 first draft: summer 1998

CHAPTER 1

ISOMETRIES AS FUNCTIONS

1.0 Functions and isometries on the plane

1.0.1 Example. Consider a **fixed** point O and the following ‘operation’: given any other point P on the plane, we send (**map**) it to a point P′ that lies on the ray going from O to P and also satisfies the equation **$|OP'| = 3 \times |OP|$** (figure 1.1). It is clear that for each point P there is precisely one point P′, the **image** of P, that satisfies the two conditions stated above. Any such process that associates precisely one image point to every point on the plane is called a **function** (or **mapping**).

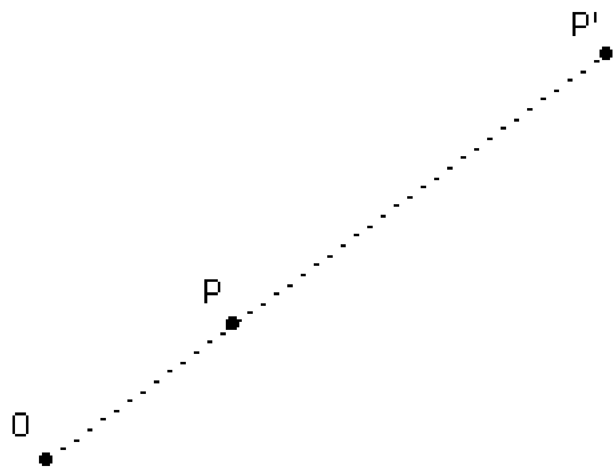

Fig. 1.1

1.0.2 Coordinates. Let us now describe the ‘blowing out’ function discussed in 1.0.1 in a different way, using the **cartesian coordinate system** and positioning O at the **origin**, (0, 0). Consider a specific point P with **coordinates** (2.5, 1.8). Looking at the **similar triangles** OPA and OP′B in figure 1.2, we see that $\frac{|P'B|}{|PA|} = \frac{|OB|}{|OA|} = \frac{|OP'|}{|OP|} = 3$, hence $|P'B| = 3 \times |PA| = 3 \times 1.8 = 5.4$ and $|OB| = 3 \times |OA|$

= 3×2.5 = 7.5. That is, the coordinates of P′ are (7.5, 5.4). In exactly the same way we can show that an arbitrary point with coordinates (x, y) is mapped to a point with coordinates (3x, 3y). We may therefore represent our function by a formula: **f(x, y) = (3x, 3y)**.

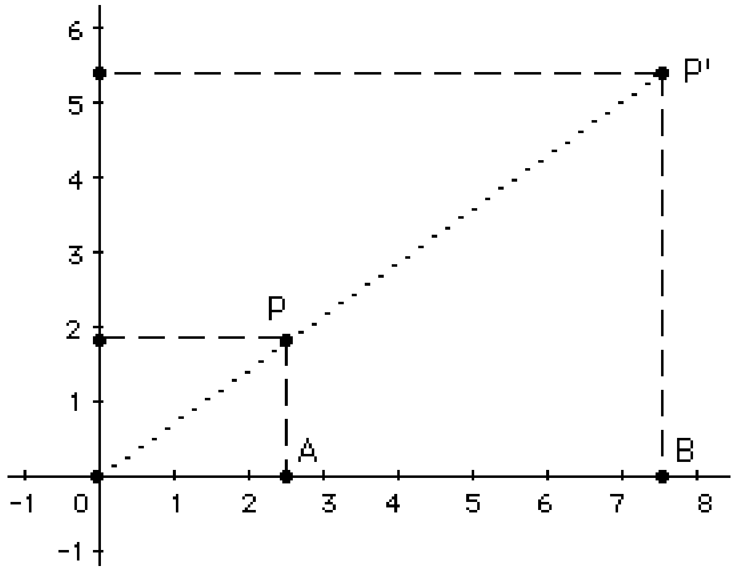

Fig. 1.2

1.0.3 Images. Let us look at the rectangle ABCD, defined by the four points A = (2, 1), B = (2, 2), C = (5, 2), D = (5, 1). What happens to it under our function? Well, it is simply mapped to a 'blown out' rectangle A′B′C′D′ -- **image** of ABCD under the 'blow out' function -- with vertices (6, 3), (6, 6), (15, 6), and (15, 3), respectively:

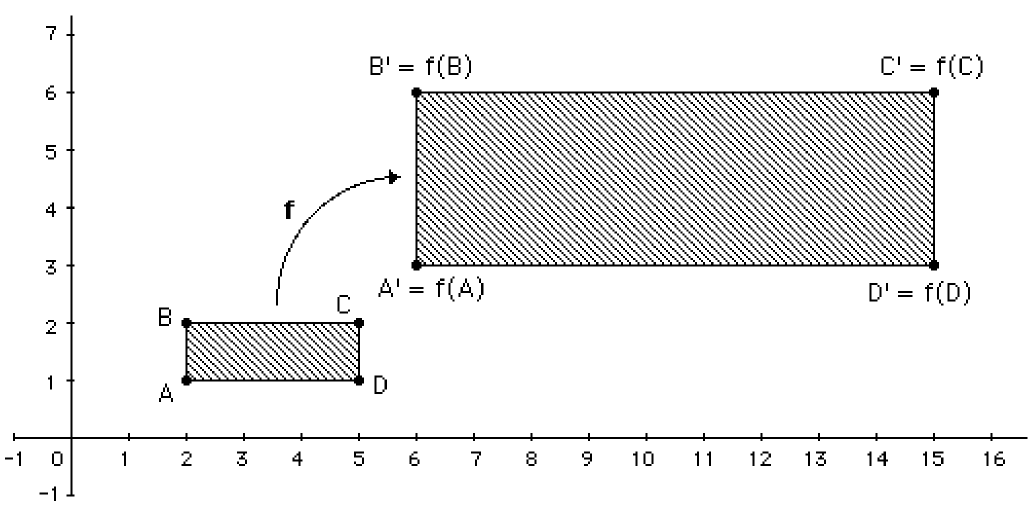

Fig. 1.3

1.0.4 More functions. One can have many more functions, formulas, and images. For example, **g(x,y) = (2x–y, x+3y)** maps ABCD to a parallelogram, while **h(x, y) = (3x+y, x–y^2+4)** maps ABCD to a semi-curvilinear quadrilateral (figure 1.4). We compute the images of A under g and h, leaving the other three vertices to you: g(A) = g(2, 1) = (2×2–1, 2+3×1) = (3, 5); h(A) = h(2, 1) = (3×2+1, 2–1^2+4) = (7, 5). (You may find more details in 1.0.8.)

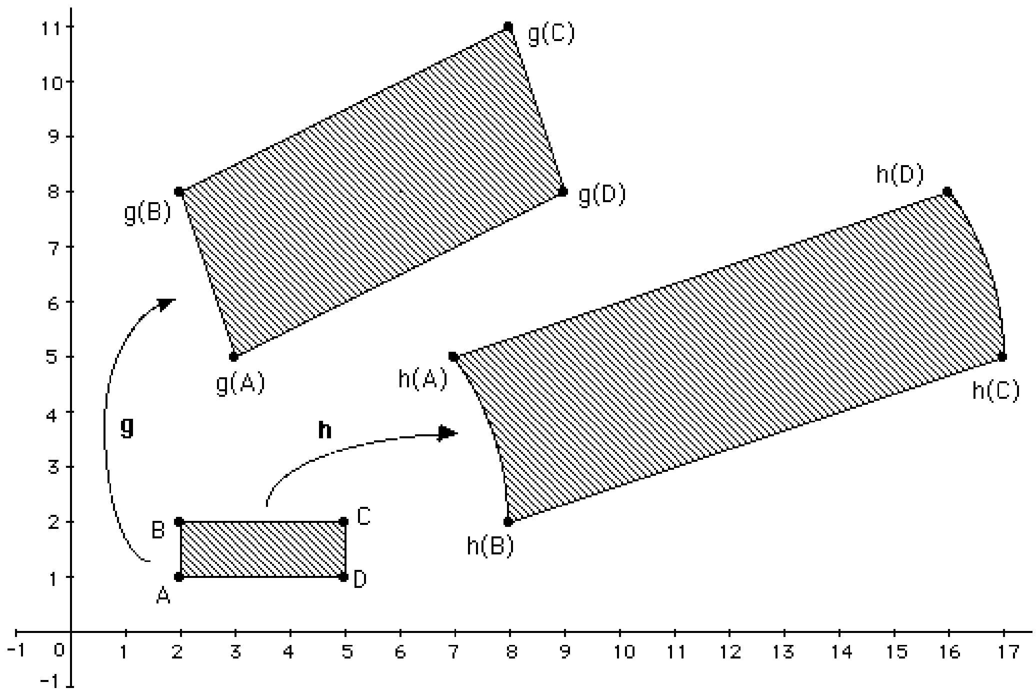

Fig. 1.4

1.0.5 Distortion and preservation. Looking at the three functions f, g, and h we have considered so far, we notice a progressive 'deterioration': f simply failed to preserve distances (mapping ABCD to a bigger rectangle), g failed to preserve right angles (but at least sent parallel lines to parallel lines), while h did not even preserve straight lines (it mapped AB and CD to curvy lines). Now that we have seen how 'bad' some (in fact most) functions can be, we may as well ask how 'good' they can get: are there any functions that preserve distances (**therefore** angles and shapes as well), satisfying **|AB| = |A′B′|** for every two points A, B on the plane?

The answer is "yes". Distance-preserving functions on the plane do exist, and we can even tell exactly what they look like: they are defined by formulas like **F(x, y) = (a′+b′x+c′y, d′+e′x+f′y)**, where a′, d′ are arbitrary, $b'^2+e'^2 = c'^2+f'^2 = 1$, and **either** f′ = b′, e′ = –c′ **or** f′ = –b′, e′ = c′! This is quite a strong claim, isn't it? Well, we will spend the rest of the chapter proving it, placing at the same time considerable emphasis on a **geometric** description of the involved functions. For the time being you may like to check what happens when b′ or c′ are equal to 0: what is the image of ABCD in these cases? (Look at specific examples involving situations like a′ = 3, b′ = 0, c′ = 1, d′ = 2, e′ = -1, f′ = 0 or a′ = -2, b′ = -1, c′ = 0, d′ = 4, e′ = 0, f′ = 1, and determine the images of A, B, C, D.)

1.0.6 What's in a name? You probably feel by now that such nice, distance and shape preserving functions like the ones mentioned above deserve to have a name of their own, don't you? Well, that name does exist and is probably Greek to you: **isometry**, from ***ison*** = "equal" and ***metron*** = "measure". The second term also lies at the root of "symmetry" = "syn" + "metron" = "plus" + "measure" (perfect measure, total harmony). In fact ancient Greek ***isometria*** simply meant "symmetry" or "equality", just like the older and more prevalent ***symmetria***. The term "isometry" with the meaning **"distance-preserving function"** entered English -- emulating somewhat earlier usage in French and German -- in 1941, with the publication of Birkhoff & Maclane's ***Survey of Modern Algebra***.

1.0.7 Isometries preserve straight lines! We claim that every isometry maps a straight line to a straight line. And, yes, one could prove this claim without even knowing (yet) what isometries look like, without having seen a single example of an isometry! In fact, one could prove that isometries preserve straight lines without even knowing for sure that isometries do exist!! This mathematical world can at times be a strange one, can't it? But how do we prove such an 'abstract' claim?

Well, a clever observation is crucial here: it suffices to show that every isometry maps three distinct **collinear** points to three distinct **collinear** points! Indeed, let's assume this 'subclaim' for now, and let's prove right below the following: every function (**not** necessarily an isometry!) that maps every three collinear points to three collinear points must also map every straight line L to (a **subset** of) a straight line L′. Once this is done, preservation of distances shows easily that the image of L actually **'fills'** L′.

Start with a straight line L and pick any two distinct points P_1, P_2 on it. These two points are mapped by our function to distinct points P'_1, P'_2 that certainly **define** a new line, call it L′. Now every other point P on L is collinear with P_1, P_2, therefore, by our subclaim above (still to be proven!), its image P′ is collinear with P'_1, P'_2, hence it lies on L′ (figure 1.5). That is, every point P on L is mapped to a point P′ on L′, hence L itself is mapped **'inside'** L′.

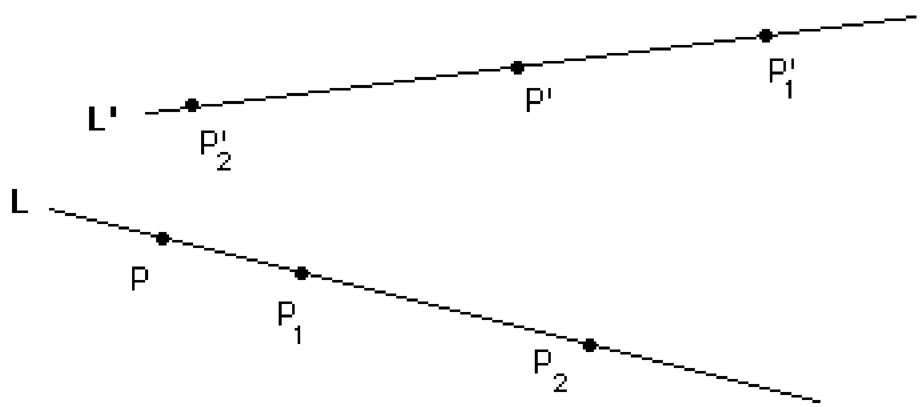

Fig. 1.5

So, how do we prove our subclaim that an isometry must always map collinear points to collinear points? Well, let A, B, C be three collinear points that are mapped to not necessarily collinear points A′, B′, C′, respectively (figure 1.6). We are dealing with an isometry, therefore |A′C′| = |AC|, |A′B′| = |AB|, and |B′C′| = |BC|. Since A, B, C are collinear, |AC| = |AB| + |BC|. But then |A′C′| = |AC| = |AB| + |BC| = |A′B′| + |B′C′|. We are forced to conclude that A′, B′, C′ must indeed be collinear: otherwise one side of the triangle A′B′C′ would be **equal** to the sum of the other two sides, violating the familiar **triangle inequality**.

