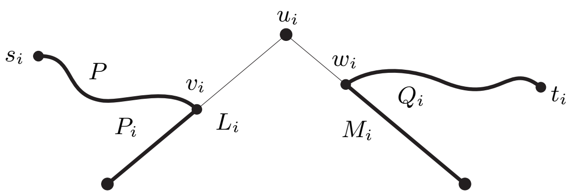

Fig. 3.5.1. Constructing an s_i–t_i path via u_i

The proof of Theorem 3.5.2 yields only an exponential upper bound for the function $f(k)$. As $2\varepsilon(G) \geqslant \delta(G) \geqslant \kappa(G)$, the following result implies the linear bound of $f(k) = 16k$:

Theorem 3.5.3. (Thomas & Wollan 2005)
Let G be a graph and $k \in \mathbb{N}$. If G is $2k$-connected and $\varepsilon(G) \geqslant 8k$, then G is k-linked.

We begin our proof of Theorem 3.5.3 with a lemma.

Lemma 3.5.4. *If $\delta(G) \geqslant 8k$ and $|G| \leqslant 16k$, then G has a k-linked subgraph.*

Proof. If G itself is k-linked there is nothing to show, so suppose not. Then we can find a set X of $2k$ vertices $s_1, \dots, s_k, t_1, \dots, t_k$ that cannot be linked in G by disjoint paths $P_i = s_i \dots t_i$. Let $\mathcal{P}$ be a set of as many such paths as possible, but all of length at most 7. If there are several such sets $\mathcal{P}$, we choose one with $|\bigcup \mathcal{P}|$ minimum. We may assume that $\mathcal{P}$ contains no path from s_1 to t_1. Let J be the subgraph of G induced by X and all the vertices on the paths in $\mathcal{P}$, and let $H := G - J$.

… that have no inner vertices in X.

Note that each vertex $v \in H$ has at most three neighbours on any given $P_i \in \mathcal{P}$: if it had four, then replacing the segment uP_iw between its first and its last neighbour on P_i by the path uvw would reduce $|\bigcup \mathcal{P}|$ and thus contradict our choice of $\mathcal{P}$. Moreover, v is not adjacent to both s_i and t_i whenever $s_i, t_i \notin \bigcup \mathcal{P}$, by the maximality of $\mathcal{P}$. Thus if $|\mathcal{P}| =: h$, then v has at most $3h + (2k - 2h)/2 \leqslant 3k$ neighbours in J. As $\delta(G) \geqslant 8k$

contrary to (1). Hence the neighbour of v_i on P is its only neighbour in $C_{i,j}$, and similarly for v_j. Thus if $C_{i,j} \neq P$, then P has an inner vertex with three identically coloured neighbours in H; let u be the first such vertex on P (Fig. 5.2.1). Since at most $\Delta - 2$ colours are used on the neighbours of u, we may recolour u. But this makes $P\mathring{u}$ into a component of $H_{i,j}$, contradicting (2).

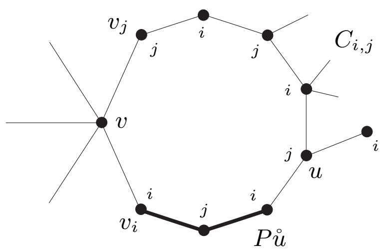

Fig. 5.2.1. The proof of (3) in Brooks's theorem

For distinct i, j, k, the paths $C_{i,j}$ and $C_{i,k}$ meet only in v_i. (4)

For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then u has two neighbours coloured j and two coloured k, so we may recolour u. In the new colouring, v_i and v_j lie in different components of $H_{i,j}$, contrary to (2).

The proof of the theorem now follows easily. If the neighbours of v are pairwise adjacent, then each has Δ neighbours in $N(v) \cup \{v\}$ already, so $G = G[N(v) \cup \{v\}] = K^{\Delta+1}$. As G is complete, there is nothing to show. We may thus assume that $v_1 v_2 \notin G$, where $v_1, \dots, v_\Delta$ derive their names from some fixed Δ-colouring c of H. Let $u \neq v_2$ be the neighbour of v_1 on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colours 1 and 3 in $C_{1,3}$, we obtain a new colouring c' of H; let v'_i, $H'_{i,j}$, $C'_{i,j}$ etc. be defined with respect to c' in the obvious way. As a neighbour of $v_1 = v'_3$, our vertex u now lies in $C'_{2,3}$, since $c'(u) = c(u) = 2$. By (4) for c, however, the path $\mathring{v}_1 C_{1,2}$ retained its original colouring, so $u \in \mathring{v}_1 C_{1,2} \subseteq C'_{1,2}$. Hence $u \in C'_{2,3} \cap C'_{1,2}$, contradicting (4) for c'. □

As we have seen, a graph G of large chromatic number must have large maximum degree: trivially at least $\chi(G) - 1$, and less trivially at least $\chi(G)$ (in most cases). What more can we say about the structure of graphs with large chromatic number?

One obvious possible cause for $\chi(G) \geqslant k$ is the presence of a K^k subgraph. This is a local property of G, compatible with arbitrary values of global invariants such as ε and κ. Hence, the assumption of $\chi(G) \geqslant k$ does not tell us anything about those invariants for G itself. It does, however, imply the existence of a subgraph where those invariants are large: by Corollary 5.2.3, G has a subgraph H with $\delta(H) \geqslant k - 1$, and hence by Theorem 1.4.3 a subgraph H' with $\kappa(H') \geqslant \lfloor \frac{1}{4}(k-1) \rfloor$. $\kappa(H') \geqslant \lceil \frac{1}{4} k \rceil$

9. Find a lower bound for the colouring number in terms of average degree.

10.$^-$ A k-chromatic graph is called *critically k-chromatic*, or just *critical*, if $\chi(G-v) < k$ for every $v \in V(G)$. Show that every k-chromatic graph has a critical k-chromatic induced subgraph, and that any such subgraph has minimum degree at least $k-1$.

11. Determine the critical 3-chromatic graphs.

12.$^+$ Show that every critical k-chromatic graph is $(k-1)$-edge-connected.

13. Given $k \in \mathbb{N}$, find a constant $c_k > 0$ such that every large enough graph G with $\alpha(G) \leqslant k$ contains a cycle of length at least $c_k\,|G|$.

14.$^-$ Find a graph G for which Brooks's theorem yields a significantly weaker bound on $\chi(G)$ than Proposition 5.2.2.

15.$^+$ Show that, in order to prove Brooks's theorem for a graph $G = (V, E)$, we may assume that $\kappa(G) \geqslant 2$ and $\Delta(G) \geqslant 3$. Prove the theorem under these assumptions, showing first the following two lemmas.

(i) Let $v_1, \dots, v_n$ be an enumeration of V. If every v_i $(i < n)$ has a neighbour v_j with $j > i$, and if $v_1v_n, v_2v_n \in E$ but $v_1v_2 \notin E$, then the greedy algorithm uses at most $\Delta(G)$ colours.

