PARA-KÄHLER MANIFOLDS OF QUASI-CONSTANT P-SECTIONAL CURVATURE *

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In the framework of para-Kähler manifolds endowed with a non-isotropic vector field ξ, we generalize the notion of constant P-sectional curvature [11] by quasi-constant P-sectional curvature, meaning that all P-planes making a certain angle with ξ have the same sectional curvature. Some characterizations and curvature properties are given.

1. Introduction

Para-Kähler manifolds are examples of symplectic, locally product and semi-Riemannian manifolds. A lot of authors gave their contributions on paracomplex geometry, as one can see in [6] and the references therein.

Definition 1.1 Let a manifold M be endowed with an almost product structure $P \neq \pm\text{Id}$, which is a $(1,1)$-tensor field such that $P^2 = \text{Id}$. We say that (M, P) (resp. (M, P, g)) is an almost product (resp. almost Hermitian) manifold, where g is a semi-Riemannian metric on M with respect to which P is skew-symmetric, that is

$$g(PX, Y) + g(X, PY) = 0, \ \forall X, Y \in \Gamma(TM). \tag{1.1}$$

Then (M, P, g) is para-Kähler if P is parallel w.r.t. the Levi-Civita connection of J. Some examples are given in [2].

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Several authors use the name of "hyperbolic" instead of "para", the first one being M. Prvanović.

The aim of the present note is to give in the para-Kählerian case a correspondent notion to the quasi-constant sectional curvature introduced in the Riemannian case in [4] (see also [10]) as well as to the quasi-constant holomorphic sectional curvature given in the Kählerian case in [3].

2. Para-Kähler manifolds of constant *P*-sectional curvature

Let $(M, P, g))$ be a para-Kähler manifold and let denote the curvature $(0,4)$-tensor field by $R(X,Y,Z,V) = g(R(X,Y)Z,V)$, $\forall X,Y,Z,V \in \Gamma(TM)$, where the Riemannian curvature $(1,3)$-tensor field associated to the Levi-Civita connection ∇ of g is given by $R = [\nabla, \nabla] - \nabla_{[\ ]}$. Then

$$\begin{aligned} R(X,Y,Z,V) &= -R(Y,X,Z,V) = -R(X,Y,V,Z) = \\ &= R(JX,JY,Z,V) \text{ and } \sum_{\sigma} R(X,Y,Z,V) = 0, \end{aligned} \tag{2.1}$$

where σ denotes the sum over all cyclic permutations.

In [11], M. Prvanović defined the following (0,4)-tensor field:

$$\begin{aligned} R_0(X,Y,Z,V) = \frac{1}{4}\{ & g(X,Z)g(Y,V) - g(X,V)g(Y,Z) - \\ & -g(X,PZ)g(Y,PV) + g(X,PV)g(Y,PZ) - \\ & -2g(X,PY)g(Z,PV)\}, \ \forall X,Y,Z,V \in \Gamma(TM). \end{aligned} \tag{2.2}$$

For any $p \in M$, a subspace $S \subset T_pM$ is called non-degenerate if g restricted to S is non-degenerate. If $\{u, v\}$ is a basis of a plane $\sigma \subset T_pM$, then σ is non-degenerate iff $g(u,u)g(v,v) - [g(u,v)]^2 \neq 0$. In this case the sectional curvature of $\sigma = \operatorname{span}\{u, v\}$ is

$$k(\sigma) = \frac{R(u,v,u,v)}{g(u,u)g(v,v) - [g(u,v)]^2}$$

From (1.1) it follows that X and PX are orthogonal for any $X \in \Gamma(TM)$. By a P-plane we mean a plane which is invariant by P. For any $p \in M$, a vector $u \in T_pM$ is isotropic provided $g(u,u) = 0$. If $u \in T_pM$ is not isotropic, then the sectional curvature $H(u)$ of the P-plane $\operatorname{span}\{u, Pu\}$ is called the P-sectional curvature defined by u. When $H(u)$ is constant, then (M, P, g) is called of constant P-sectional curvature, or a para-Kähler space form.

The following result is known, [7] and [11].