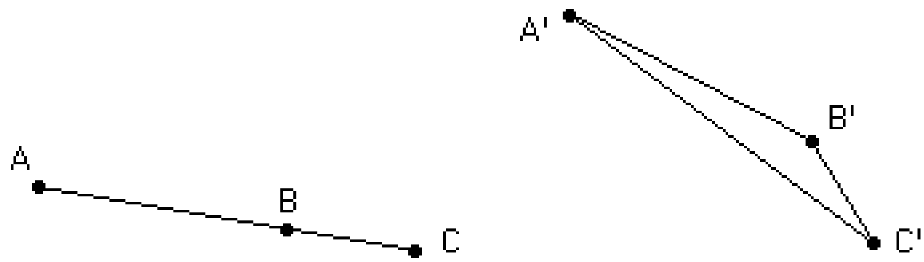

Fig. 1.6

1.0.8 Practical (drawing) issues. As you will experience in the coming sections, the fact that isometries map straight lines to straight lines makes life a whole lot easier: to draw the image of a straight line segment, for example, all you have to do is determine the images of the two **endpoints** and then **connect** them with a straight line segment. On our part, we will be repeatedly applying this principle throughout this chapter without specifically mentioning it.

On the other hand, determining the image of either a straight segment under a function that is not an isometry or a curvy segment under any function requires more work: one needs to determine the images of several points between the two endpoints and then connect them with a **rough sketch**. This is, for example, how h(AB) has been determined in 1.0.4: h(2, 1) = (7, 5), h(2, 1.2) = (7.2, 4.56), h(2, 1.4) = (7.4, 4.04), h(2, 1.6) = (7.6, 3.44), h(2, 1.8) = (7.8, 2.76), h(2, 2) = (8, 2). This is indeed a lot of work, especially when not done on a computer! Luckily, most images in this book are determined geometrically rather than algebraically; more to the point, most shapes under consideration will be quite simple geometrically, defined by straight lines.

1.0.9* How about parallel lines? Now that you have seen why isometries must map straight lines to straight lines, could you go one step further and prove that isometries must also map parallel lines to parallel lines? You can do this arguing **by contradiction**: suppose that parallel lines L_1, L_2 are mapped by an isometry to non-parallel lines L_1', L_2' intersecting each other at point K; can you then notice something impossible that happened to those **distinct** points K_1, K_2 (on L_1, L_2, respectively) that got mapped to K?

And if you are truly adventurous, can you prove, perhaps by contradiction, that whenever a function (not necessarily an isometry!) maps straight lines to straight lines it must also map parallel lines to parallel lines?

1.1 Translation

1.1.1 Example. Consider the triangles ABC, A′B′C′ below:

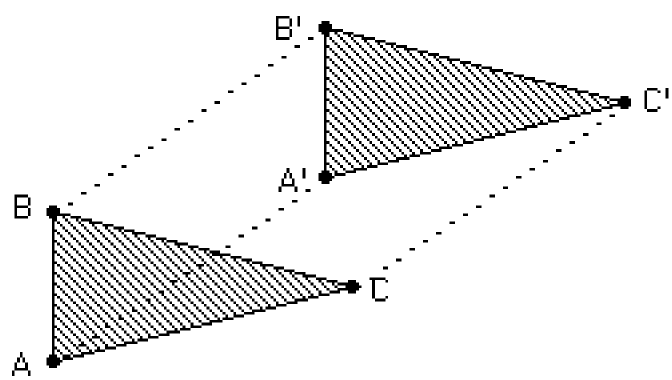

Fig. 1.7

Not only they are congruent to each other, but they also happen to be **'parallel'** to each other: AB, BC, and CA are parallel to A′B′, B′C′, and C′A′, respectively. This is a rather special situation, and what lies behind it is a **vector**.

1.1.2 Vectors. Familiar as it might be from Physics, a vector is a hard-to-define entity. It basically stands for a uniform motion that takes place all over the plane: every single point moves in the same **direction** (the vector's direction -- but have a look at 1.1.5, too) and by the same **distance** (the vector's length). In figure 1.7, for example, it is easy to see that every point of the triangle ABC has moved in the same southwest to northeast (SW-NE) direction by the same distance. We represent this motion by the ‘arrow’ below and call it a **translation** -- “transferring (ABC) to (A′B′C′)”, in the same way a **text** is **transferred** from one language **to** another -- defined by the vector $\vec{\mathbf{v}}$:

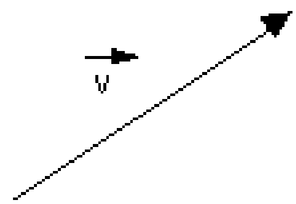

Fig. 1.8

Comparing figures 1.7 and 1.8, we easily conclude that every vector uniquely defines a translation and vice-versa. Notice also that the triangle A′B′C′ moves back to the triangle ABC by a translation **opposite** of the SW-NE one we already discussed, a translation defined by a NE-SW vector of **equal** length:

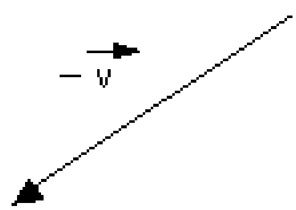

Fig. 1.9

1.1.3 It’s an isometry! While figure 1.7 makes it ‘obvious’ that every translation does preserve distances, it would be nice to actually have a proof of this claim. All we need to do is to show that if points A and B move by the **same** vector $\vec{v}$ to image points A′, B′ then **|A′B′| = |AB|.** But this is easy: as AA′ and BB′ are ‘by definition’ **parallel** and **equal** to each other, AB and A′B′ are by necessity the opposite, therefore equal (and parallel), sides of a **parallelogram**:

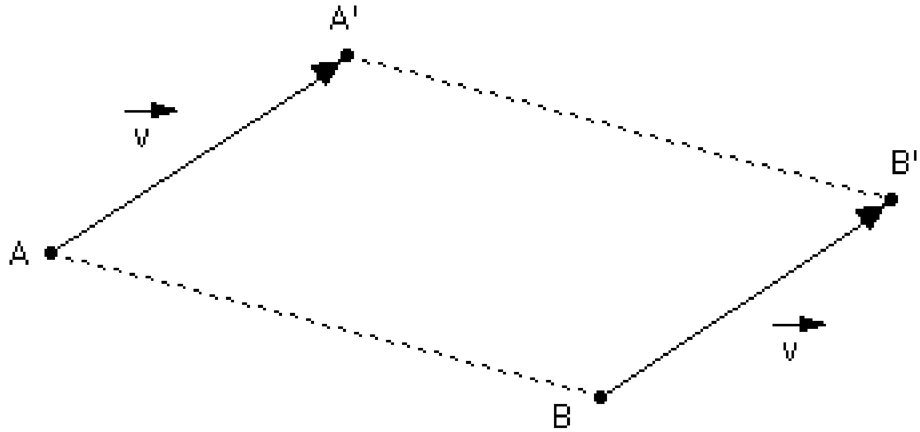

Fig. 1.10

1.1.4 Coordinates. Let us revisit 1.1.1, placing now figure 1.7 in a cartesian coordinate system, so that the coordinates of A, B, C and the **approximate** coordinates of A′, B′, C′ are as shown below:

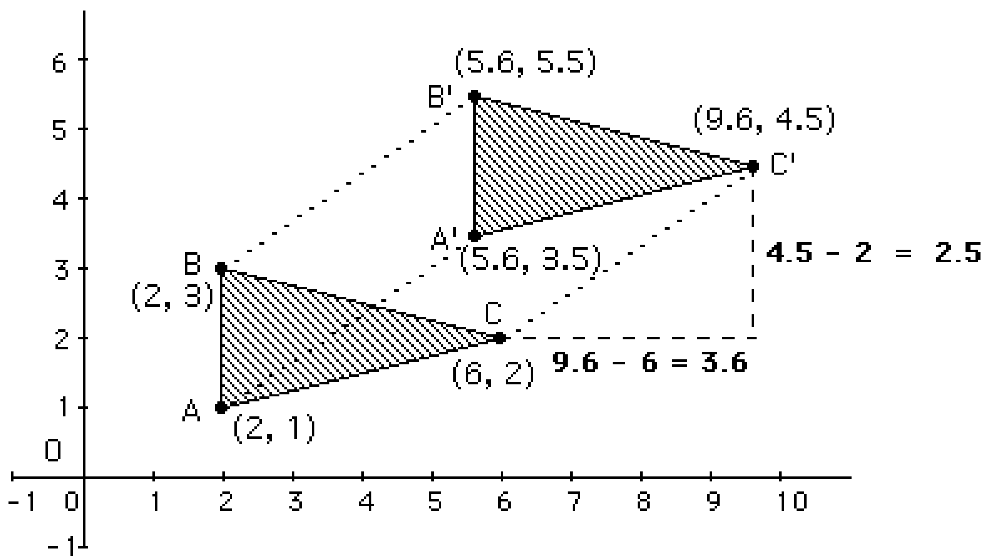

Fig. 1.11

It doesn’t take long to realize that our translation simply **adds** approximately 3.6 units to the x-coordinate of every point and approximately 2.5 units to the y-coordinate of every point; this is explicitly shown in figure 1.11 for C and C′. We call these two numbers **coordinates** of the translation vector, which we may now write as $\vec{\mathbf{v}} \approx$ **<3.6, 2.5>**. By the **Pythagorean Theorem**, the vector's length is approximately $\sqrt{3.6^2 + 2.5^2} \approx 4.3$. All this is further clarified in figure 1.12, which in particular shows how the translation vector's coordinates are determined by the image O′ of **(0, 0)**:

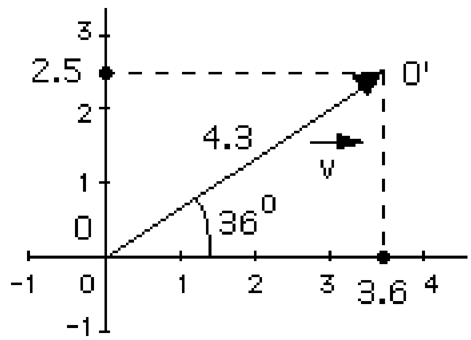

Fig. 1.12

That is, we may represent this translation employing the formula **T(x, y) = (3.6+x, 2.5+y)**. More generally, every translation on the plane may be represented by a formula of the form **T(x, y) = (a+x, b+y)**. Conversely, each formula of the form T(x, y) = (a+x, b+y) represents a translation defined by the vector **<a, b>**; sometimes we may even denote the translation itself by <a, b>. Observe that the opposite of the translation defined by the vector <a, b> is simply defined by the vector **<−a, −b>**. For example, the opposite translation of <3.6, 2.5> that we discussed in 1.1.2 is **<−3.6, −2.5>**.

1.1.5 <u>'Determining' a vector.</u> While figure 1.12 provides sufficient illustration on the relation between a vector's length, direction, and coordinates, just a bit of **Trigonometry** makes everything so much clearer! Indeed, since the vector $\vec{v}$ of length 4.3 makes a **vector-angle** of about 36^0 with the **positive** x-axis, its x-coordinate and y-coordinate are given by $4.3 \times \cos 36^0 \approx 4.3 \times .81 \approx 3.48$ and $4.3 \times \sin 36^0 \approx 4.3 \times .59 \approx 2.53$, respectively. While absolute precision has not been achieved, <3.48, 2.53> is indeed very close to <3.6, 2.5>. The quotient $2.53/3.48 \approx .73$ is the vector's **slope**, which is another quantitative way of describing the vector's direction. (Those who know a bit more know of course that this slope is equal to approximately $\tan 36^0$.)

Beware at this point of a simple, yet important, fact: while the two distinct vectors <3.6, 2.5> and <–3.6, –2.5> share the same direction, their slopes being equal (2.5/3.6 = (–2.5)/(–3.6)), they are of opposite **sense**, going opposite ways (as figures 1.8 & 1.9 demonstrate). Moreover, as we shall see in 1.4.7, **opposite** vectors have **distinct** vector-angles, in this case 36^0 and 216^0, respectively. So, it is important to remember that for every slope/direction there exist two distinct, and opposite of each other, senses. Notice at this point that any two vectors of equal length and same direction **and** sense are one and the same, while any two vectors of equal length and same direction might be either one and the same or opposite of each other.

1.2 Reflection

1.2.1 Mirrors create equals. Anyone who has ever successfully looked into a mirror is aware of this simple, as well as deep, natural phenomenon. Moreover, the closer you stand to a mirror, the closer you see your image in it -- another simple truth that even your cat is likely to be painfully aware of! In fact your **mirror image** lies precisely as far ‘inside’ the mirror as far away from it you stand: a fact used by many restaurants, bars, etc, to ‘double’ their perceived space. As mirrors or calm ponds cannot be included in books, we need a more abstract way of illustrating such natural observations, and we must indeed invent a ‘paper equivalent’ of a mirror!

1.2.2 Reflection axes. In order to ‘touch’ your mirror image inside a mirror you need to extend your hand toward the mirror **straight ahead**, so that it makes a **right angle** with the mirror, right? Well, this simple observation, together with the ones made in 1.2.1, helps us come up with the needed representation of a mirror on paper. The image P′ of a point P under **reflection** about the **axis** (mirror) L is found as figure 1.13 indicates:

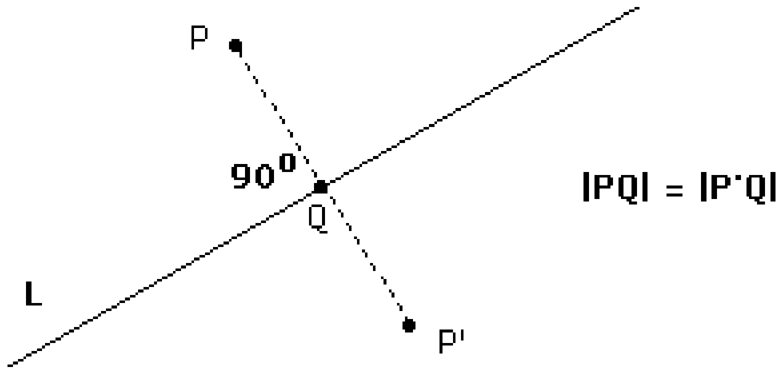

Fig. 1.13

That is, we get the same effect on P as if an actual mirror had somehow been placed perpendicularly to this page along L: you may of course go through such an experiment and see what happens!