(ii) If G is not complete and v_n has maximum degree in G, then v_n has neighbours v_1, v_2 as in (i).

16.$^+$ Show that the following statements are equivalent for a graph G:

(i) $\chi(G) \leqslant k$;

(ii) G has an orientation without directed paths of length $k-1$; *length k*

(iii) G has an acyclic such orientation (one without directed cycles).

17. Given a graph G and $k \in \mathbb{N}$, let $P_G(k)$ denote the number of vertex colourings $V(G) \to \{1, \dots, k\}$. Show that P_G is a polynomial in k of degree $n := |G|$, in which the coefficient of k^n is 1 and the coefficient of k^{n-1} is $-\|G\|$. (P_G is called the *chromatic polynomial* of G.)

(Hint. Apply induction on $\|G\|$.)

18.$^+$ Determine the class of all graphs G for which $P_G(k) = k\,(k-1)^{n-1}$. (As in the previous exercise, let $n := |G|$, and let P_G denote the chromatic polynomial of G.)

19. In the definition of k-constructible graphs, replace the axiom (ii) by

(ii)$'$ *Every supergraph of a k-constructible graph is k-constructible*;

and the axiom (iii) by

(iii)$'$ *If G is a graph with vertices x, y_1, y_2 such that $y_1y_2 \in E(G)$ but $xy_1, xy_2 \notin E(G)$, and if both $G + xy_1$ and $G + xy_2$ are k-constructible, then G is k-constructible.*

Show that a graph is k-constructible with respect to this new definition if and only if its chromatic number is at least k.

Proof. We prove the theorem with $c = 10$. Let G be a graph of average degree at least $10r^2$. By Theorem 1.4.3 with $k := r^2$, G has an r^2-connected subgraph H with $\varepsilon(H) > \varepsilon(G) - r^2 \geqslant 4r^2$. To find a TK^r in H, we start by picking r vertices as branch vertices, and $r-1$ neighbours of each of these as some initial subdividing vertices. These are r^2 vertices in total, so as $\delta(H) \geqslant \kappa(H) \geqslant r^2$ they can be chosen distinct. Now all that remains is to link up the subdividing vertices in pairs, by disjoint paths in H corresponding to the edges of the K^r of which we wish to find a subdivision. Such paths exist, because H is $\frac{1}{2}r^2$-linked by Theorem 3.5.3. □

The proof of Thm 7.2.1 fails in the last line if r is odd, since then $r^2/2$ is not an integer. Here are three alternatives:

1. Repeat the proof as stated, but with $c = 11$. Then $\varepsilon(H)$ is large enough to make H $\lceil r^2/2 \rceil$-linked.

2. Start with $c = 10$ and H as stated but link the r branch vertices directly, using Ex. 24, Ch. 3.

3. Start with $c = 10$ and H as stated. Let H' be the graph obtained from H by deleting the r chosen branch vertices. Their $r(r-1)$ neighbours can be linked in H' as required. Indeed, H' is k-linked by Theorem 3.5.3 for $k = \frac{1}{2}r(r-1)$ ($\in \mathbb{N}$), as $\kappa(H') \geqslant \kappa(H) - r \geqslant r(r-1)$ and $\varepsilon(H') \geqslant \varepsilon(H) - r \geqslant 4r(r-1)$.

For small r, one can try to determine the exact number of edges needed to force a TK^r subgraph on n vertices. For $r = 4$, this number is $2n-2$; see Corollary 7.3.2. For $r = 5$, plane triangulations yield a lower bound of $3n-5$ (Corollary 4.2.10). The converse, that $3n-5$ edges do force a TK^5—not just either a TK^5 or a $TK_{3,3}$, as they do by Corollary 4.2.10 and Kuratowski's theorem—is already a difficult theorem (Mader 1998).

Let us now turn from topological minors to general minors. The average degree needed to force a K^r minor is known almost precisely. Thomason (2001) determined, asymptotically, the smallest constant c that makes the following theorem true as $\alpha + o(1)$, where $o(1)$ stands for a function of r tending to zero as $r \to \infty$ and $\alpha = 0.53131\ldots$ is an explicit constant.

Theorem 7.2.2. (Kostochka 1982)
There exists a constant $c \in \mathbb{R}$ such that, for every $r \in \mathbb{N}$, every graph G of average degree $d(G) \geqslant c\,r\sqrt{\log r}$ contains K^r as a minor. Up to the value of c, this bound is best possible as a function of r.

The easier implication of the theorem, the fact that in general an average degree of $c\,r\sqrt{\log r}$ is needed to force a K^r minor, follows from considering random graphs, to be introduced in Chapter 11. The converse implication, that this average degree suffices, is proved by methods not dissimilar to the proof of Theorem 3.5.3.

Rather than proving Theorem 7.2.2, therefore, we devote the remainder of this section to another striking aspect of forcing minors: that we can force a K^r minor in a graph simply by raising its girth (as long as we do not merely subdivide edges). At first glance, this may seem almost paradoxical. But it looks more plausible if, rather than trying to force a K^r minor directly, we instead try to force a minor just of large minimum or average degree—which suffices by Theorem 7.2.2. For if the girth g of a graph is large then the ball $\{\, v \mid d(x,v) < \lfloor g/2 \rfloor \,\}$ around a vertex x induces a tree with many leaves, each of which sends all but one of its incident edges away from the tree. Contracting enough disjoint

15.$^-$ Prove the Erdős-Sós conjecture for the case when the tree considered is a star.

16. Prove the Erdős-Sós conjecture for the case when the tree considered is a path.

(Hint. Use Exercise 7 of Chapter 1.)

17.$^+$ For which trees T is there a function $f: \mathbb{N} \to \mathbb{N}$ tending to infinity, such that every graph G with $\chi(G) < f(d(G))$ contains an induced copy of T? (In other words: can we force the chromatic number up by raising the average degree, as long as T does not occur as an induced subgraph? Or, as in Gyárfás's conjecture: will a large average degree force an induced copy of T if the chromatic number is kept small?)

18. Given two graph invariants i_1 and i_2, write $i_1 \leqslant i_2$ if we can force i_2 arbitrarily high on a subgraph of G by making $i_1(G)$ large enough. (Formally: write $i_1 \leqslant i_2$ if there exists a function $f: \mathbb{N} \to \mathbb{N}$ such that, given any $k \in \mathbb{N}$, every graph G with $i_1(G) \geqslant f(k)$ has a subgraph H with $i_2(H) \geqslant k$.) If $i_1 \leqslant i_2$ as well as $i_1 \geqslant i_2$, write $i_1 \sim i_2$. Show that this is an equivalence relation for graph invariants, and sort the following invariants into equivalence classes ordered by $<$: minimum degree; average degree; connectivity; arboricity; chromatic number; colouring number; choice number; $\max\{ r \mid K^r \subseteq G \}$; $\max\{ r \mid TK^r \subseteq G \}$; $\max\{ r \mid K^r \preccurlyeq G \}$; $\min \max d^+(v)$, where the maximum is taken over all vertices v of the graph, and the minimum over all its orientations.