Theorem 2.1 *Let (M, P, g) be a para-Kähler manifold. Then for each*

$p \in M$, there exists $c(p) \in \mathbb{R}$ satisfying $H(u) = c(p)$ for any non-isotropic $u \in T_pM$ iff the Riemannian curvature R satisfies $R = cR_0$, where c is a function defined by $p \to c(p)$.

A Schur-type result is also valid. For the classification of para-Kähler space forms see [7], [8].

Theorem 2.2 [9] *A para-Kähler manifold M of dimension ≥ 4 is a para–Kähler space form iff M is Einstein and has zero Bochner flat.*

3. Algebraic calculus

In this section we denote by (M, P, g, ξ) a para-Kähler manifold endowed with a unit vector field ξ and we work in T_pM, where $p \in M$ is a fixed arbitrary point. Let $\sigma = \text{span}\{u, Pu\}$ be a P-plane. In particular, let denote $\varepsilon = \text{span}\{\xi_p, P\xi_p\}$. For any $\theta \in [0, \pi/2]$, let $P(\xi_p, \theta)$ denote the set of all P-planes in T_pM, making the angle θ with ξ_p. For instance $P(\xi_p, 0) = \{\varepsilon\}$.

Proposition 3.1 *If u is a non-isotropic vector in T_pM, then the angle of $\sigma = \text{span}\{u, Pu\}$ with ξ_p coincides with the angle of u with ε.*

Proof. We may assume that u is unitary. Then

$$\begin{aligned} &\not< (\xi_p, \sigma) = \theta \Longleftrightarrow [g(\xi_p, u)]^2 + [g(\xi_p, Pu)]^2 = \cos^2\theta \Longleftrightarrow \\ &\Longleftrightarrow [g(\xi_p, u)]^2 + [g(P\xi_p, u)]^2 = \cos^2\theta \Longleftrightarrow \theta = \not< (u, \varepsilon). \end{aligned} \tag{3.1}$$

Example 3.1 Let $(\mathbb{R}^{2n}, < >)$, $n \geq 2$, be the pseudo-Euclidean space, where $< x, y >= x^1y^1 + ... + x^ny^n - x^{n+1}y^{n+1} - ... - x^{2n}y^{2n}$, w.r.t. the standard frame $\{e_i\}_{i=\overline{1,2n}}$. Let P be the product structure defined such that $P(e_i) = e_{n+i}$, $i = \overline{1,n}$. If $\xi = e_1$ and $\sigma = \text{span}\{u, Pu\}$, where $u = (\sqrt{2}/2)(e_1 + e_2)$, then $\not< (\xi, \sigma) = \pi/4$.

From (3.1), for any $\theta \in [0, \pi/2]$, we have:

$$\begin{aligned} &P(\xi_p, \theta) = \{\text{span}\{u, P\}/u = \cos\theta[\cos\varphi \cdot \xi_p + \sin\varphi \cdot P\xi_p] + \sin\theta \cdot \ell, \\ &\forall\varphi \in \mathbb{R},\ \ell \text{ is a unit vector orthogonal to } \varepsilon\}. \end{aligned} \tag{3.2}$$

Lemma 3.2 *Let $\theta \in (0, \pi/2)$ and suppose $H(u)$ is the same for any vector u with $\not< (u, \varepsilon) = \theta$. Then any non-degenerate plane containing ξ_p and* *orthogonal to* $P\xi_p$ *is of constant curvature* s *iff* $H(\ell)$ *is constant for any* $\ell \in \varepsilon^{\perp}$. *In that case, if* u *and* ℓ *are unit vectors, we have*

(3.3) $$H(u) = \cos^4\theta \cdot H(\xi_p) + 8\cos^2\theta\sin^2\theta \cdot s + \sin^4\theta \cdot H(\ell).$$

This relation is trivial for $\theta = 0$ *or* $\pi/2$.

Proof. Let u and ℓ be unit vectors. From (2.1) we compute:

$$H(u) = R(u, Ju, Ju, u) = \cos^4\theta \cdot H(\xi_p) +$$
$$+4R(v(\varphi), Pv(\varphi), Pv(\varphi), w) + 6R(v(\varphi), Pw, Pw, v(\varphi)) +$$
$$+4R(v(\varphi), Pw, Pw, w) + 2R(v(\varphi), w, w, v(\varphi)) + \sin^4\theta \cdot H(\ell),$$

where $u = v(\varphi) + w$, with $v(\varphi) = \cos\theta[\cos\varphi \cdot \xi_p + \sin\varphi \cdot P\xi_p]$ and $w = \sin\theta \cdot \ell$, $\forall \varphi \in \mathbb{R}$.