1.2.3 Images. Let us return to triangle ABC of figure 1.7 and try to find its image under reflection by the straight line L in figure 1.14. We do that simply by determining the images A′, B′, C′ of vertices A, B, C and then connecting them to obtain the image triangle A′B′C′:

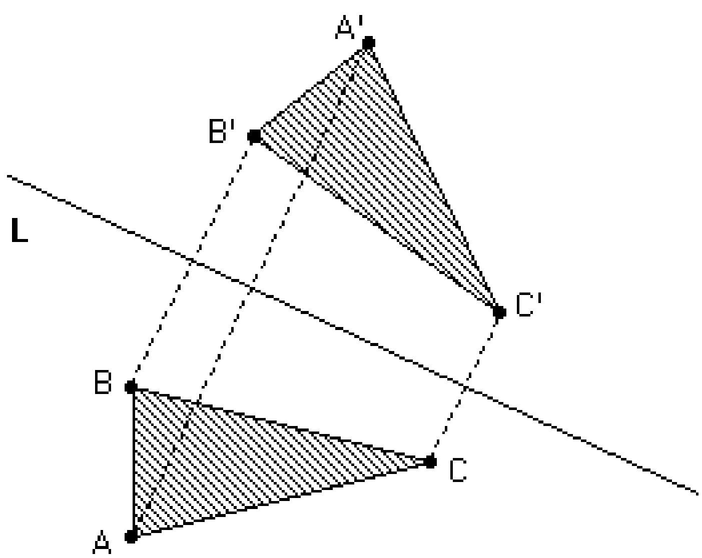

Fig. 1.14

1.2.4 It's an isometry! The two triangles in figure 1.14 certainly look congruent. This might not be as obvious as it was in the case of the two triangles of figure 1.7 -- we will elaborate on this in section 3.0 -- but, having three pairs of seemingly equal sides, ABC and A′B′C′ have to be congruent. How do we show that **|BC| = |B′C′|**, for example?

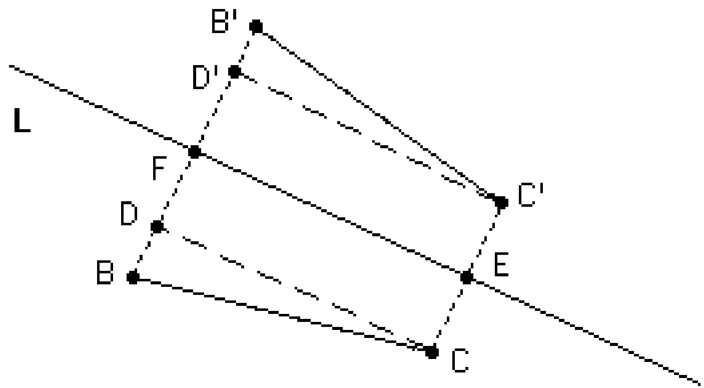

Fig. 1.15

Well, all we need to do is draw segments CD, C′D′ (both **parallel** to L and **perpendicular** to BB′, CC′) and notice, with the help of the **rectangles** FECD and FEC′D′ (figure 1.15), that |DF| = |CE| = |C′E| = |D′F|, therefore |DB| = |BF| – |DF| = |B′F| – |D′F| = |D′B′|, while |DC| = |FE| = |D′C′|: it follows that the two **right triangles** DBC and D′B′C′ are **congruent** (because |DB| = |D′B′| and |DC| = |D′C′|), hence |BC| = |B′C′|.

1.2.5 Coordinates. Let us now place triangles ABC, A′B′C′ and the axis L in a cartesian coordinate system (figure 1.16) and see what happens! You may use your straightedge to estimate the coordinates of A′, B′, and C′ and verify that (2, 1), (2, 3), and (6, 2) got mapped to **approximately** (5.1, 7.7), (3.6, 6.4), and (6.9, 4), respectively.

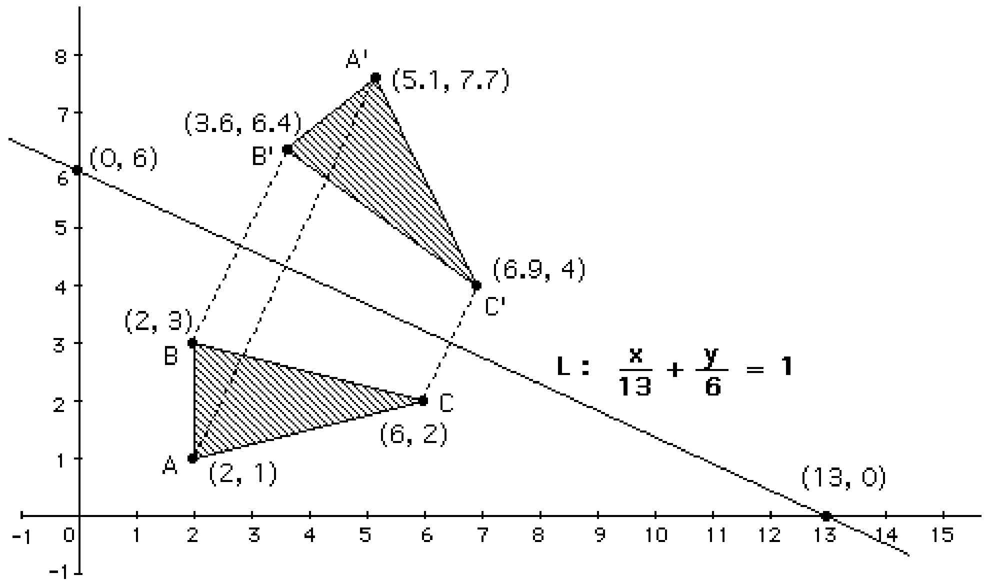

Fig. 1.16

Unlike in the case of translation, there is no obvious **algebraic** way of describing the transformation of coordinates observed above. This is of course no reason for giving up on determining the magic formula, if any, that lies behind this transformation of coordinates.

1.2.6 The reflection formula. Let L be a straight line with equation **ax + by = c** and M(x, y) be the mirror image of an arbitrary point (x, y) under reflection about L. Then ('magic formula')

M(x, y) =

$$= \left(\frac{2ac}{a^2+b^2} + \frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y,\quad \frac{2bc}{a^2+b^2} - \frac{2ab}{a^2+b^2}x - \frac{b^2-a^2}{a^2+b^2}y\right)$$

Proof*: Let **(x′, y′)** be the coordinates of M(x, y) and (x_1, y_1) be the coordinates of the **midpoint Q** of the segment connecting (x, y) and (x′, y′); Q lies, of course, **on** the mirror L (figure 1.17), while $x_1 = \frac{x+x'}{2}$ and $y_1 = \frac{y+y'}{2}$.

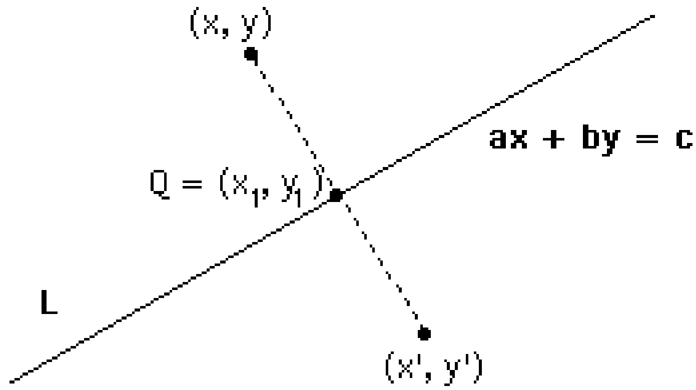

Fig. 1.17

Since $(x_1, y_1) = (\frac{x+x'}{2}, \frac{y+y'}{2})$ lies on the line $ax + by = c$, we obtain $a(\frac{x+x'}{2})+b(\frac{y+y'}{2}) = c$, therefore $a(x+x')+b(y+y') = 2c$, $ax+ax'+by+by' = 2c$, and, finally, $\mathbf{ax'+by' = 2c-ax-by}$ (I).

Next, observe that the line $ax + by = c$ (or, in equivalent form, and assuming $\mathbf{b \neq 0}$, $\mathbf{y = -\frac{a}{b}x + \frac{c}{b}}$) and the segment connecting (x, y) and (x', y') have **slopes** that are **negative reciprocals** of each other: indeed the line and the segment are **perpendicular** to each other. Since the line's slope is $\mathbf{-\frac{a}{b}}$ and the segment's slope is $\mathbf{\frac{y'-y}{x'-x}}$, we conclude that $\frac{y'-y}{x'-x} = \frac{b}{a}$, hence $a(y'-y) = b(x'-x)$, $ay'-ay = bx'-bx$ and, finally, $\mathbf{bx'-ay' = bx-ay}$ (II).

Multiplying now (I) by **a** and (II) by **b** and adding the two products we get $(a^2x'+aby')+(b^2x'-aby') = (2ac-a^2x-aby)+(b^2x-aby)$, therefore $(a^2+b^2)x' = 2ac+(b^2-a^2)x-2aby$ and $\mathbf{x' = \frac{2ac}{a^2+b^2} + \frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y}$ (III). Very similarly (multiplying (I) by **b** and (II) by **–a**, etc) we see that $\mathbf{y' = \frac{2bc}{a^2+b^2} - \frac{2ab}{a^2+b^2}x - \frac{b^2-a^2}{a^2+b^2}y}$ (IV). Observe now that (III) and (IV) together yield the reflection formula we wished to establish.

When **b = 0**, one can see **directly** that the image of (x, y) under reflection about the **vertical** line $\mathbf{x = \frac{c}{a}}$ is $M(x, y) = \mathbf{(\frac{2c}{a} - x,\ y)}$; this observation does prove the reflection formula in the special case b = 0.

1.2.7 Let's check it out! Does an application of the reflection formula obtained in 1.2.6 confirm our empirical observations and coordinate **estimates** in 1.2.5? Well, it better, otherwise there is something wrong somewhere! Let's see: in figure 1.16 the reflection axis L passes through **(13, 0)** (x-intercept) and **(0, 6)** (y-intercept), hence its equation is $\mathbf{\frac{x}{13} + \frac{y}{6} = 1}$ or **6x + 13y = 78**; this leads to **a = 6**, **b = 13**, and **c = 78**, so that $a^2+b^2 = 6^2+13^2 = 36+169 = 205$, $b^2-a^2 = 13^2-6^2 = 169-36 = 133$, $2ab = 2 \times 13 \times 6 = 156$, $2ac = 2 \times 6 \times 78 = 936$ and $2bc = 2 \times 13 \times 78 = 2{,}028$. The reflection formula tells us that the image of a point (x, y) is given by

$$M(x, y) = (\frac{2ac}{a^2+b^2} + \frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y,\ \frac{2bc}{a^2+b^2} - \frac{2ab}{a^2+b^2}x - \frac{b^2-a^2}{a^2+b^2}y) =$$

$$= (\frac{936}{205} + \frac{133}{205}x - \frac{156}{205}y,\ \frac{2028}{205} - \frac{156}{205}x - \frac{133}{205}y) \approx$$

$$\approx \mathbf{(4.56 + .65x - .76y,\ 9.89 - .76x - .65y)}.$$

Therefore the image of, say, **(6, 2)** 'predicted' by our formula is $(4.56 + .65 \times 6 - .76 \times 2,\ 9.89 - .76 \times 6 - .65 \times 2) = (4.56+3.9-1.52,\ 9.89-4.56-1.3) =$ **(6.94, 4.03)**, which is marvelously close to our estimates in figure 1.16! (This is the check applied to C and C′; make sure you verify the reflection formula for A, A′ and B, B′, too.)

1.2.8 Crossing a mirror? One important aspect of reflection you will have to get used to is the fact that, unlike in the real world, a mirror can **cross** through an object (set) reflected about it, and vice versa. To find the image of a set that does intersect a mirror, all you have to do is apply the 'natural rules' outlined in 1.2.1 and 1.2.2 without worrying about 'physical realities'. Here is an example:

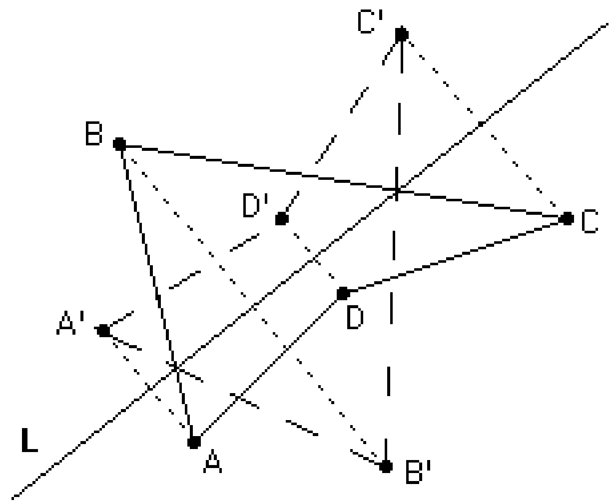

Fig. 1.18

Beware of ‘mirror-crossing’ cases where the mirror image falls back on the original set point by point: in such cases, which will become important in the rest of this book, we say that the set in question has an **internal mirror** and **mirror symmetry**. The following English letters have mirror symmetry: A, B, C, D, E, H, I, M, O, T, U, V, W, X, Y.