19.$^+$ Prove, from first principles, the theorem of Wagner (1964) that every graph of chromatic number at least 2^r contains K^r as a minor.

(Hint. Use induction on r. For the induction step, contract a connected subgraph chosen so that the remaining graph still needs at least half as many colours as the given graph.)

20. Let G be a graph of average degree at least 2^{r-2}. By considering the neighbourhood of a vertex in a minimal minor $H \preccurlyeq G$ with $\varepsilon(H) \geqslant \varepsilon(G)$, prove Mader's (1967) theorem that $G \succcurlyeq K^r$.

21.$^-$ Derive Wagner's theorem (Ex. 19) from Mader's theorem (Ex. 20).

22.$^+$ Given a graph G with $\varepsilon(G) \geqslant k \in \mathbb{N}$, find a minor $H \preccurlyeq G$ such that both $\delta(H) \geqslant k$ and $\delta(H) \geqslant |H|/2$.

23.$^+$ Find a constant c such that every graph with n vertices and at least $n + 2k(\log k + \log\log k + c)$ edges contains k edge-disjoint cycles (for all $k \in \mathbb{N}$). Deduce an edge-analogue of the Erdős-Pósa theorem (2.3.2).

(Hint. Assuming $\delta \geqslant 3$, delete the edges of a short cycle and apply induction. The calculations are similar to the proof of Lemma 2.3.1.)

24.$^-$ Use Exercise 22 of Chapter 3 to reduce the constant c in Theorem 7.2.1 from 10 to 5.

Exercise 24 should read:

24. (i) Find the flaw in the last line of the proof of Thm 7.2.1 given in the text.

(ii) Correct the proof by joining the branch vertices directly, using Exercise 24 of Chapter 3.

25.$^+$ Show that any function h as in Lemma 3.5.1 satisfies the inequality $h(r) > \frac{1}{8}r^2$ for all even r, and hence that Theorem 7.2.1 is best possible up to the value of the constant c.

Let $n \in \mathbb{N}$ be given. If $n = 0$, choose any ray from $\mathcal{R}_0$ as Q_0, and put $\mathcal{R}_1 := \mathcal{R}_0 \setminus \{ Q_0 \}$. Then conditions (1)–(5) hold for $n = 0$.

Suppose now that $n \geqslant 1$, and consider a ray $R_n^0 \in \mathcal{R}_n$. By (4), R_n^0 is disjoint from

$$H := Q_0 \cup \ldots \cup Q_{n-1} \cup \bigcup_{i=1}^{n-1} \mathcal{P}_{n-1}(Q_i)\,.$$

$$\bigcup_{i=1}^{n-1} \bigcup \mathcal{P}_{n-1}(Q_i)$$

By the choice of $\mathcal{R}_0$ and (4), we know that $R_n^0 \in \omega$. As also $Q_0 \in \omega$, there exists an infinite set $\mathcal{P}$ of disjoint R_n^0–H paths. If possible, we choose $\mathcal{P}$ so that $\bigcup \mathcal{P} \cap \bigcup \mathcal{P}_{n-1}(Q_i) = \emptyset$ for all $i \leqslant n-1$. We may then further choose $\mathcal{P}$ so that $\bigcup \mathcal{P} \cap Q_i \neq \emptyset$ for only one i, since by (1) the Q_i are disjoint for different i. We define $p(n)$ as this i, and put $\mathcal{P}_n(Q_j) := \mathcal{P}_{n-1}(Q_j)$ for all $j \leqslant n-1$.

If $\mathcal{P}$ cannot be chosen in this way, we may choose it so that all its vertices in H lie in $\bigcup \mathcal{P}_{n-1}(Q_i)$ for the same i, since by (3) the graphs $\bigcup \mathcal{P}_{n-1}(Q_i)$ are disjoint for different i. We can then find infinite disjoint subsets $\mathcal{P}_n(Q_i)$ of $\mathcal{P}_{n-1}(Q_i)$ and $\mathcal{P}'$ of $\mathcal{P}$. We continue infinitely many of the paths in $\mathcal{P}'$ along paths from $\mathcal{P}_{n-1}(Q_i) \setminus \mathcal{P}_n(Q_i)$ to Q_i or to $Q_{p(i)}$, to obtain an infinite set $\mathcal{P}''$ of disjoint R_n^0–Q_i or R_n^0–$Q_{p(i)}$ paths, and define $p(n)$ as i or as $p(i)$ accordingly. The paths in $\mathcal{P}''$ then avoid $\bigcup \mathcal{P}_n(Q_j)$ for all $j \leqslant n-1$ (with $\mathcal{P}_n(Q_j) := \mathcal{P}_{n-1}(Q_j)$ for $j \neq i$) and Q_j for all $j \neq p(n)$. We rename $\mathcal{P}''$ as $\mathcal{P}$, to simplify notation.

In either case, we have now defined $\mathcal{P}_n(Q_i)$ for all $i < n$ so as to satisfy (5) for n, chosen $p(n)$ as in (2), and found an infinite set $\mathcal{P}$ of disjoint R_n^0–$Q_{p(n)}$ paths that avoid all other Q_j and all the sets $\mathcal{P}_n(Q_i)$. All that can prevent us from choosing R_n^0 as Q_n and $\mathcal{P}$ as $\mathcal{P}_n(Q_n)$ and $\mathcal{R}_{n+1} \leqslant \mathcal{R}_n \setminus \{ R_n^0 \}$ is condition (4): if $\mathcal{P}$ meets all but finitely many rays in $\mathcal{R}_n$ infinitely, we cannot find an infinite set $\mathcal{R}_{n+1} \leqslant \mathcal{R}_n$ of rays avoiding $\mathcal{P}$.

However, we may now assume the following:

Whenever $R \in \mathcal{R}_n$ and $\mathcal{P}' \leqslant \mathcal{P}$ is an infinite set of R–$Q_{p(n)}$ paths, there is a ray $R' \neq R$ in $\mathcal{R}_n$ that meets $\mathcal{P}'$ infinitely. $(*)$

For if $(*)$ failed, we could choose R as Q_n and $\mathcal{P}'$ as $\mathcal{P}_n(Q_n)$, and select from every ray $R' \neq R$ in $\mathcal{R}_n$ a tail avoiding $\mathcal{P}'$ to form $\mathcal{R}_{n+1}$. This would satisfy conditions (1)–(5) for n.