Since $H(u)$ is the same for any $\varphi \in \mathbb{R}$, we replace φ by $\varphi + \pi$, which yields $v(\varphi) = -v(\varphi + \pi)$ and from (2.1) we obtain

$$H(u) = \cos^4\theta \cdot H(\xi_p) + 6R(v(\varphi), Pw, Pw, v(\varphi)) +$$
$$+2R(v(\varphi), w, w, v(\varphi)) + \sin^4\theta \cdot H(\ell).$$

If we replace φ by $\varphi + \pi/2$, then $v(\varphi + \pi/2) = Pv(\varphi)$ and from (2.1) we have:

$$H(u) = \cos^4\theta \cdot H(\xi_p) + 6R(v(\varphi), w, w, v(\varphi)) +$$
$$+2R(v(\varphi), Pw, Pw, v(\varphi))) + \sin^4\theta \cdot H(\ell).$$

The last two relations yields $R(v(\varphi), Pw, Pw, v(\varphi)) = R(v(\varphi), w, w, v(\varphi))$, which leads to

$$H(u) = \cos^4\theta \cdot H(\xi_p) + 8R(v(\varphi), w, w, v(\varphi)) + \sin^4\theta \cdot H(\ell).$$

As $R(v(\varphi), w, w, v(\varphi))$ is the same for any $\varphi \in \mathbb{R}$, then from (2.1) we obtain

$$R(v(\varphi), w, w, v(\varphi)) = \cos^2\theta \cdot \sin^2\theta \cdot R(\xi_p, \ell, \ell, \xi_p)$$

which yield (3.3) and the rest of the statement follows. The proof is complete.

By the help of the previous lemma, we obtain the following characterization.

Proposition 3.3 *Assume* $H(\ell)$ *is the same for any* $\ell \in \varepsilon^{\perp}$ *and let* $\theta \in (0, \pi/2)$. *Then* $H(u)$ *is constant for any unit vector* $u \in T_pM$ *with* $\measuredangle(u, \varepsilon) = \theta$ *iff there exist* $c_o(p), c_1(p), c_2(p) \in \mathbb{R}$ *such that*

(3.4) $$H(u) = c_0(p) + c_1(p)\cos^2\theta + c_2(p)\cos^4\theta.$$

4. Quasi-constant P-sectional curvature

We extend the notion of constant holomorphic sectional curvature to the following:

Definition 4.1 A para-Kähler manifold (M, P, g, ξ) is of quasi-constant P-sectional curvature if for any $p \in M$ and $\theta \in [0, \pi/2]$, the P-sectional curvature $H(u)$ is the same for any $u \in T_pM$, with $\not\measuredangle (u, \varepsilon) = \theta$.

From Proposition 3.3, we obtain

Proposition 4.2 *A para-Kähler manifold (M, P, g, ξ) is of quasi-constant P-sectional curvature iff there exist c_0, c_1, c_2 functions on M such that*

$$H(u) = c_0(p) + c_1(p)\cos^2\theta + c_2(\theta)\cos^4\theta, \tag{4.1}$$

for any unit vector $u \in T_pM$ with $\not\measuredangle (u, \varepsilon) = \theta,\ \theta \in [0, \pi/2],\ p \in M$.