1.3 Rotation

1.3.1 How about a ‘time game’? Suppose that it is 9:40 (PM or AM) now that you are reading this paragraph. Take your watch in your

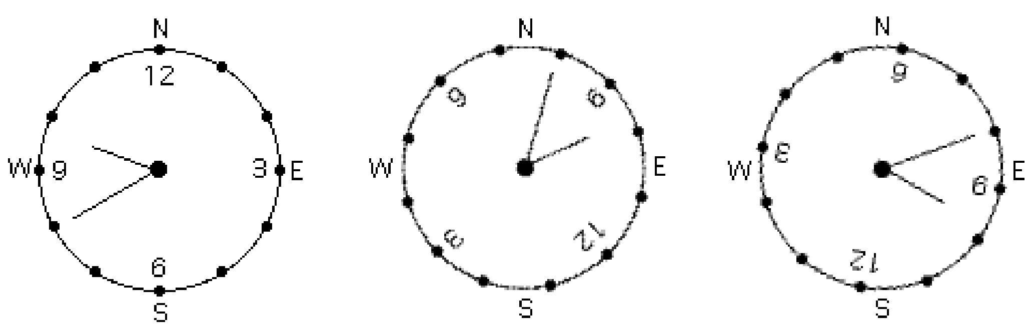

Fig. 1.19

hands, stop the time at 9:40 and start to slowly **turn** it around. You may do this on a piece of paper where you have already marked the positions of 12, 3, 6, and 9 as if on a compass (N, E, S, W); you may of course mark the other hours in between as well. There are positions where, as figure 1.19 demonstrates, your turning watch's hands will approximately show 2:00 or 4:10, right? Could you tell how much you should turn your watch in order to 'attain' these times? Well, while a precise answer would require (just like the determination of the exact attainable times) some serious mathematical thinking, you should be able to handle the question as posed: a **clockwise** ('screwing') turn by 130^0 will make the watch show a time close to 2:00 (2:01′:49″ to be precise!), while a clockwise turn by 195^0 will 'change the time' from 9:40 to approximately 4:10 (in fact 4:12′:44″, but you don't have to worry about that right now).

1.3.2 <u>What happened to the watch?</u> While the 'time game' described in 1.3.1 can indeed get quite complicated, especially if played with precision, some simple facts about the stopped watch's 'condition' during its turning around are simple and indisputable: its **center** remained fixed, the **angle** between the two hands remained the same (50^0), and, last but not least, the **distance** between the tips of the two hands never changed. It seems that our watch-turning game preserves distances: could in fact be some kind of isometry, then?

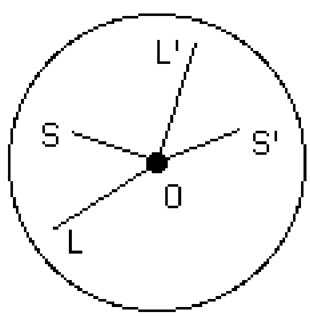

Fig. 1.20

Figure 1.20 describes the change of time from 9:40 to approximately 2:00 during our game: S and L represent the position of the tips of the watch's short and long hand, respectively, at 9:40, while S′ and L′ represent the position of those tips at about 2:00. One thing that cannot be missed is the fact that **both** angles LOL′ and SOS′ are equal to about 130^0. What really happened to the tips, and hands, and the entire watch in fact, is a clockwise **rotation** by an angle of approximately 130^0 about the point O (the watch's center).

1.3.3 How does rotation work? Even though figure 1.20 says it all, it would be useful to offer another example here, this time of a **counterclockwise** ('unscrewing') rotation **by** 70^0 **about** the center K shown in figure 1.21. How do we find the image P′ of any given point P under this rotation? We simply draw KP, measure it either with a ruler or with a compass, then 'build' a 70^0 angle 'to the left hand' of KP with the help of a protractor, and finally pick a point P′ on the angle's 'new' leg so that **|KP′| = |KP|**. That's all!

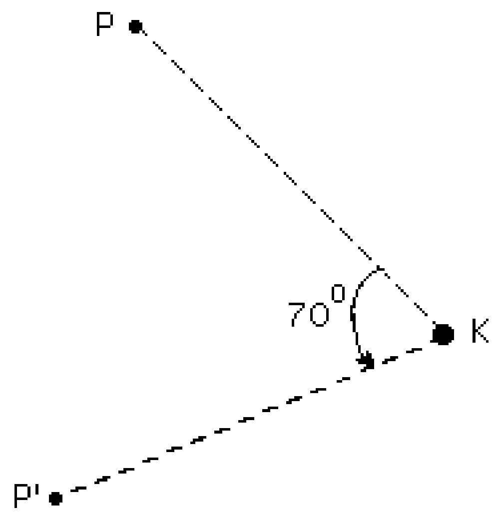

Fig. 1.21

1.3.4 It's an isometry! Let us now verify our conjecture in 1.3.2 and prove that every rotation is indeed an isometry. We return to our watch example and prove that **|LS| = |L′S′|**, which says that the distance between the two images L′, S′ is equal to the distance between the two original points L, S; the general case is proven in exactly the same way.

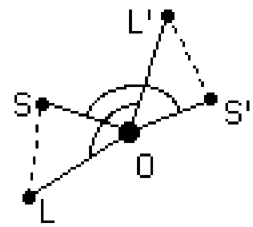

Fig. 1.22

Let us look at the two triangles OLS, OL′S′ (figure 1.22): they have two pairs of equal sides as **|OS| = |OS′|** (short hands) and **|OL| = |OL′|** (long hands). If we show the in-between angles ∠LOS and ∠L′OS′ to be equal, then the two triangles are congruent and, of course, |LS| = |L′S′|. But the equality of the two angles follows easily: ∠LOS = ∠LOL′–∠SOL′ = ∠SOS′ – ∠SOL′ = ∠L′OS′. (Note: **∠LOL′ = ∠SOS′ ≈ 130^0**.)

1.3.5 Images. Now that we know how rotation works, let us find the image of triangle ABC from 1.1.1 under rotation about the center K in figure 1.23 and by the counterclockwise 70^0 angle of 1.3.3. We do this by finding the images A′, B′, C′ of A, B, C (figure 1.23):

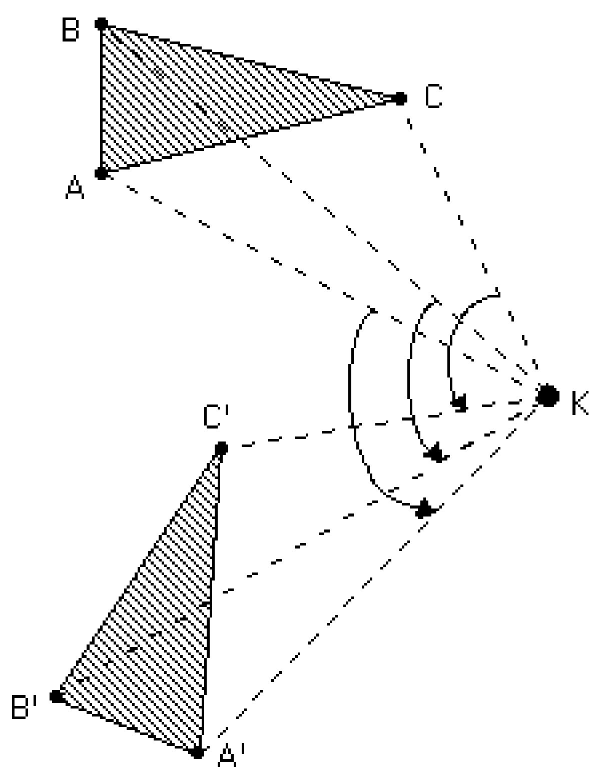

Fig. 1.23

1.3.6 Coordinates. Let us now place the triangles ABC, A′B′C′ of 1.3.5 in a cartesian coordinate system as shown in figure 1.24, so that the coordinates of A, B, C will again be (2, 1), (2, 3), (6, 2), respectively, those of the rotation center will be (8, −2). Estimating the coordinates of A′, B′, and C′ as in 1.2.5, we find them to be **approximately** (3.2, −6.8), (1.4, −6), and (3.6, −2.7), respectively.

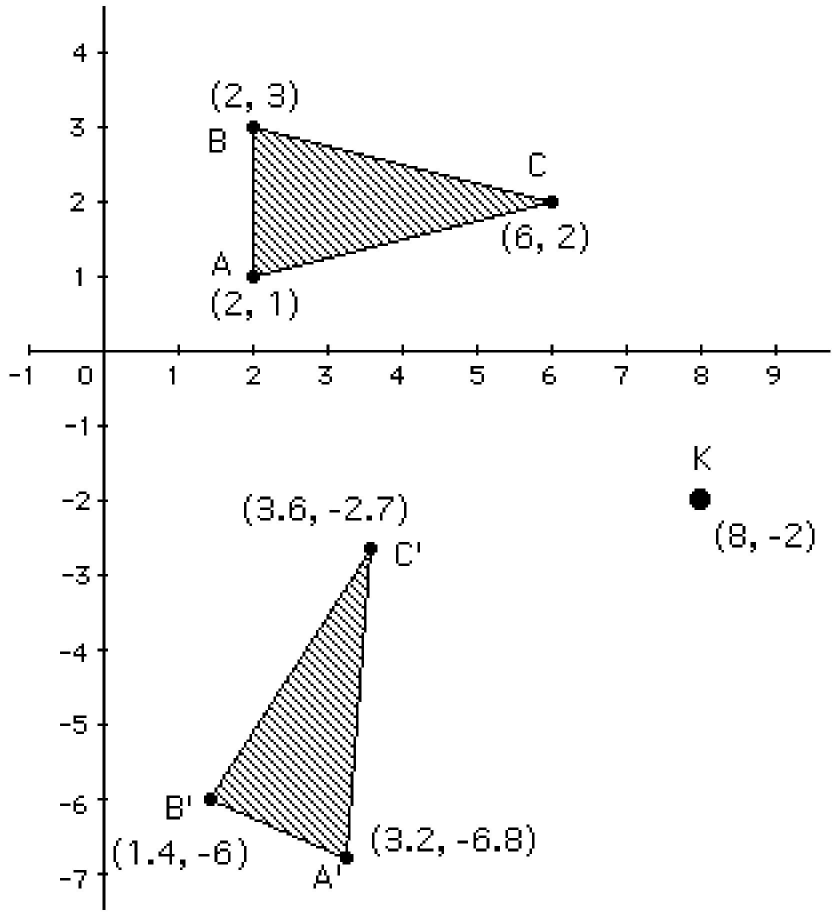

Fig. 1.24

Exactly as in 1.2.5, you might wonder whether there could possibly exist a ‘magic formula’ that could ‘predict’ the coordinates estimated above. Such a formula does exist, but its derivation is even harder than that of the reflection formula in 1.2.6 and can only be understood with some knowledge of Trigonometry.

1.3.7 The rotation formula. Let **R(x, y) = (x′, y′)** be the image of an arbitrary point (x, y) under rotation about the **rotation**

center K = (a, b) by a **rotation angle** ϕ. Then either

$$x' = (1-\cos\phi)a - (\sin\phi)b + (\cos\phi)x + (\sin\phi)y,$$

$$y' = (\sin\phi)a + (1-\cos\phi)b - (\sin\phi)x + (\cos\phi)y$$

(in case ϕ is **clockwise**) or

$$x' = (1-\cos\phi)a + (\sin\phi)b + (\cos\phi)x - (\sin\phi)y,$$

$$y' = -(\sin\phi)a + (1-\cos\phi)b + (\sin\phi)x + (\cos\phi)y$$

(in case ϕ is **counterclockwise**).

These formulas are indeed complicated, almost hard to believe, aren't they? Well, for a quick check you may like to verify that, no matter what ϕ is, both formulas yield $x' = a$ and $y' = b$ when $x = a$ and $y = b$, that is, **R(a, b) = (a, b)**: indeed the center of every rotation remains **invariant** under that rotation! Moreover, the two formulas are really one and the same: mathematicians tend to view clockwise angles as **'negative'**; so, substituting ϕ by $-\phi$ in the second formula yields the first one via $\mathbf{\cos(-\phi) = \cos\phi}$, $\mathbf{\sin(-\phi) = -\sin\phi}$.

Proof*: We offer a complete proof for the case of **clockwise** ϕ and a basic hint for the very similar case of counterclockwise ϕ. Once again, some familiarity with basic trigonometric functions and identities will be assumed; do not get discouraged if this proof seems too hard for you, and do not hesitate to ask for some help!

Let $P = (x, y)$ be a point that clockwise rotation by angle ϕ about center $K = (a, b)$ maps to a point $P' = (x', y')$, as shown in figure 1.25. Notice, referring to figure 1.25 always, that $\mathbf{|KP'| = |KP|}$, $|GK| = |CA| = |OA| - |OC|$, and $|GP| = |BD| = |OD| - |OB|$; moreover, $\mathbf{\theta' = 180^0-\theta-\phi}$, hence $\cos\theta' = -\cos(\theta+\phi) = -\cos\theta\cos\phi + \sin\theta\sin\phi$ and $\sin\theta' = \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

Now $x' = |OE| = |OA| + |AE| = |OA| + |KH| = |OA| + \mathbf{|KP'|\cos\theta' =}$
$= |OA| - \mathbf{|KP|\cos\theta}\cos\phi + \mathbf{|KP|\sin\theta}\sin\phi = |OA| - |GK|\cos\phi + |GP|\sin\phi =$
$= |OA| - (|OA| - |OC|)\cos\phi + (|OD| - |OB|)\sin\phi =$
$= a - (a - x)\cos\phi + (y - b)\sin\phi =$

= (1–cosϕ)a – (sinϕ)b + (cosϕ)x + (sinϕ)y, as claimed.