Consider the paths in $\mathcal{P}$ as linearly ordered by the natural order of their starting vertices on R_n^0. This induces an ordering on every $\mathcal{P}' \leqslant \mathcal{P}$. If $\mathcal{P}'$ is a set of R–$Q_{p(n)}$ paths for some ray R, we shall call this ordering of $\mathcal{P}'$ *compatible* with R if the ordering it induces on the first vertices of its paths coincides with the natural ordering of those vertices on R.

Using assumption $(*)$, let us choose two sequences $R_n^0, R_n^1, \ldots$ and $\mathcal{P}^0 \geqslant \mathcal{P}^1 \geqslant \ldots$ such that every R_n^k is a tail of a ray in $\mathcal{R}_n$ and each $\mathcal{P}^k$ is an infinite set of R_n^k–$Q_{p(n)}$ paths whose ordering is compatible

More precisely, if $\leqslant$ is a graph relation (such as the minor, topological minor, subgraph, or induced subgraph relation up to isomorphism), we call a countable graph G^* *universal* in $\mathcal{P}$ (for $\leqslant$) if $G^* \in \mathcal{P}$ and $G \leqslant G^*$ for every countable graph $G \in \mathcal{P}$.

Is there a graph that is universal in the class of *all* countable graphs? Suppose a graph R has the following property:

Whenever U and W are disjoint finite sets of vertices in R, there exists a vertex $v \in R - U - W$ that is adjacent in R to all the vertices in U but to none in W. $(*)$

Then R is universal even for the strongest of all graph relations, the induced subgraph relation. Indeed, in order to embed a given countable graph G in R we just map its vertices $v_1, v_2, \ldots$ to R inductively, making sure that v_n gets mapped to a vertex $v \in R$ adjacent to the images of all the neighbours of v_n in $G[v_1, \ldots, v_n]$ but not adjacent to the image of any non-neighbour of v_n in $G[v_1, \ldots, v_n]$. Clearly, this map is an isomorphism between G and the subgraph of R induced by its image.

Theorem 8.3.1. (Erdős and Rényi 1963)
There exists a unique countable graph R with property $(*)$.

Proof. To prove existence, we construct a graph R with property $(*)$ inductively. Let $R_0 := K^1$. For all $n \in \mathbb{N}$, let R_{n+1} be obtained from R_n by adding for every set $U \subseteq V(R_n)$ a new vertex v joined to all the vertices in U but to none outside U. (In particular, the new vertices form an independent set in R_{n+1}.) Clearly $R := \bigcup_{n \in \mathbb{N}} R_n$ has property $(*)$.

To prove uniqueness, let $R = (V, E)$ and $R' = (V', E')$ be two graphs with property $(*)$, each given with a fixed vertex enumeration. We construct a bijection $\varphi: V \to V'$ in an infinite sequence of steps, defining $\varphi(v)$ for one new vertex $v \in V$ at each step.

At every odd step we look at the first vertex v in the enumeration of V for which $\varphi(v)$ has not yet been defined. Let U be the set of those of its neighbours u in R for which $\varphi(u)$ has already been defined. This is a finite set. Using $(*)$ for R', find a vertex $v' \in V'$ that is adjacent in R' to all the vertices in $\varphi(U)$ but to no other vertex in the image of φ (which, so far, is still a finite set). Put $\varphi(v) := v'$.

The vertex v' should also lie outside the image of φ.

At even steps in the definition process we do the same thing with the roles of R and R' interchanged: we look at the first vertex v' in the enumeration of V' that does not yet lie in the image of φ, and set $\varphi(v) = v'$ for a vertex v that matches the adjacencies and non-adjacencies of v' among the vertices for which φ (resp. φ^{-1}) has already been defined.

new vertex v

By our minimum choices of v and v', the bijection gets defined on all of V and all of V', and it is clearly an isomorphism. □

Given a set S of vertices in a graph G, let us write $\mathcal{C}'_{G-S}$ for the set of factor-critical components of $G-S$, and G'_S for the bipartite graph with vertex set $S \cup \mathcal{C}'_{G-S}$ and edge set $\{ sC \mid \exists c \in C : sc \in E(G) \}$.

Definition: *$G \neq \emptyset$ is factor-critical if G has no 1-factor but $G - v$ does for every vertex $v \in G$.*

Theorem 8.4.11. (Aharoni 1988)
A graph G has a 1-factor if and only if, for every set $S \subseteq V(G)$, the set $\mathcal{C}'_{G-S}$ is matchable to S in G'_S.

Applied to a finite graph, Theorem 8.4.11 implies Tutte's 1-factor theorem (2.2.1): if $\mathcal{C}'_{G-S}$ is not matchable to S in G'_S, then by the marriage theorem there is a subset S' of S that sends edges to more than $|S'|$ components in $\mathcal{C}'_{G-S}$ that are also components of $G-S'$, and these components are odd because they are factor-critical.

Theorems 8.4.8 and 8.4.11 also imply an infinite version of the Gallai-Edmonds theorem (2.2.3):

Corollary 8.4.12. *Every graph $G=(V,E)$ has a set S of vertices that is matchable to $\mathcal{C}'_{G-S}$ in G'_S and such that every component of $G-S$ not in $\mathcal{C}'_{G-S}$ has a 1-factor. Given any such set S, the graph G has a 1-factor if and only if $\mathcal{C}'_{G-S}$ is matchable to S in G'_S.*

Proof. Given a pair (S,M) where $S \subseteq V$ and M is a matching of S in G'_S, and given another such pair (S',M'), write $(S,M) \leqslant (S',M')$ if

$$S \subseteq S' \subseteq V \smallsetminus \bigcup \{ V(C) \mid C \in \mathcal{C}'_{G-S} \}$$

and $M \subseteq M'$. Since $\mathcal{C}'_{G-S} \subseteq \mathcal{C}'_{G-S'}$ for any such S and S', Zorn's lemma implies that there is a maximal such pair (S,M).

For the first statement, we have to show that every component C of $G-S$ that is not in $\mathcal{C}'_{G-S}$ has a 1-factor. If it does not, then by Theorem 8.4.11 there is a set $T \subseteq V(C)$ such that $\mathcal{C}'_{C-T}$ is not matchable to T in C'_T. By Corollary 8.4.9, this means that $\mathcal{C}'_{C-T}$ has a subset $\mathcal{C}$ that is not matchable in C'_T to the set $T' \subseteq T$ of its neighbours, while T' is matchable to $\mathcal{C}$; let M' be such a matching. Then $(S,M) < (S \cup T', M \cup M')$, contradicting the maximality of (S,M).

Of the second statement, only the backward implication is non-trivial. Our assumptions now are that $\mathcal{C}'_{G-S}$ is matchable to S in G'_S and vice versa (by the choice of S), so Proposition 8.4.6 yields that G'_S has a 1-factor. This defines a matching of S in G that picks one vertex x_C from every component $C \in \mathcal{C}'_{G-S}$ and leaves the other components of $G-S$ untouched. Adding to this matching a 1-factor of $C - x_C$ for every $C \in \mathcal{C}'_{G-S}$ and a 1-factor of every other component of $G-S$, we obtain the desired 1-factor of G. $\square$

its closure together with all inner points of $C(v)$–S_n edges. Then $G[S_n]$ and these $\hat{C}(v)$ together partition $|G|$.

We wish to prove that, for some n, each of the sets $\hat{C}(v)$ with $v \in D_n$ is contained in some $O(v) \in \mathcal{O}$. For then we can take a finite subcover of $\mathcal{O}$ for $G[S_n]$ (which is compact, being a finite union of edges and vertices), and add to it these finitely many sets $O(v)$ to obtain the desired finite subcover for $|G|$.

Suppose there is no such n. Then for each n the set V_n of vertices $v \in D_n$ such that no set from $\mathcal{O}$ contains $\hat{C}(v)$ is non-empty. Moreover, for every neighbour $u \in D_{n-1}$ of $v \in V_n$ we have $C(v) \subseteq C(u)$ because $S_{n-1} \subseteq S_n$, and hence $u \in V_{n-1}$; let $f(v)$ be such a vertex u. By the infinity lemma (8.1.2) there is a ray $R = v_0 v_1 \ldots$ with $v_n \in V_n$ for all n. Let ω be its end, and let $O \in \mathcal{O}$ contain ω. Since O is open, it contains a basic open neighbourhood of ω: there exist a finite set $S \subseteq V$ and $\epsilon > 0$ such that $\hat{C}_\epsilon(S, \omega) \subseteq O$. Now choose n large enough that S_n contains S and all its neighbours. Then $\hat{C}(v_n)$ lies inside a component of $G - S$. $C(v_n)$
As $C(v_n)$ contains $v_n R \in \omega$, this component must be $C(S, \omega)$. Thus

$$\hat{C}(v_n) \subseteq C(S, \omega) \subseteq O \in \mathcal{O}\,,$$ $\hat{C}_\epsilon(S,\omega)$

contradicting the fact that $v_n \in V_n$. □

If G has a vertex of infinite degree then $|G|$ cannot be compact. (Why not?) But $\Omega(G)$ can be compact; see Exercise 61 for when it is.

What else can we say about the space $|G|$ in general? For example, is it metrizable? Using a normal spanning tree T of G, it is indeed not difficult to define a metric on $|G|$ that induces its topology. But not every connected graph has a normal spanning tree, and it is not easy to determine which graphs do. Surprisingly, though, it is possible conversely to deduce the existence of a normal spanning tree just from the assumption that the subspace $V \cup \Omega$ of $|G|$ is metric. Thus whenever $|G|$ is metrizable, a natural metric can be made visible in this simple structural way:

Theorem 8.5.2. *For a connected graph G, the space $|G|$ is metrizable if and only if G has a normal spanning tree.*

The proof of Theorem 8.5.2 is indicated in Exercises 30 and 63.

Our next aim is to review, or newly define, some topological notions of paths and connectedness, of cycles, and of spanning trees. By substituting these topological notions with respect to $|G|$ for the corresponding graph-theoretical notions with respect to G, one can extend to locally finite graphs a number of theorems about paths, cycles and spanning trees in finite graphs that would not otherwise extend. We shall do this, as a case in point, for the tree-packing theorem of Nash-Williams and

9.$^-$ Theorem 8.1.3 implies that there exists an $\mathbb{N}\to\mathbb{N}$ function f_χ such that, for every $k \in \mathbb{N}$, every infinite graph of chromatic number at least $f_\chi(k)$ has a finite subgraph of chromatic number at least k. (Namely, let f_χ be the identity on $\mathbb{N}$.) Are there similar functions f_δ and f_κ for the minimum degree and connectivity?

10. Prove Theorem 8.1.3 for countable graphs using the fact that, in this case, the topological space X defined in the second proof of the theorem is sequentially compact. (Thus, every infinite sequence of points in X has a convergent subsequence: there is an $x \in X$ such that every neighbourhood of X contains a tail of the subsequence.) *of x*

11.$^+$ Show that, given $k \in \mathbb{N}$ and an edge e in a graph G, there are only finitely many bonds in G that consist of exactly k edges and contain e.

12.$^-$ Extend Theorem 2.4.4 to infinite graphs.

13. Rephrase Gallai's cycle-cocycle partition theorem (Ex. 35, Ch. 1) in terms of degrees, and extend the equivalent version to locally finite graphs.

14. Prove Theorem 8.4.8 for locally finite graphs. Does your proof extend to arbitrary countable graphs?

15. Extend the marriage theorem to locally finite graphs, but show that it fails for countable graphs with infinite degrees.

16.$^+$ Show that a locally finite graph G has a 1-factor if and only if, for every finite set $S \subseteq V(G)$, the graph $G-S$ has at most $|S|$ odd (finite) components. Find a counterexample that is not locally finite.

17.$^+$ Extend Kuratowski's theorem to countable graphs.

18.$^-$ A vertex $v \in G$ is said to *dominate* an end ω of G if any of the following three assertions holds; show that they are equivalent.

(i) For some ray $R \in \omega$ there is an infinite v–R fan in G.

(ii) For every ray $R \in \omega$ there is an infinite v–R fan in G.

(iii) No finite subset of $V(G-v)$ separates v from a ray in ω.

19. Show that a graph G contains a $TK^{\aleph_0}$ if and only if some end of G is dominated by infinitely many vertices.

20. Construct a countable graph with uncountably many thick ends.

21. Show that a countable tree has uncountably many ends if and only if it contains a subdivision of the binary tree T_2.

22. A graph $G = (V,E)$ is called *bounded* if for every vertex labelling $\ell\colon V \to \mathbb{N}$ there exists a function $f\colon \mathbb{N}\to\mathbb{N}$ that exceeds the labelling along any ray in G eventually. (Formally: for every ray $v_1v_2\ldots$ in G there exists an n_0 such that $f(n) > \ell(v_n)$ for every $n > n_0$.) Prove the following assertions:

(i) The ray is bounded.

(ii) Every locally finite connected graph is bounded.

The question is whether a locally finite factor-critical graph can be infinite.
$G \neq \emptyset$ is factor-critical if G has no 1-factor but $G - v$ does for every vertex $v \in G$.

56. Construct a locally finite factor-critical graph (or prove that none exists).

57.$^-$ Let G be a countable graph whose finite subgraphs are all perfect. Show that G is weakly perfect but not necessarily perfect.

58.$^+$ Let G be the incomparability graph of the binary tree. (Thus, $V(G) = V(T_2)$, and two vertices are adjacent if and only if they are incomparable in the tree-order of T_2.) Show that G is perfect but not strongly perfect.