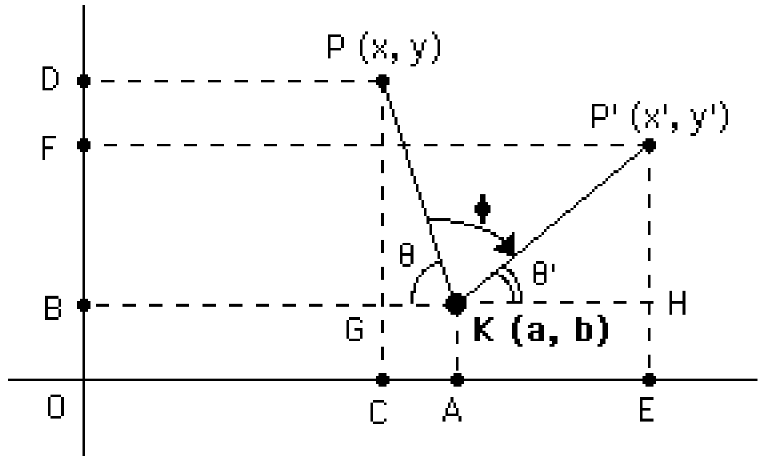

Fig. 1.25

Similarly, y′ = |OF| = |OB| + |BF| = |OB| + |HP′| = |OB| + **|KP′|sinθ′ =**
= |OB| + **|KP|sinθ**cosϕ + **|KP|cosθ**sinϕ = |OB| + |GP|cosϕ + |GK|sinϕ =
= |OB| + (|OD| – |OB|)cosϕ + (|OA| – |OC|)sinϕ =
= b + (y – b)cosϕ + (a – x)sinϕ =
= (sinϕ)a + (1–cosϕ)b – (sinϕ)x + (cosϕ)y, as claimed.

When ϕ happens to be **counterclockwise**, figure 1.25 changes into figure 1.26 below: now **θ′ = θ–ϕ**, hence cosθ′ = cosθcosϕ + sinθsinϕ and sinθ′ = sinθcosϕ – cosθsinϕ; |GK| = |OA| – |OC| and |GP| = = |OD| – |OB| remain valid. You should be able to fill in the details and derive the claimed formulas for x′ and y′.

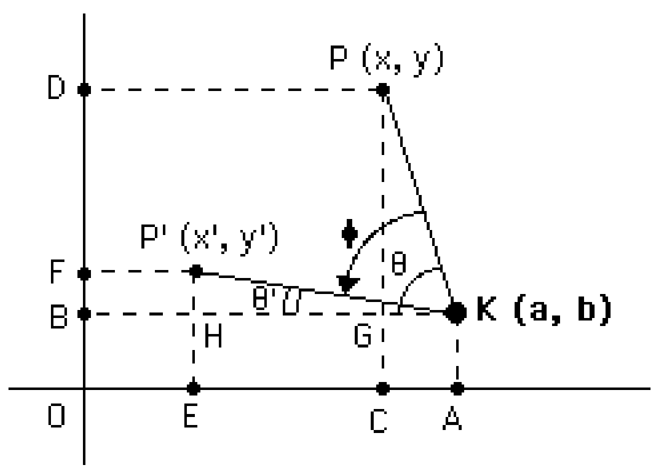

Fig. 1.26

There are in fact more cases to investigate, like when ϕ is counterclockwise and **larger** than θ, for example. Similar arguments left to you as exercises do work in all such cases, and our rotation formula works always.

1.3.8 Let's check it out! Let us now return to figure 1.24, augmented by A″B″C″, the **clockwise** image of ABC (figure 1.27). As we did in the cases of translation (1.1.4) and reflection (1.2.7), we would like to verify that **geometrical** estimates (1.3.6) and **algebraic** formulas (1.3.7) are in full agreement with each other.

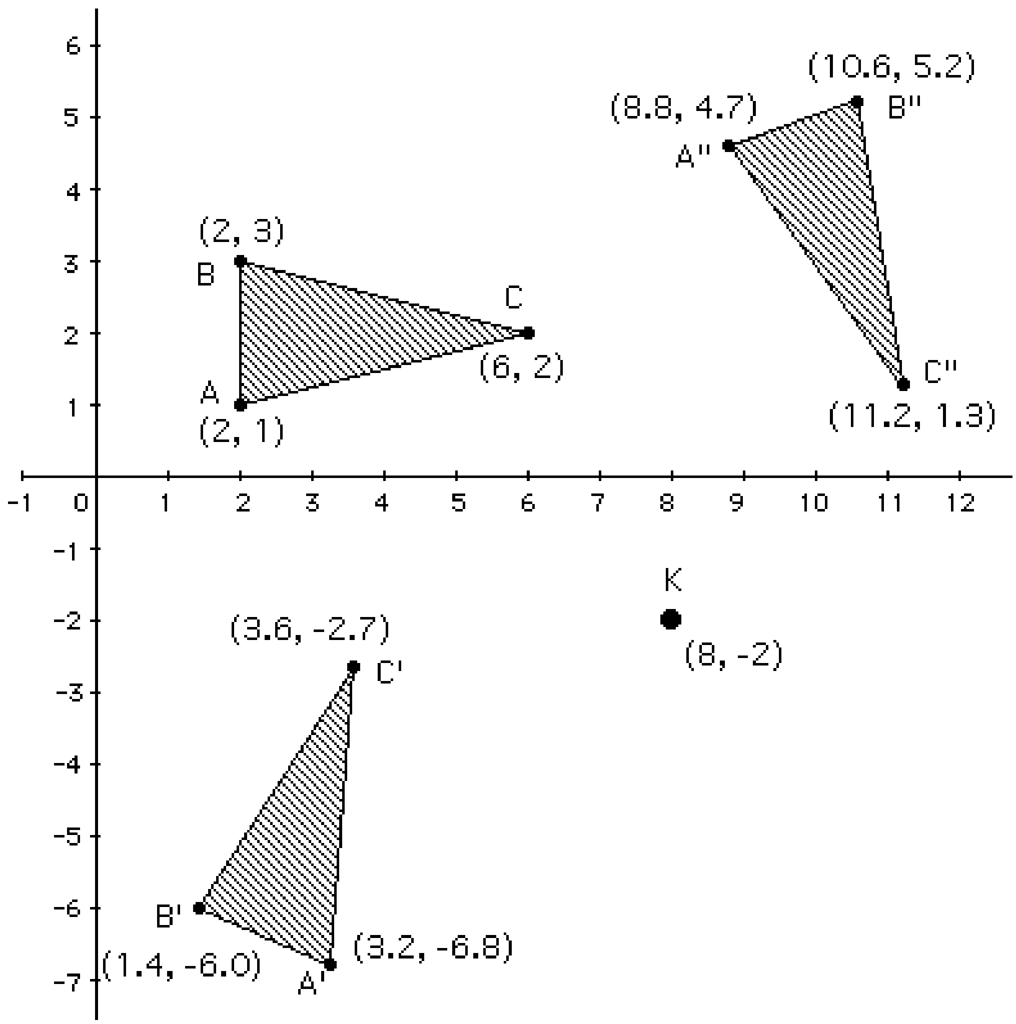

Fig. 1.27

Let's see: with $\cos 70^0 \approx .34$ and $\sin 70^0 \approx .94$, the

counterclockwise rotation formula for A = **(2, 1)** yields

$x' \approx (1-.34)\times 8 + .94\times(-2) + .34\times 2 - .94\times 1 = 5.28-1.88+.68-.94 =$ 3.14 and
$y' \approx -.94\times 8 + (1-.34)\times(-2) + .94\times 2 + .34\times 1 = -7.52-1.32+1.88+.34 = -6.62$,

therefore **R(2, 1) ≈ (3.1, –6.6)**, which is quite close indeed to that (3.2, –6.8) estimate in 1.3.6. Perhaps we could have achieved some greater precision with the use of more precise drawing and instruments, but such great precision will probably not be possible when you take your exam anyway...

Let us now see how things work out for A″, the **clockwise** image of A = **(2, 1)**. Coordinate estimates (figure 1.27) indicate that A″ ≈ (8.8, 4.7). The rotation formula yields

$x'' \approx (1-.34)\times 8 - .94\times(-2) + .34\times 2 + .94\times 1 = 5.28+1.88+.68+.94 = = 8.78$ and
$y'' \approx .94\times 8 + (1-.34)\times(-2) - .94\times 2 + .34\times 1 = 7.52-1.32-1.88+.34 = = 4.66$,

therefore **R(2, 1) ≈ (8.8, 4.7)**: our estimate (in fact our drawing) worked perfectly this time -- it happens!

By the way, a closer look at the preceding two examples should help you understand why the image of an **arbitrary** point (x, y) under rotation by 70^0 about (8, –2) is approximately **(7.16 + .34x + .94y, 6.2 – .94x + .34y)** in the clockwise case and **(3.4 + .34x – .94y, –8.84 + .94x + .34y)** in the counterclockwise case.

You should now get a bit more practice by near-matching formula outcomes and geometrical estimates for B′, C′, B″, and C″, redrawing the image triangles A′B′C′ and A″B″C″ in case you are not happy with our drawing: good luck!

1.3.9 'Interior' centers. Just as the reflection axis is allowed to cross a set that is reflected about it (1.2.8), the rotation center K could very well be **inside** a set rotated about it. Here is an example:

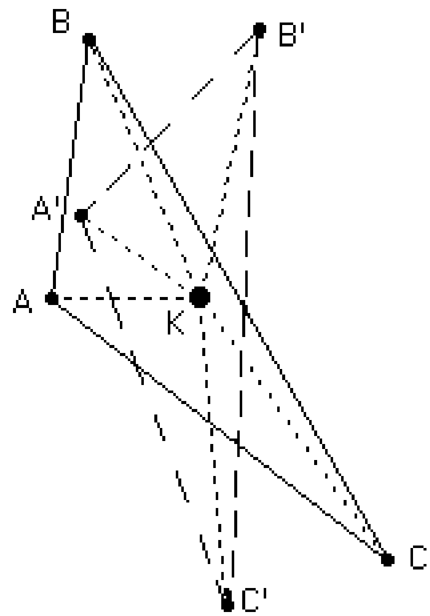

Fig. 1.28

When, in the case of such ‘internal centers’, the image falls, point by point, back on the original, we say that the figure in question has an **internal turn** and **rotational symmetry**. The following English letters have rotational symmetry (of 180^0, see 1.3.10): H, I, O, S, X, Z.

1.3.10 <u>A ‘straight’ rotation.</u> We have seen how important it is to know whether a rotation is clockwise or counterclockwise. There is however precisely one angle for which the distinction between clockwise and counterclockwise does not matter at all, and that is the **180^0** angle: **regardless** of which way the point P is rotated about the rotation center K, we end up with an image point P′ on the **extension** of the segment PK such that **|KP′| = |KP|** (figure 1.29).

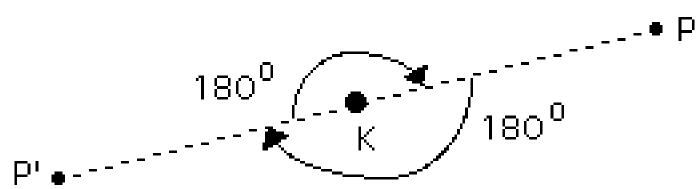

Fig. 1.29

This very special rotation by 180^0 is also known as **half turn** or **point reflection** -- we will be using either term at will -- and is destined to become very important in chapter 2 and beyond. For the time being, here is an example of half turn applied to the quadrilateral of figure 1.18:

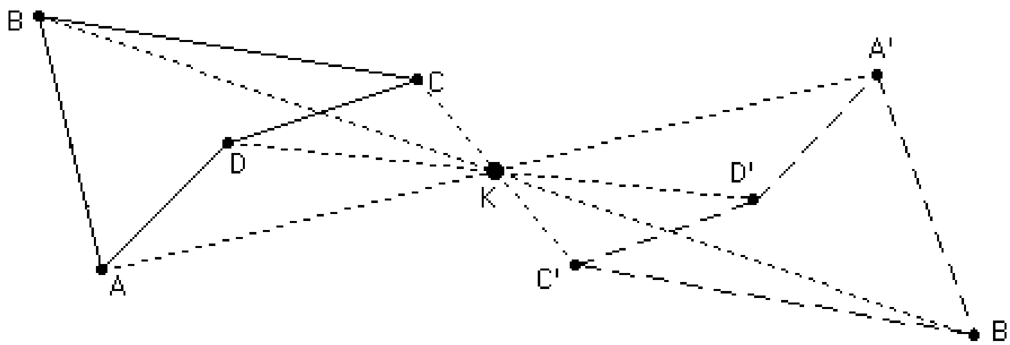

Fig.1.30

Notice here an important property of point reflection: it always maps a straight line segment to a straight line segment (equal and) **parallel** to it. You may confirm this with the help of figure 1.30 and/or a simple geometrical proof.

1.4 Glide reflection

1.4.1 <u>Is it a 'new' isometry?</u> Our fourth and last planar isometry is at the same time the least 'intuitive' -- you will truly understand it only after going through the next three chapters -- and the easiest one to introduce: how can this happen? The answer is simple: it is the **'combination'** of two already described isometries, translation and reflection, but it is not so clear in the beginning why anyone would ever bother to combine them!