59. Let G be a graph, $X \subseteq V(G)$, and $R \in \omega \in \Omega(G)$. Show that G contains a comb with spine R and teeth in X if and only if $\omega \in \overline{X}$.

60. Give an independent proof of Proposition 8.5.1 using sequential compactness and the infinity lemma.

61.$^+$ Let G be a connected countable graph that is not locally finite. Show that $|G|$ is not compact, but that $\Omega(G)$ is compact if and only if for every finite set $S \subseteq V(G)$ only finitely many components of $G - S$ contain a ray.

62. Given graphs $H \subseteq G$, let $\eta\colon \Omega(H) \to \Omega(G)$ assign to every end of H the unique end of G containing it as a subset (of rays). For the following questions, assume that H is connected and $V(H) = V(G)$.
   (i) Show that η need not be injective. Must it be surjective?
   (ii) Investigate how η relates the subspace $\Omega(H)$ of $|H|$ to its image in $|G|$. Is η always continuous? Is it open? Do the answers to these questions change if η is known to be injective?
   (iii) A spanning tree is called *end-faithful* if η is bijective, and *topologically end-faithful* if η is a homeomorphism. Show that every connected countable graph has a topologically end-faithful spanning tree.

63.$^+$ Let G be a connected graph. Assuming that G has a normal spanning tree, define a metric on $|G|$ that induces its usual topology. Conversely, use Jung's theorem of Exercise 30 to show that if $V \cup \Omega \subseteq |G|$ is metrizable then G has a normal spanning tree.

64.$^+$ (for topologists) In a locally compact, connected, and locally connected Hausdorff space X, consider sequences $U_1 \supseteq U_2 \supseteq \ldots$ of open, nonempty, connected subsets with compact frontiers such that $\bigcap_{i \in \mathbb{N}} \overline{U_i} = \emptyset$. Call such a sequence *equivalent* to another such sequence if every set of one sequence contains some set of the other, and vice versa. Note that this is indeed an equivalence relation, and call its classes the *Freudenthal ends* of X. Now add these to the space X, and define a natural topology on the extended space $\hat{X}$ that makes it homeomorphic to $|X|$ if X is a graph, by a homeomorphism that is the identity on X.

65. Let F be a set of edges in a locally finite graph G, and let $A := \overline{\bigcup F}$ be its closure in $|G|$. Show that F is a circuit if and only if, for every two edges $e, e' \in F$, the set $A \setminus \mathring{e}$ is connected but $A \setminus (\mathring{e} \cup \mathring{e}')$ is disconnected in $|G|$.

Let us call our tree-decomposition $(T, \mathcal{V})$ of G *linked*, or *lean*,[4] if it satisfies the following condition:

(T4) Given $t_1, t_2 \in T$ and vertex sets $Z_1 \subseteq V_{t_1}$ and $Z_2 \subseteq V_{t_2}$ such that $|Z_1| = |Z_2| =: k$, either G contains k disjoint Z_1–Z_2 paths or there exists an edge $tt' \in t_1 T t_2$ with $V_t \cap V_{t'} < k$. $|V_t \cap V_{t'}|$

The 'branches' in a lean tree-decomposition are thus stripped of any bulk not necessary to maintain their connecting qualities: if a branch is thick (i.e. the separators $V_t \cap V_{t'}$ along a path in T are large), then G is highly connected along this branch. For $t_1 = t_2$, (T4) says that the parts themselves are no larger than their 'external connectivity' in G requires; cf. Lemma 12.4.5 and Exercise 35.

In our quest for tree-decompositions into 'small' parts, we now have two criteria to choose between: the global 'worst case' criterion of width, which ensures that T is nontrivial (unless G is complete) but says nothing about the tree-likeness of G among parts other than the largest, and the more subtle local criterion of leanness, which ensures tree-likeness everywhere along T but might be difficult to achieve except with trivial or near-trivial T. Surprisingly, though, it is always possible to find a tree-decomposition that is optimal with respect to both criteria at once:

Theorem 12.3.10. (Thomas 1990)
Every graph G has a lean tree-decomposition of width $\text{tw}(G)$.

There is now a short proof of Theorem 12.3.10; see the notes. The fact that this theorem gives us a useful property of minimum-width tree-decompositions 'for free' has made it a valuable tool wherever tree-decompositions are applied.

The tree-decomposition $(T, \mathcal{V})$ of G is called *simplicial* if all the separators $V_{t_1} \cap V_{t_2}$ induce complete subgraphs in G. This assumption can enable us to lift assertions about the parts of the decomposition to G itself. For example, if all the parts in a simplicial tree-decomposition of G are k-colourable, then so is G (proof?). The same applies to the property of not containing a K^r minor for some fixed r. Algorithmically, it is easy to obtain a simplicial tree-decomposition of a given graph into irreducible parts. Indeed, all we have to do is split the graph recursively along complete separators; if these are always chosen minimal, then the set of parts obtained will even be unique (Exercise 27).

Conversely, if G can be constructed recursively from a set $\mathcal{H}$ of graphs by pasting along complete subgraphs, then G has a simplicial tree-decomposition into elements of $\mathcal{H}$. For example, by Wagner's Theorem 7.3.4, any graph without a K^5 minor has a supergraph with a simplicial tree-decomposition into plane triangulations and copies of the

[4] depending on which of the two dual aspects of (T4) we wish to emphasize

Let H be a graph, S a surface, and $k \in \mathbb{N}$. We say that H is *k-nearly embeddable* in S if H has a set X of at most k vertices such that $H - X$ can be written as $H_0 \cup H_1 \cup \ldots \cup H_k$ so that

(N1) there exists an embedding $\sigma \colon H_0 \hookrightarrow S - k$ that maps only vertices to cuffs and no vertex to the root of a cuff;

(N2) the graphs $H_1, \ldots, H_k$ are pairwise disjoint (and may be empty), and $H_0 \cap H_i = \sigma^{-1}(C_i)$ for each i;

(N3) every H_i with $i \geqslant 1$ has a linear decomposition $(V_z^i)_{z \in C_i \cap \sigma(H_0)}$ of width at most k such that $z \in V_z^i$ for all z. *Replace z by $\sigma^{-1}(z)$.*

Here, then, is the structure theorem for the graphs without a K^n minor:

Theorem 12.4.11. (Robertson & Seymour 2003)
For every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that every graph G not containing K^n as a minor has a tree-decomposition whose torsos are k-nearly embeddable in a surface in which K^n is not embeddable. $n \geqslant 5$

Note that there are only finitely many surfaces in which K^n is not embeddable. The set of those surfaces in the statement of Theorem 12.4.11 could therefore be replaced by just two surfaces: the orientable and the non-orientable surface of maximum genus in this set. Note also that the separators $V_t \cap V_{t'}$ in the tree-decomposition of G (for edges tt' of the decomposition tree) have bounded size, e.g. at most $2k + n$, because they induce complete subgraphs in the torsos and these are k-nearly embeddable in one of those two surfaces.

We remark that Theorem 12.4.11 has only a qualitative converse: graphs that admit a decomposition as described can clearly have a K^n minor, but there exists an integer r depending only on n such that none of them has a K^r minor.

Theorem 12.4.11, as stated above, is true also for infinite graphs (Diestel & Thomas 1999). There are also structure theorems for excluding infinite minors, and we state two of these.

First, the structure theorem for excluding $K^{\aleph_0}$. Call a graph H *nearly planar* if H has a finite set X of vertices such that $H - X$ can be written as $H_0 \cup H_1$ so that (N1–2) hold with $S = S^2$ (the sphere) and $k = 1$, while (N3) holds with $k = |X|$. (In other words, deleting a bounded number of vertices makes H planar except for a subgraph of bounded linear width sewn on to the unique cuff of $S^2 - 1$.) A tree-decomposition $(T, (V_t)_{t \in T})$ of a graph G has *finite adhesion* if for every edge $tt' \in T$ the set $V_t \cap V_{t'}$ is finite and for every infinite path $t_1 t_2 \ldots$ in T the value of $\liminf_{i \to \infty} |V_{t_i} \cap V_{t_{i+1}}|$ is finite.

Unlike its counterpart for K^n, the excluded-$K^{\aleph_0}$ structure theorem has a direct converse. It thus characterizes the graphs without a $K^{\aleph_0}$ minor, as follows:

36.$^+$ (continued)

Find an $\mathbb{N} \to \mathbb{N}^2$ function $k \mapsto (h, \ell)$ such that every graph with an externally ℓ-connected set of h vertices contains a bramble of order at least k. Deduce the weakening of Theorem 12.3.9 that, given k, every graph of large enough tree-width contains a bramble of order at least k.

A *tangle of order* $k \in \mathbb{N}$ in a graph $G = (V, E)$ is a set $\mathcal{T}$ of ordered pairs (A, B) of subsets of V satisfying the following conditions.

($\mathcal{T}$1) For every $(A, B) \in \mathcal{T}$, the 2-set $\{A, B\}$ is a separation in G or order $< k$.

($\mathcal{T}$2) For every separation $\{A, B\}$ of order $< k$ in G, at least one of (A, B), (B, A) is an element of $\mathcal{T}$.

($\mathcal{T}$3) If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq V$.

($\mathcal{T}$4) No $(A, B) \in \mathcal{T}$ is such that $A = V$.

Condition ($\mathcal{T}$4) is redundant

(It makes sense in the original definition of a tangle, where separations are pairs of subgraphs, not vertex sets, and ($\mathcal{T}$3) refers to G while ($\mathcal{T}$4) refers to V.)

37. Deduce from Exercise 35 that every graph of tree-width at least $4k$ has a tangle of order k.

38. Extend Corollary 12.4.10 as follows. Let H be a connected planar graph, let $\mathcal{X}$ be any set of connected graphs including H, and let $\mathcal{H} := \{MX \mid X \in \mathcal{X}\}$. Show that $\mathcal{H}$ has the Erdős-Pósa property, witnessed by the same function f as defined in the proof of Corollary 12.4.10. Explain how it is possible that f depends on H but not on any of the other graphs in $\mathcal{X}$.

39.$^+$ Show that, for every non-planar graph H, the class MH fails to have the Erdős-Pósa property.

(Hint. Embed H in a surface S, and consider only graphs embedded in S.)

40.$^+$ Extend Corollary 12.4.10 to disconnected graphs H, or find a counterexample.

41.$^+$ Show that the four ingredients to the structure of the graphs in $\text{Forb}_{\preccurlyeq}(K^n)$ as described in Theorem 12.4.11—tree-decomposition, an *apex set* X, genus, and *vortices* $H_1, \ldots, H_k$—are all needed to capture all the graphs in $\text{Forb}_{\preccurlyeq}(K^n)$. More precisely, find examples of graphs in $\text{Forb}_{\preccurlyeq}(K^n)$ showing that Theorem 12.4.11 becomes false if we require in addition that the tree-decomposition has only one part, or that X is always empty, or that S is always the sphere, or that $H_1, \ldots, H_k$ are always empty. No exact proofs are required.

42. Without using the minor theorem, show that the chromatic number of the graphs in any $\preccurlyeq$-antichain is bounded.

43. Let S_g denote the surface obtained from the sphere by adding g handles. Find a lower bound for $|\mathcal{K}_{\mathcal{P}(S)}|$ in terms of g.

(Hint. The smallest g such that a given graph can be embedded in S_g is its *orientable genus*. Use the theorem that the orientable genus of a graph is equal to the sum of the genera of its blocks.)

44. Deduce the graph minor theorem from the self-minor conjecture.

the classification theorem, but to form a picture[2] let us see what the above operations mean. To *add a handle* to a surface S, we remove two open discs whose closures in S are disjoint, and identify[3] their boundary circles with the circles $S^1 \times \{0\}$ and $S^1 \times \{1\}$ of a copy of $S^1 \times [0,1]$ disjoint from S. To *add a crosscap*, we remove one open disc, and then identify opposite points on its boundary circle in pairs.

Since S^1 has two possible orientations, two copies of S^1 can be identified in two essentially different ways. The corresponding two ways of adding a handle yield different new surfaces. For the classification one only uses one of these, the way that preserves the orientability of the surface (as in Figure B.1).

In order to see that these operations do indeed give new surfaces, we have to check that every identification point ends up with a neighbourhood homeomorphic to $\mathbb{R}^2$. To do this rigorously, let us first look at circles more generally.

A *cylinder* is the product space $S^1 \times [0,1]$, or any space homeomorphic to it. Its *middle circle* is the circle $S^1 \times \{\frac{1}{2}\}$. A *Möbius strip* is any space homeomorphic to the product space $[0,1] \times [0,1]$ after identification of $(1,y)$ with $(0,1-y)$ for all $y \in [0,1]$. Its *middle circle* is the set $\{(x,\frac{1}{2}) \mid 0 < x < 1\} \cup \{p\}$, where p is the point resulting from the identification of $(1,\frac{1}{2})$ with $(0,\frac{1}{2})$. It can be shown[4] that every circle C in a surface S is the middle circle of a suitable cylinder or Möbius strip N in S, which can be chosen small enough to avoid any given compact subset of $S \setminus C$. If this *strip neighbourhood* is a cylinder, then $N \setminus C$ has two components and we call C *two-sided*; if it is a Möbius strip, then $N \setminus C$ has only one component and we call C *one-sided*.

Using small neighbourhoods inside a strip neighbourhood of the (two-sided) boundary circle of the disc or discs we removed from S in order to attach a crosscap or handle, one can show easily that both operations do produce new surfaces.