1.4.2 <u>Axes and vectors.</u> All we need to describe the new isometry is, as hinted above, a reflection axis L and a translation vector $\vec{v}$ **parallel** to L: the image of an arbitrary point P is now found either by first reflecting about L to P_L and then **gliding**

(translating) along $\vec{v}$ to P′ or by first gliding along $\vec{v}$ to P_v and then reflecting about L to P′ (figure 1.31). That is, the order in which the two operations are performed does **not** affect the final outcome P′, the image of P under **glide reflection G = (L, $\vec{v}$)** by L **and** $\vec{v}$. We view glide reflection as a **'deferred reflection'** and use **dotted** lines for L (a 'half' (glided) mirror) and $\vec{v}$ (a 'half' (mirrored) glide) in order to stress their interdependence:

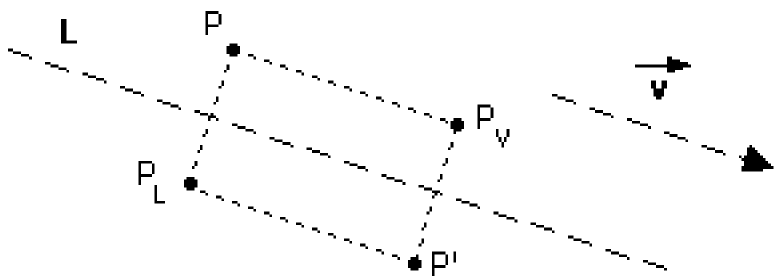

Fig. 1.31

Why do these two isometries, a reflection and a translation parallel to each other, **commute**? Figure 1.31 (and **rectangle** $\mathbf{PP_LP'P_V}$ in particular) makes that 'obvious', but it is worth stressing the role of parallelism: since $\vec{v}$ is parallel to L, the final image of P is bound to lie on a line parallel to L and at a distance from L **equal** to the distance from P to L, regardless of the order in which we performed the two operations. Observe at this point that a reflection and a translation **not** parallel to each other do **not** commute (figure 1.32);

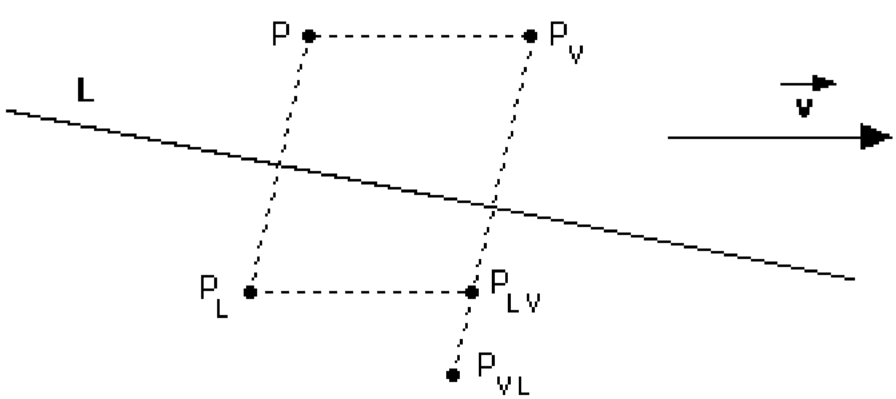

Fig. 1.32

that is generally the case whenever one tries to 'combine' any two isometries, as we will see in chapter 7. On the other hand, leaving something unchanged twice certainly preserves it: the combined effect of every two isometries is still an isometry, as each of the two isometries preserves all distances; in particular every glide reflection is indeed an isometry.

1.4.3 Images. Figure 1.33 demonstrates how one determines the image of the pentagon **S** under a glide reflection **G** = (**L**, $\vec{\mathbf{v}}$), as well as the commutativity between reflection and translation; the relation among the three isometries (translation, reflection, glide reflection) and the respective three images (**T(S)**, **M(S)**, **G(S)**) is shown clearly:

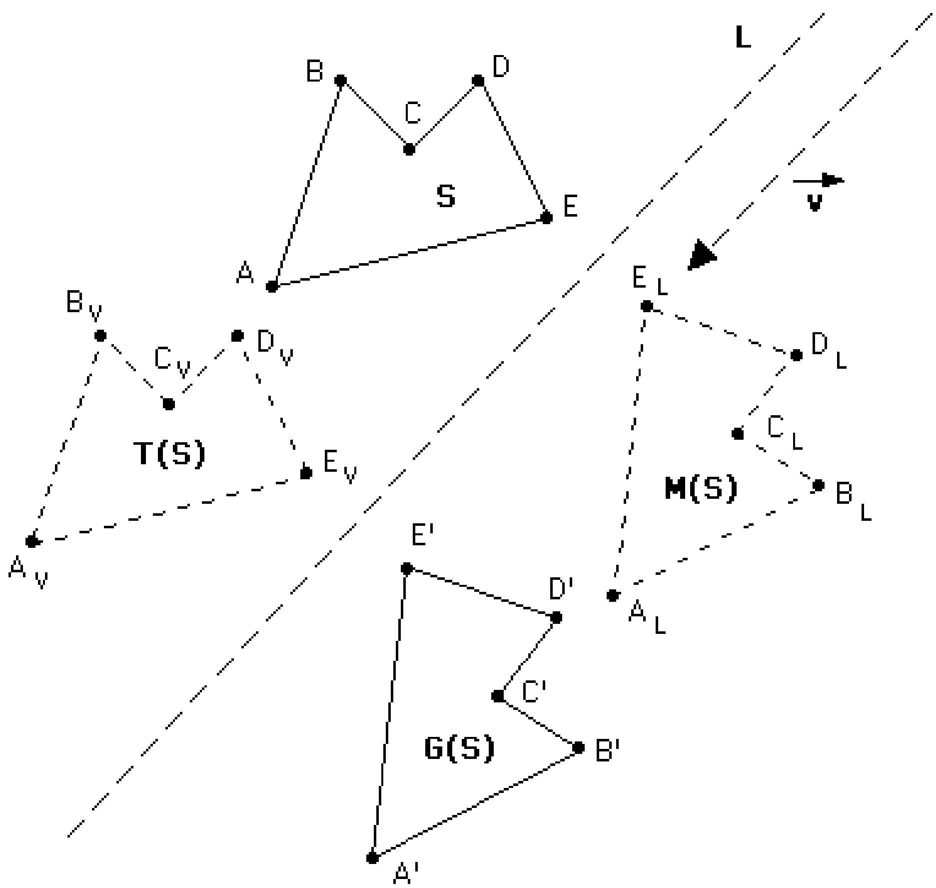

Fig. 1.33

1.4.4 Two opposite glide reflections. Let us once again revisit triangle ABC of figure 1.11, as well as the reflection axis **L** of figure 1.16, adding two vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ (figure 1.34): these are of **equal length** and of the **same direction** (parallel to L), but of **opposite sense**. The two vectors create two glide reflections **opposite** of each other, $\mathbf{G_1} = (\mathbf{L}, \vec{\mathbf{v}}_1)$ and $\mathbf{G_2} = (\mathbf{L}, \vec{\mathbf{v}}_2)$; the images A′B′C′, A″B″C″ of ABC under $\mathbf{G_1}$, $\mathbf{G_2}$, respectively, are shown below:

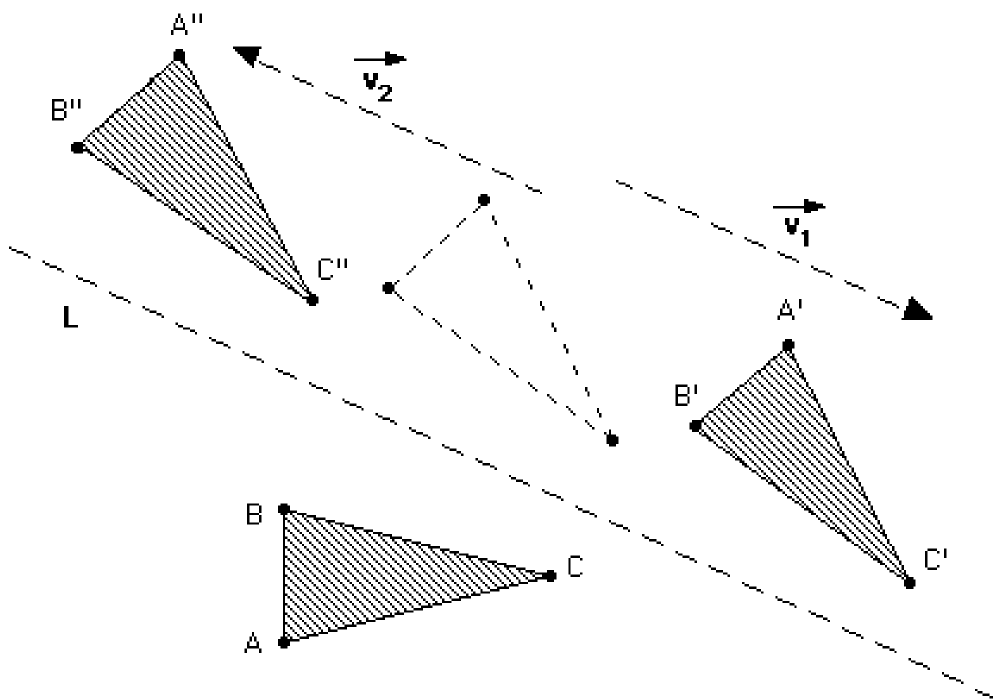

Fig. 1.34

What would have happened in case we **successively** applied $\mathbf{G_1}$ followed by $\mathbf{G_2}$ (or $\mathbf{G_2}$ followed by $\mathbf{G_1}$) to ABC? It shouldn't take you that long to realize that we would have gone first to A′B′C′ (or A″B″C″) and then back where we started from, ABC. This is why $\mathbf{G_1}$ and $\mathbf{G_2}$ are called **inverses** of each other: they simply cancel each other's effect, just as the two translations of 1.1.4 and the two rotations of 1.3.8 do. Notice by the way that every **reflection** is the **inverse of itself**, and the same holds for every **half turn**.

1.4.5 The glide reflection formula. Deriving a formula for the coordinates of the image point G(x, y) under a glide reflection is not that challenging in view of the work we have done in sections 1.1 and 1.2. We will in fact offer two formulas, one here (based on 1.1.4 and 1.2.6) and one in 1.4.7 (based on 1.1.5 and 1.2.6): unless you still have problems with basic Trigonometry you will probably find the formula in 1.4.7 easier to use, so you may certainly choose to read that section **first**.

Avoiding Trigonometry for now, let $\mathbf{ax + by = c}$ be the equation of the glide reflection axis L and let **<A, B>** be the glide reflection vector parallel to L. The slope **B/A** of <A, B> must be **equal** to the slope of L, which is **–a/b** (see 1.2.6); so we may and do write <A, B> as <bs, –as>, where **s** is a **parameter** that depends on the **vector's length S** via

$$S = \sqrt{A^2+B^2} = \sqrt{(bs)^2+(as)^2} = |s|\sqrt{a^2+b^2}.$$

That is, **<A, B> = <bs, –as>**, where $\mathbf{s = \pm\frac{S}{\sqrt{a^2+b^2}}}$. In the case of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ of 1.4.4, a = 6, b = 13, S = 5 (see 1.2.7 and figures 1.16 & 1.35), so $s = \pm 5/\sqrt{6^2+13^2} \approx \pm .35$; now $\mathbf{s = +.35}$ yields $\vec{\mathbf{v}}_1 \approx \langle 13 \times .35, -6 \times .35\rangle = \mathbf{\langle 4.55, -2.1\rangle}$, while $\mathbf{s = -.35}$ leads to $\vec{\mathbf{v}}_2 \approx \langle -13 \times .35, 6 \times .35\rangle = \mathbf{\langle -4.55, 2.1\rangle}$. We obtain approximately the same coordinates for $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ (like **<4.6, –2.1>** and **<–4.6, 2.1>**, as in figure 1.35) following the procedures outlined in figures 1.11 or 1.12.

Now we combine the **reflection** formula from 1.2.6 and the **translation** formula from 1.1.4 to obtain **G(x, y) = (x′, y′)**, where

$$\mathbf{x' = bs + \frac{2ac}{a^2+b^2} + \frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y,}$$

$$y' = -as + \frac{2bc}{a^2+b^2} - \frac{2ab}{a^2+b^2}x - \frac{b^2-a^2}{a^2+b^2}y.$$

So, all we did was to apply the reflection first and then **add** the translation effect coordinatewise! We still had to do a bit of work, of course, and that was the determination of the translation vector's **coordinates**.

1.4.6 Let's check it out! The game is perfectly familiar by now: we redraw figure 1.34 in a cartesian coordinate system, estimate the coordinates of points and vectors alike (figure 1.35), and use this numerical input to confirm the validity of the glide reflection formula. The work has been largely done in 1.2.7 (where we computed the quotients $\frac{b^2-a^2}{a^2+b^2}$, $\frac{2ab}{a^2+b^2}$, $\frac{2ac}{a^2+b^2}$, and $\frac{2bc}{a^2+b^2}$ for a = 6, b = 13, and c = 78 in order to derive the reflection part of the formula) and 1.4.5 (where we determined s and the two vectors of length 5 that are parallel to the axis 6x + 13y = 78). Combining everything, we obtain

$\mathbf{G_1(x,\ y)}$ = (4.55 + 4.56 + .65x – .76y, –2.1 + 9.89 – .76x – .65y) = **(9.11 + .65x – .76y, 7.79 – .76x – .65y)** and

$\mathbf{G_2(x,\ y)}$ = (–4.55 + 4.56 + .65x – .76y, 2.1 + 9.89 – .76x – .65y) = **(0.01 + .65x – .76y, 11.99 – .76x – .65y)**.

Applying these formulas to A and B, respectively, we obtain $\mathbf{G_1(2,\ 1) = (9.65,\ 5.62)}$ for A′ and $\mathbf{G_2(2,\ 3) = (-.97,\ 8.52)}$ for B″, which are quite close to our geometrical estimates below:

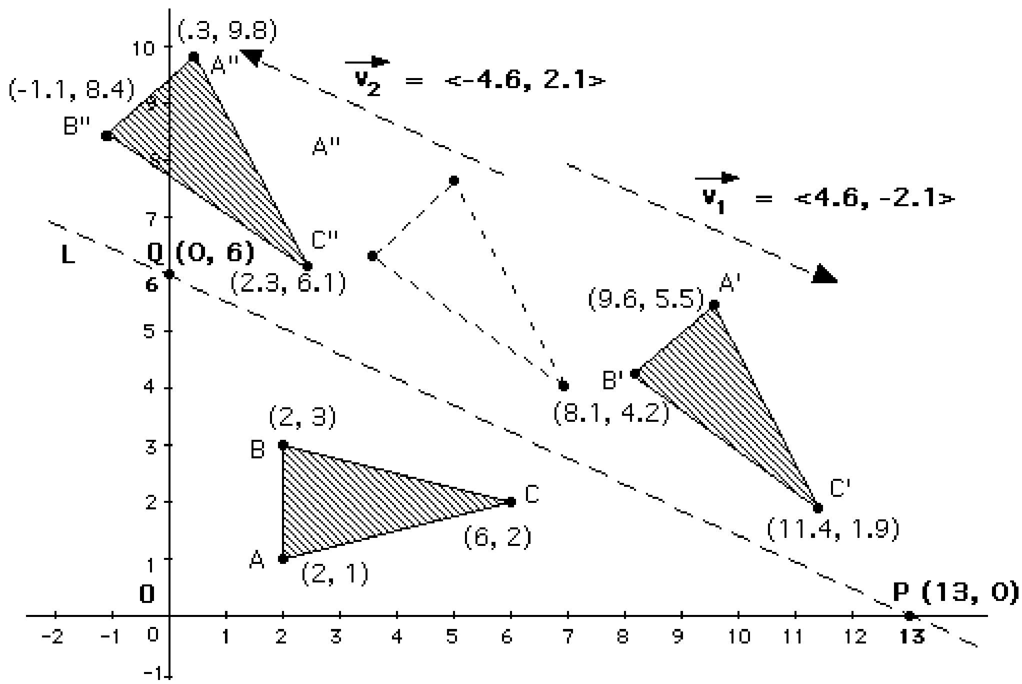

Fig. 1.35

1.4.7 <u>Alternative formula.</u> You certainly know that, more often than not, there is a gap between theory and practice. In the present context we point out that, while, in theory, the formula in 1.4.5 nicely expresses the glide reflection vector's coordinates in terms of the glide reflection axis' equation's coefficients, in practice determining the parameter **s** (and its **sign**) is quite complicated. It turns out that, as we promised in 1.4.5, Trigonometry offers a quick rescue.