Since S is connected, $S \setminus C$ cannot have more components than $N \setminus C$. If $S \setminus C$ has two components, we call C a *separating* circle in S; if it has only one, then C is *non-separating*. While one-sided circles are obviously non-separating, two-sided circles can be either separating or non-separating. For example, the middle circle of a cylinder added to S as a 'handle' is a two-sided non-separating circle in the new surface obtained. When S' is obtained from S by adding a crosscap in place of a disc D, then every arc in S that runs half-way round the boundary circle of D becomes a one-sided circle in S'.

The classification theorem thus has the following corollary:

Lemma B.1. *Every surface other than the sphere contains a non-separating circle.*

[2] Compare also Figure B.1.

[3] This is made precise by the *identification topology*, whose formal definition can be found in any topology book.

[4] In principle, the strip neighbourhood N is constructed as in the proof of Lemma 4.2.2, using the compactness of C. However since we are not in a piecewise linear setting now, the construction is considerably more complicated.

67. Recall that, in S^1, every point has a neighbourhood basis consisting of arcs in $\mathbb{R}^2$. Can you show that every arc in C that links two ends must meet an edge? If not, can you show that it meets a vertex? If not, remember the proof of Lemma 8.5.5.

68. Exercise 26.

69. Enumerate the double rays D and D_ℓ in one infinite sequence, and inductively define partial homeomorphisms between these D_ℓ and suitable segments of S^1. When this is done, extend the partial homeomorphism on the union of all the double rays to the ends of G so as to make the final map continuous.

70. The main assertion to be proved is that every subspace C satisfying the conditions is a circle. Let $A \subseteq C$ be an arc linking two vertices x_0 and y_0. If v is any vertex in $C \setminus A$, the arc-connectedness of C yields a v–A arc in C, which has a first point on A. By the degree condition assumed, this must be x_0 or y_0. Starting from an enumeration $v_0, v_1, \dots$ of the vertices in C, construct a 2-way infinite sequence $\dots x_{-2}, x_{-1}, x_0, y_0, y_1, y_2 \dots$ of vertices such that C contains arcs A_i linking x_{-i-1} to x_{-i} and B_i linking y_i to y_{i+1} for all $i \in \mathbb{N}$, so that the union U of A and all these arcs is a homeomorphic copy of $(0,1)$ in C. Use the connectedness of its 'tails' to show that these converge to unique ends in C. Deduce from the degree assumptions that these two ends coincide, and that $\overline{U} = C$ is a circle.

71. Use Lemma 8.5.4. You may also use that every circle contains an edge.

72.$^-$ Show that if a topological spanning tree is homeomorphic to a space $|T|$ with T a tree, but does not itself have this form, it contains an end which this homeomorphism maps to a point in T (i.e., not to an end). Can you find a topological spanning tree for which this is impossible?

73. Start with a maximal set of disjoint rays.

74.$^+$ Given a point $\omega \in \overline{A} \setminus A$, pick a sequence $v_1, v_2, \dots$ of vertices in A that converges to ω, and arcs $A_n \subseteq A$ from v_n to v_{n+1}. Then use the infinity lemma to concatenate suitable portions of the A_n to form a continuous function $\alpha\colon [0,1] \to |G|$ that maps $[0,1)$ to A and 1 to ω. You may use the fact that the image of such a function α contains an arc from $\alpha(0) \in A$ to $\alpha(1) = \omega$.

74.$^+$ Given a point $\omega \in \overline{A} \setminus A$ and a sequence $x_1, x_2, \dots$ of points in A converging to ω, one can find x_n–x_{n+1} arcs in A and concatenate these to one continuous function $\alpha\colon [0,1) \to A$. Now map 1 to ω. What additional properties of the A_n do you need so that the extended map $[0,1] \to |G|$ is continuous at 1? (You may use the fact that, if it is, then its image contains an arc from $\alpha(0) \in A$ to $\alpha(1) = \omega$.) To ensure that the A_n have these properties, it may help to choose the x_n as vertices.

75. Recall that non-separating induced cycles of a plane graph are face boundaries.

76.$^-$ How can $\overline{T}$ fail to be a topological spanning tree?

77. Find the circuits greedily, making sure all edges are captured.

78. Check thinness. For an alternative proof, use Theorem 8.5.8 (i) instead of (ii).

8.5.8 (iii)

79.$^+$ For the 'only if' part, use a theorem from the text. The task in the 'if' part is to combine the edge-disjoint circles from Theorem 8.5.8 (ii) into a single continuous image of S^1. Start with one of those circles, and incorporate the others step by step. Check that the 'limit map' $\sigma\colon S^1 \to |G|$ is continuous (and defined) on all of S^1.

5. What would be the measure of the set $\{G\}$ for a fixed G?

6. Consider the complementary properties.

7.$^-$ $\mathcal{P}_{2,1}$.

8. Apply Lemma 11.3.2.

9. Induction on $|H|$ with the aid of Exercise 6.

10. Imitate the proof of Lemma 11.2.1.

11. Imitate the proof of Proposition 11.3.1. To bound the probabilities involved, use the inequality $1 - x \leqslant e^{-x}$ as in the proof of Lemma 11.2.1.

12.$^+$ (i) Calculate the expected number of isolated vertices, and apply Lemma 11.4.2 as in the proof of Theorem 11.4.3.

(ii) Linearity.

13.$^+$ Chapter 7.2, the proof of Erdős's theorem, and a bit of Chebyshev.

14. For the first problem modify an increasing property slightly, so that it ceases to be increasing but keeps its threshold function. For the second, look for an increasing property whose probability does not really depend on p.

Interchange 'first' and 'second'.

15.$^-$ Permutations of $V(H)$.

16.$^-$ This is a result from the text in disguise.

17.$^-$ Balance.

18. For $p/t \to 0$ apply Lemmas 11.1.4 and 11.1.5. For $p/t \to \infty$ apply Corollary 11.4.4.

19. There are only finitely many trees of order k.

20.$^+$ Show first that no such threshold function $t = t(n)$ can tend to zero as $n \to \infty$. Then use Exercise 11.

21.$^+$ Examine the various steps in the proof of Theorem 11.4.3, identify the two points where it now fails, and repair them. While the first part requires a slightly different tack as a consequence, the second adapts more mechanically.

Hints for Chapter 12

1.$^-$ Antisymmetry.

2. For the backward implication, assume first that A has an infinite antichain; this case is easier. The proof for other case is not quite as obvious but similar; note that $A = \mathbb{Z}$ is not a counterexample.

3. To prove Proposition 12.1.1, consider an infinite sequence in which every strictly decreasing subsequence is finite. How does the last element of a maximal decreasing subsequence compare with the elements that come after it? For Corollary 12.1.2, start by proving that at least one element forms a good pair with infinitely many later elements.