Indeed, going back to 1.1.5, we recall that every vector may be written as **$\langle S\cdot\cos\theta, S\cdot\sin\theta\rangle$**, where **S** is the vector's length and **θ** is the vector-angle, that is the **counterclockwise** angle between the vector and the **positive** x-axis. In the case of the two opposite gliding vectors $\vec{v}_1$, $\vec{v}_2$ of 1.4.5, our method is fully illustrated in figure 1.36:

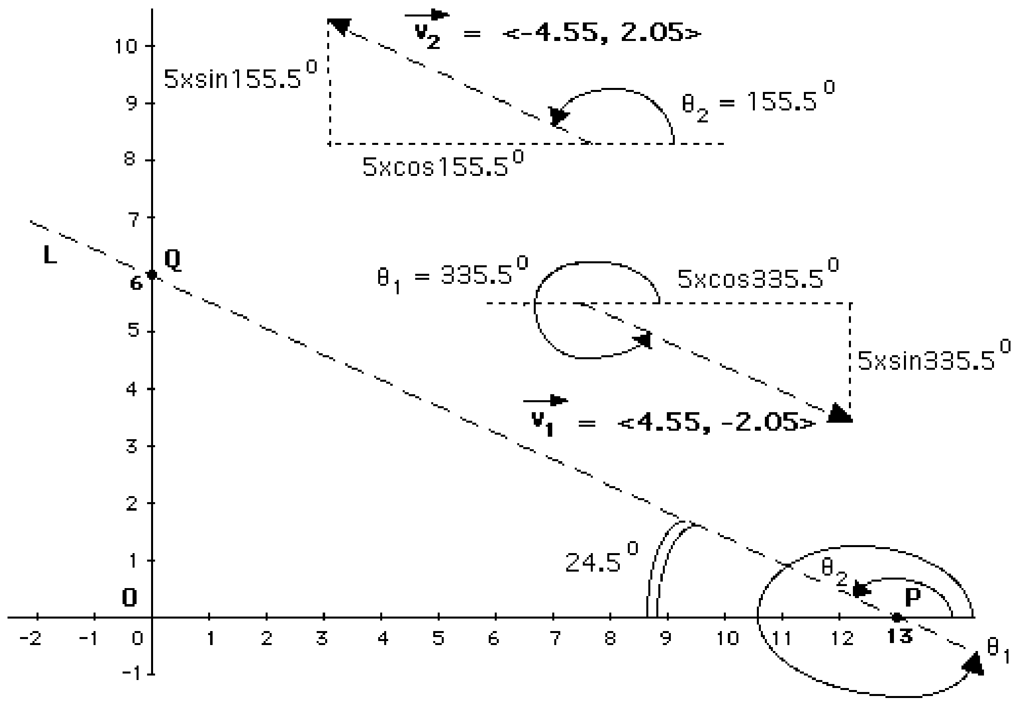

Fig. 1.36

That is, the fact that the glide reflection vectors $\vec{v}_1$, $\vec{v}_2$ are **parallel** to the glide reflection axis L **reduces** the angle's measurement, for both vectors, to simply measuring the **counterclockwise** angle between L and the **positive** x-axis (as shown in figure 1.36). As the vector-angles for $\vec{v}_2$ and $\vec{v}_1$ are $\boldsymbol{\theta_2} \approx 180^0 - 24.5^0 = 155.5^0$ and $\boldsymbol{\theta_1} = \theta_2 + 180^0 \approx 155.5^0 + 180^0 = 335.5^0$, we obtain $\cos\theta_2 \approx \cos(155.5^0) \approx -.91$ and $\sin\theta_2 \approx \sin(155.5^0) \approx .41$, so that $\cos\theta_1 = \cos(\theta_2+180^0) = -\cos\theta_2 \approx .91$ and $\sin\theta_1 = \sin(\theta_2+180^0) = -\sin\theta_2 \approx -.41$. It follows, with S = 5, that $\boldsymbol{\vec{v}_1 = <5\cdot\cos\theta_1,\ 5\cdot\sin\theta_1>} \approx <5 \times (.91),\ 5 \times (-.41)> = <4.55, -2.05>$ and $\boldsymbol{\vec{v}_2 = <5\cdot\cos\theta_2,\ 5\cdot\sin\theta_2>} \approx <5 \times (-.91),\ 5 \times (.41)> = <-4.55, 2.05>$: these are indeed very close to the vectors determined in 1.4.5.

From here on all there is to be done is to add the translation effect (as computed above) to the reflection effect (as determined in 1.2.6), obtaining **G(x, y) = (x′, y′)** with

$$x' = S\cdot\cos\theta + \frac{2ac}{a^2+b^2} + \frac{b^2-a^2}{a^2+b^2}x - \frac{2ab}{a^2+b^2}y,$$

$$y' = S\cdot\sin\theta + \frac{2bc}{a^2+b^2} - \frac{2ab}{a^2+b^2}x - \frac{b^2-a^2}{a^2+b^2}y,$$

where **S** is the gliding vector's length, θ is the gliding vector's vector-angle (as discussed right above and also in 1.1.5), and, again, **ax + by = c** is the equation of the glide reflection axis L. We leave it to you to check that, with vector-angles θ_1 and θ_2 for $G_1 = (L, \vec{v}_1)$ and $G_2 = (L, \vec{v}_2)$, respectively,

$$G_1(x, y) \approx (9.11 + .65x - .76y,\ 7.84 - .76x - .65y) \text{ and}$$

$$G_2(x, y) \approx (0.01 + .65x - .76y,\ 11.94 - .76x - .65y):$$

these formulas are certainly very close to those in 1.4.6.

Of course, those with a strong Trigonometry background should have no trouble seeing the **connection** between 1.4.6 and 1.4.7: indeed $\cos155.5^0 = -\cos24.5^0 \approx -\frac{|OP|}{|PQ|} = -\frac{13}{\sqrt{13^2+6^2}} \approx -.908$ and $\sin155.5^0 = \sin24.5^0 \approx \frac{|OQ|}{|PQ|} = \frac{6}{\sqrt{13^2+6^2}} \approx .419$. Moreover, they would know that the coordinates of P and Q yield a more **exact** value for the vector-angle via $\cos^{-1}(.908) \approx \sin^{-1}(.419) \approx \tan^{-1}(6/13) \approx 24.77^0$.

1.4.8 Reflections as glide reflections. Trivial as it might seem to you right now, this is a fact that is worth keeping in mind: every reflection may be seen as a 'degenerate' glide reflection the gliding vector of which has length **zero**. Indeed setting either **s = 0** in the glide reflection formula of 1.4.5 or **S = 0** in the glide reflection formula of 1.4.7 yields the reflection formula of 1.2.6.

1.5* Why precisely four planar isometries?

1.5.1 An old claim revisited. Back in 1.0.5 we promised to show that every isometry on the plane can be expressed via a formula like $F(x, y) = (a'+b'x+c'y, d'+e'x+f'y)$, where a' and d' are arbitrary, $\mathbf{b'^2+e'^2 = c'^2+f'^2 = 1}$, and either $\mathbf{f' = b'}$, $\mathbf{e' = -c'}$ or $\mathbf{f' = -b'}$, $\mathbf{e' = c'}$. Before we establish this claim (and more) in 1.5.4, let us prove that every function on the plane defined by such a formula is indeed an isometry. We do this using the **distance formula**: given any two points (x_1, y_1), (x_2, y_2), the distance between their images, $F(x_1, y_1) = (a'+b'x_1+c'y_1, d'+e'x_1+f'y_1)$ and $F(x_2, y_2) = (a'+b'x_2+c'y_2, d'+e'x_2+f'y_2)$, is

$$\sqrt{((a'+b'x_1+c'y_1)-(a'+b'x_2+c'y_2))^2 + ((d'+e'x_1+f'y_1)-(d'+e'x_2+f'y_2))^2}$$
$$= \sqrt{((b'(x_1-x_2) + c'(y_1-y_2))^2 + ((e'(x_1-x_2) + f'(y_1-y_2))^2}$$
$$= \sqrt{(b'^2+e'^2)(x_1-x_2)^2 + 2(b'c'+e'f')(x_1-x_2)(y_1-y_2) + (c'^2+f'^2)(y_1-y_2)^2}$$
$$= \sqrt{(x_1-x_2)^2 + 2(b'c'-b'c')(x_1-x_2)(y_1-y_2) + (y_1-y_2)^2}$$
$$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\text{, the distance between } (x_1, y_1) \text{ and } (x_2, y_2).$$

Notice at this point that, once (and if) we know that all isometries are **linear**, that is of the form $F(x, y) = (a'+b'x+c'y, d'+e'x+f'y)$, then it is not too difficult to show that they must be of the form conjectured in 1.0.5 (and restated above): you might be able to do this using the fact that all **three distances** among (1, 0), (0, 1), and (0, 0) must be preserved. But how do we show that every isometry is linear? One possible way to do that would be to first recall that every isometry maps straight lines to straight lines (1.0.7) and then try to prove that every planar function that preserves straight lines must indeed be linear: the latter happens to be true, but it's a real **theorem** the proof of which lies beyond the scope of this book.

We can actually show that every isometry is linear following a more direct path: first we record the particular way (**linear formula)** in which each one of the four isometries already studied is linear (1.5.2); then we show that every linear function expressed by one of the four linear formulas **must** actually be one of the four isometries already studied (1.5.3); and finally we prove that **every** isometry is expressed by one of the four linear formulas (1.5.4). That is, we will manage to show that all isometries are linear and must be one of the four isometries already studied ... **at the same time**!

1.5.2 Our bag of isometries. In the previous four sections we studied four planar functions (translation, reflection, rotation, and glide reflection) and showed each one of them to be an isometry. Our proof was purely geometrical in all four cases. Now we can provide algebraic proofs using the lemma we just established in 1.5.1! We do this by going back to the formulas derived in 1.1.4, 1.2.6, 1.3.7, and 1.4.5 and simply verifying that each of them satisfies the **isometry conditions** of 1.5.1:

Translation: $f' = b' = 1$, $e' = -c' = 0$, $a' = a$, $d' = b$.

Reflection: $f' = -b' = -\dfrac{b^2-a^2}{a^2+b^2}$, $e' = c' = -\dfrac{2ab}{a^2+b^2}$, $a' = \dfrac{2ac}{a^2+b^2}$, $d' = \dfrac{2bc}{a^2+b^2}$; $(b^2-a^2)^2 + (2ab)^2 = (a^2+b^2)^2$ implies $b'^2+e'^2 = c'^2+f'^2 = 1$.

Rotation: $f' = b' = \cos\phi$, $e' = -c' = \pm\sin\phi$, $a' = (1-\cos\phi)a + (\pm\sin\phi)b$, $d' = -(\pm\sin\phi)a + (1-\cos\phi)b$; $\cos^2\phi + \sin^2\phi = 1$ yields $b'^2+e'^2 = c'^2+f'^2 = 1$.

Glide reflection: $f' = -b' = -\dfrac{b^2-a^2}{a^2+b^2}$, $e' = c' = -\dfrac{2ab}{a^2+b^2}$,
$a' = bs + \dfrac{2ac}{a^2+b^2}$, $d' = -as + \dfrac{2bc}{a^2+b^2}$

1.5.3 'Going backwards'. The formulas summarized in 1.5.2 allow us to characterize any linear function $F(x, y) = (a'+b'x+c'y, d'+e'x+f'y)$ satisfying the isometry conditions of 1.5.1 as one of the four types of isometries we have encountered in this chapter; omitting the technical details involved (like solutions of 2×2 **linear systems**), we present the results as follows:

(I) A linear function $F(x, y) = (a'+b'x+c'y, d'+e'x+f'y)$ satisfying $\mathbf{f' = b' \neq \pm 1}$, $\mathbf{e' = -c' \neq 0}$, and $\mathbf{b'^2+e'^2 = c'^2+f'^2 = 1}$ is a **rotation** by (angle) $\mathbf{\cos^{-1}(b')}$ about (center) $\left[\dfrac{(1-b')a'+c'd'}{2(1-b')}, \dfrac{-a'c'+(1-b')d'}{2(1-b')}\right]$, clockwise if $\mathbf{c' < 0}$ and counterclockwise if $\mathbf{c' > 0}$; this rotation becomes a **half turn** about $\left(\dfrac{a'}{2}, \dfrac{d'}{2}\right)$ when $\mathbf{c' = 0}$, $\mathbf{b' = -1}$, and is reduced to a **translation** by $\mathbf{\langle a', d' \rangle}$ when $\mathbf{c' = 0}$, $\mathbf{b' = 1}$.

(II) A linear function $F(x, y) = (a'+b'x+c'y, d'+e'x+f'y)$ satisfying $\mathbf{f' = -b' \neq \pm 1}$, $\mathbf{e' = c' \neq 0}$, and $\mathbf{b'^2+e'^2 = c'^2+f'^2 = 1}$ is a **glide reflection** about (axis) $\mathbf{2(1-b')x - 2c'y = a'(1-b')-c'd'}$ by (vector) $\left\langle \dfrac{a'c'+(1-b')d'}{(1-b')^2+c'^2}\cdot c', \dfrac{a'c'+(1-b')d'}{(1-b')^2+c'^2}\cdot(1-b')\right\rangle$ when $\mathbf{(1-b')^2 + c'^2 \neq 0}$ and about (axis) $\mathbf{y = \dfrac{d'}{2}}$ by (vector) $\mathbf{\langle a', 0\rangle}$ when $\mathbf{c' = 0}$, $\mathbf{b' = 1}$; this glide reflection is reduced to a **reflection** when $\mathbf{a'c' + (1-b')d' = 0}$ (first case) or $\mathbf{a' = 0}$ (second case).

You should probably try to verify the validity of these claims and formulas by revisiting our old examples, like 1.2.7 (where $a' \approx 4.56$, $b' \approx .65$, $c' \approx -.76$, and $d' \approx 9.89$ do indeed satisfy the **reflection condition** $a'c' + (1-b')d' = 0$) or 1.3.8 (where either $a' \approx 7.16$, $b' \approx .34$, $c' \approx .94$, $d' \approx 6.2$ or $a' \approx 3.4$, $b' \approx .34$, $c' \approx -.94$, $d' \approx -8.84$ do indeed yield the **rotation center** via $\left[\dfrac{(1-b')a'+c'd'}{2(1-b')}, \dfrac{-a'c'+(1-b')d'}{2(1-b')}\right] = (8, -2)$).
More to the point, you may substitute a', b', c', d' by the 'general' values provided by the formulas in 1.5.2, and see what happens!

With these important observations (on the nature of the linear formulas associated with each one of the four known isometries) at hand, we are now ready to demonstrate why **every** planar isometry must be one of the four familiar ones: this is the kind of result that mathematicians affectionately call **classification theorem**.

1.5.4 Isometries and circles. Let us begin with a fundamental observation: every planar isometry is bound to map a circle of radius r to a circle of radius r. Indeed if the center O is mapped to O′, then every image point P′ must satisfy **|O′P′| = |OP| = r**. Consider now a fixed isometry that maps the unit circle, $x^2 + y^2 = 1$, to the circle $(x-a')^2 + (y-d')^2 = 1$, and the point $P_1 = (1, 0)$ to a point P'_1 (figure 1.37). Isometries map straight lines to straight lines (1.0.7), so the x-axis OP_1 is mapped to a line $O'P'_1$ that makes a **counterclockwise** angle ϕ with the **positive x-axis** (figure 1.37). Consider now the points **P = (r, 0)** and **Q = (rcosθ, rsinθ)** on the circle $x^2 + y^2 = r^2$, which is mapped to the circle $(x-a')^2 + (y-d')^2 = r^2$. Since P lies on OP_1, it must be mapped to the **unique** point P′ on the intersection of $O'P'_1$ and $(x-a')^2 + (y-d')^2 = r^2$ that satisfies $|P'P'_1| = |PP_1|$ (figure 1.37).

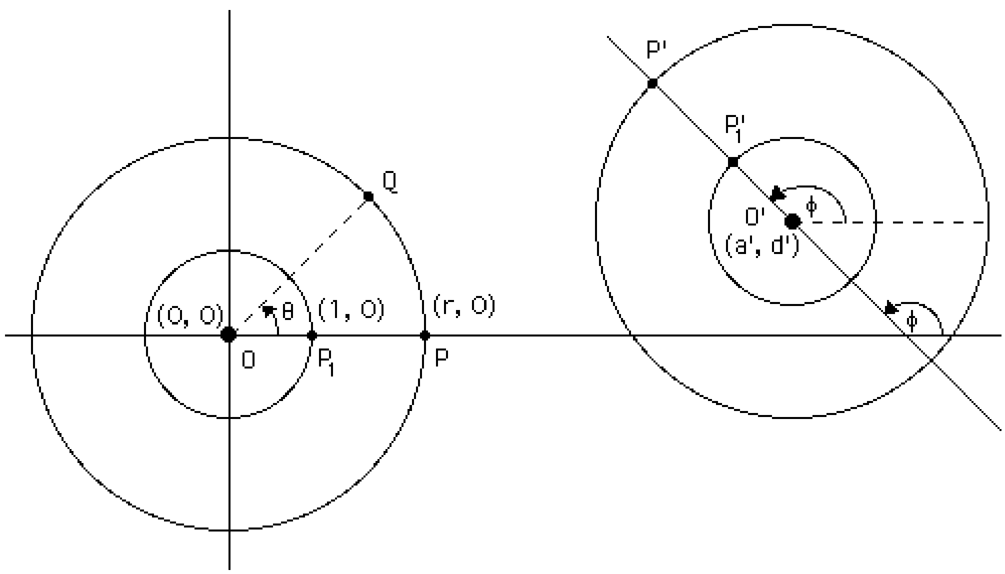

Fig. 1.37

The critical question is: where is Q mapped? Obviously to a point Q′ on $(x-a')^2 + (y-d')^2 = r^2$ such that **|P′Q′| = |PQ|**. But there isn't that much room on a circle, is there? If you are standing at P′ facing O′ and wish to move to **any** other point **on** the circle at a **given** distance from P′, how many choices do you have altogether? Precisely **two**: either you move **'to your left hand'** (making a

clockwise angle θ with O′P′) or you move **'to your right hand'** (making a **counterclockwise** angle θ with O′P′); these two possibilities are shown in figures 1.38 & 1.39, respectively. Moreover, it shouldn't take you long to realize that **all** points on $x^2 + y^2 = r^2$ are 'isometrically forced' to follow the fate of Q: we cannot have some points going clockwise and some points going counterclockwise!

In the first ('clockwise') case, $Q = (x, y) = (r\cos\theta, r\sin\theta)$ is mapped (figures 1.37 & 1.38, see also 1.3.7 and figure 1.25) to

$\mathbf{Q' = (a'+r\cos(\phi-\theta), d'+r\sin(\phi-\theta))}$ =

$= (a'+r\cos\phi\cos\theta+r\sin\phi\sin\theta, d'+r\sin\phi\cos\theta-r\cos\phi\sin\theta) =$

$= (a'+(\cos\phi)(r\cos\theta)+(\sin\phi)(r\sin\theta), d'+(\sin\phi)(r\cos\theta)+(-\cos\phi)(r\sin\theta))$

$= (a'+(\cos\phi)x+(\sin\phi)y, d'+(\sin\phi)x+(-\cos\phi)y) =$

$= (a'+b'x+c'y, d'+e'x+f'y)$, where $f' = -b' = -\cos\phi$, $e' = c' = \sin\phi$, $b'^2+e'^2 = c'^2+f'^2 = (\cos\phi)^2+(\sin\phi)^2 = 1$.

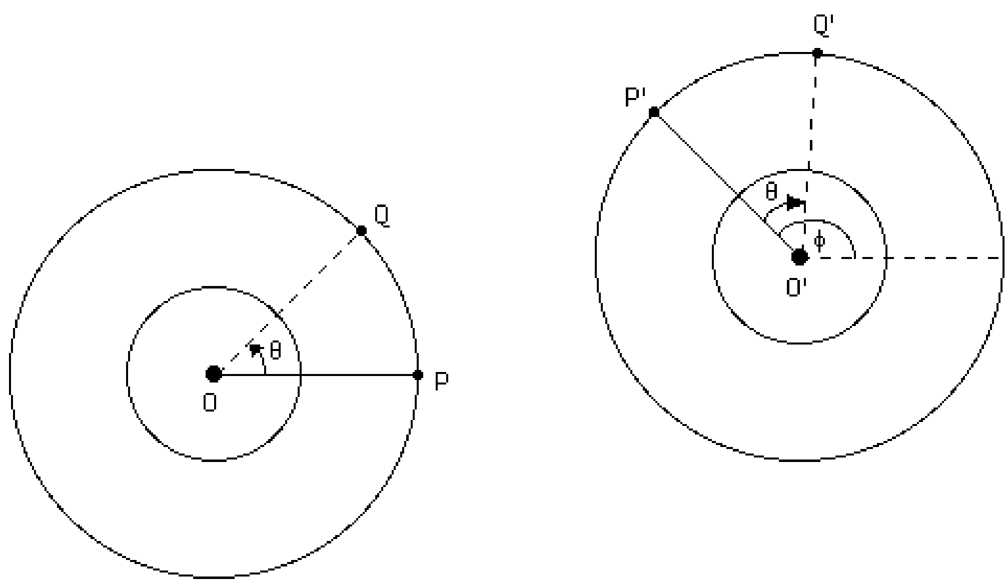

Fig. 1.38

The whole argument holds for **every r and every θ** (hence taking care of every single point (x, y) on the plane!) and is indeed very similar to what we did when we established the rotation formula in 1.3.7. At first you might even think that our isometry is in fact a rotation, but a careful look at the list of isometries in 1.5.2 shows otherwise: while rotations (and translations) satisfy

$f' = b'$ and $e' = -c'$, our isometry satisfies $\mathbf{f' = -b'}$ and $\mathbf{e' = c'}$, just as reflections and glide reflections do! To summarize, our isometry has to be either a **reflection** (in the special case $\mathbf{\tan\phi = \dfrac{2a'd'}{a'^2-d'^2}}$, as it follows from the conditions given in 1.5.3) or, far more likely, a **glide reflection** -- essentially because it maps 'counterclockwise circles' (think of the P-to-Q arc) to 'clockwise circles' (think of the P′-to-Q′ arc), formally because of our observations in 1.5.3.

In the second ('counterclockwise') case, $Q = (x, y) = (r\cos\theta, r\sin\theta)$ is mapped (figures 1.37 & 1.39, see also 1.3.7 and figure 1.26) to

$\mathbf{Q' = (a'+r\cos(\phi+\theta), d'+r\sin(\phi+\theta))}$ =

$= (a'+r\cos\phi\cos\theta-r\sin\phi\sin\theta, d'+r\sin\phi\cos\theta+r\cos\phi\sin\theta) =$

$= (a'+(\cos\phi)(r\cos\theta)+(-\sin\phi)(r\sin\theta), d'+(\sin\phi)(r\cos\theta)+(\cos\phi)(r\sin\theta))$

$= (a'+(\cos\phi)x+(-\sin\phi)y, d'+(\sin\phi)x+(\cos\phi)y) =$

$= (a'+b'x+c'y, d'+e'x+f'y)$, where $f' = b' = \cos\phi$, $e' = -c' = \sin\phi$, $b'^2+e'^2 = c'^2+f'^2 = (\cos\phi)^2+(\sin\phi)^2 = 1$.

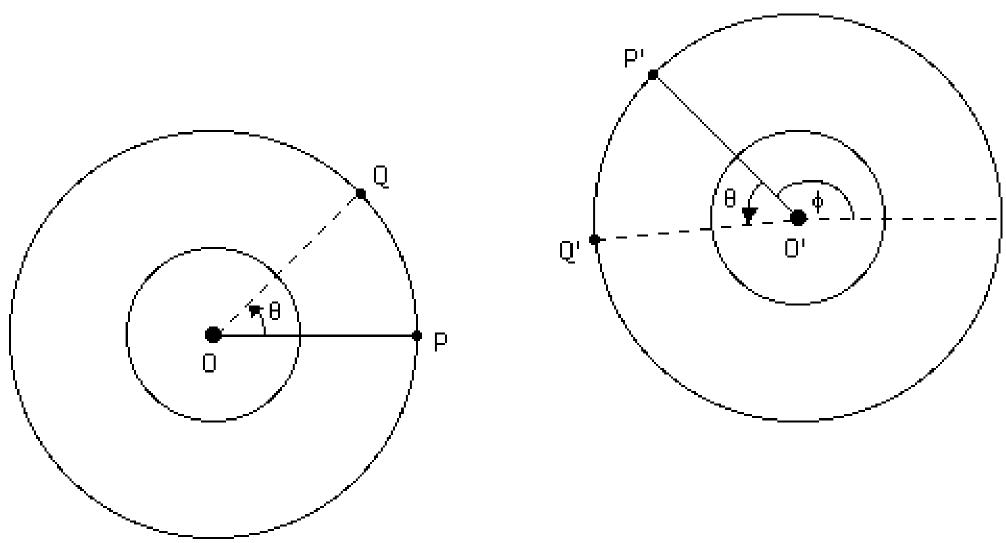

Fig.1.39

As in the first case, these computations are valid for **all r and all θ** and cover the entire plane. But this time our isometry maps 'counterclockwise circles' to 'counterclockwise circles' (think, as in figure 1.38, of the P-to-Q and P′-to-Q′ arcs) and 'looks identical' to a rotation! Is it one? Referring to 1.5.3 again, we see that yes, this time it is indeed a **rotation** (by angle ϕ), unless of course $\boldsymbol{\phi = 0^0}$, in which case $f' = b' = 1$, $e' = -c' = 0$ and our isometry is the **translation** <a′, d′> (which does **not** rotate circles at all)!

To summarize, we have shown that every planar isometry maps circles to circles and does so either **reversing circular orientation** (in which case it must be a glide reflection, or possibly a reflection) or **preserving circular orientation** (in which case it must be a rotation, or possibly a translation). We ended up both proving our claim from 1.0.5 about isometries being linear and classifying them! This is not the only way to classify isometries: probably it is not even the easiest one, see for example section 7.2. But it is a rather neat way to do it, at least for those with some familiarity with Precalculus. And those with greater such familiarity could even have more fun, like trying to determine the axis and vector (in the case of a glide reflection) or the center (in the case of a rotation) in terms of a′, d′, and ϕ (and in the spirit of 1.5.3), for example!

Postscript: It is possible to combine our ‘circular’ approach above with ideas from chapter 7 in order to provide a completely geometrical classification of isometries (not only of the plane but of space as well): please check ***Isometries Come in Circles*** at http://www.oswego.edu/~baloglou/103/circle-isometries.pdf.

first draft: summer 1998