Let $\eta \in \Gamma(T^*M)$ denote the dual form of ξ, i.e. $\eta(X) = g(X, \xi)$, $\forall X \in \Gamma(TM)$. We define the following $(0, 4)$-tensor fields:

$$\begin{aligned} R_1(X,Y,Z,V) &= g(S(X,Y,Z),V) + g(S(PX,PY,Z),V); \\ R_2(X,Y,Z,V) &= [\eta(X)\eta(PY) - \eta(PX)\eta(Y)][\eta(PZ)\eta(V) - \eta(Z)\eta(PV)]. \end{aligned} \tag{4.2}$$

where $S(X, Y, Z)) = P(X, Y, Z) - P(Y, X, Z)$, and

$$\begin{aligned} &P(X,Y,Z) = \frac{1}{8}\{\eta(Y)\eta(Z)X + \eta(X)\eta(PZ)PY + \\ &+ \eta(X)\eta(PY)PZ + g(Y,Z)\eta(X)\xi + g(X,PZ)\eta(Y)P\xi + \\ &+ \frac{1}{2}\, g(X,PY)[\eta(PZ)\xi + \eta(Z)P\xi]\},\ \forall X,Y,Z,V \in \Gamma(TM). \end{aligned}$$

Theorem 4.3 *A para-Kähler manifold (M, P, g, ξ) is of quasi-constant P-sectional curvature iff there exist c_0, c_1, c_2 functions on M which express the $(0, 4)$-curvature tensor field by:*

$$R = c_0R_0 + c_1R_1 + c_2R_2. \tag{4.3}$$

Proof. We show (4.3) punctually. For any $p \in M$ and any unit vector $u \in T_pM$, we obtain from (3.1) and (4.1):

$$\begin{aligned} &R(u,Pu,Pu,u) = c_0(p) + c_1(p)[\eta^2(u) + \eta^2(Pu)] + \\ &+ c_2(p)[\eta^2(u) + \eta^2(Pu)]^2 = c_0(p)R_0(u,Pu,Pu,u) + \\ &+ c_1(p)R_1(u,Pu,Pu,u) + c_2(p)R_2(u,Pu,Pu,u), \end{aligned}$$

which proves (4.3).

Theorem 4.4 *A para-Kähler manifold* (M, P, g, ξ) *is of quasi-constant* P*-sectional curvature iff*

(A) $R(\xi, P\xi) \in \varepsilon$
(B) $R(\ell, P\ell)\ell \in \xi^{\perp},\ \forall \ell \in \varepsilon^{\perp}$
(C) There exist c_0, c_1 *functions on* M *s.t. the sectional curvature of any plane containing* ξ_p *and orthogonal to* $P\xi_p$ *is* $(2c_0(p)+c_1(p))/8$ *and* $H(\ell) = c_0(p)$, $\forall \ell \in \xi_p^{\perp}$, $p \in M$.

Proof. From (4.3) follow (A)-(C). Conversely, let $p \in M$, $\theta \in [0, \pi/2]$ and a unit vector $u \in T_pM$ with $\not< (u, \varepsilon) = \theta$.

To compute $H(u)$ we apply (3.2), (2.1), (A)-(C) and from (2.1) and (C) we use $R(\xi, P\ell, \ell, \xi) = 0$, $\forall \ell \in \xi^{\perp}$. For $c_2(p) = H(\xi_p) - c_0(p) - c_1(p)$, the relation (4.1) is verified, which complete the proof.

5. Examples

1. On $\mathbb{R}^{2m+1}$ with the coordinates $(x_1, ..., x_m, y_1, ..., y_m, z)$ we consider the vector field $\xi = 2\dfrac{\partial}{\partial z}$ and the 1-form $\eta = \left(dz - \sum_{i=1}^{m} y_i dx_i\right)/2$. On $M = \mathbb{R}^{2m+1} \times (0, \infty)$ we take the metric $G = t^2 \left[\eta \otimes \eta + \sum_{i=1}^{m}(dx_i^2 - dy_i^2)/4\right] - dt^2$ and the product structure P defined such that $P(e_h) = e_{m+h}$, $P(e_{m+h}) = e_h$, $h = \overline{1, m}$, $P(\xi) = \dfrac{d}{dt}$ and $P\left(\dfrac{d}{dt}\right) = \xi$. Then $(M, P, G, \xi/t^2)$ is a para-Kähler manifold of quasi-constant P-sectional curvature.

2. Let N be a para-Kähler manifold of constant P-sectional curvature k and let $S_1 \times S_2$ be a product surface endowed with the canonical product structure and the metric $g_1 - g_2$, where g_i is Riemannian on S_i, $i = \overline{1, 2}$. The manifold $N \times (S_1 \times S_2)$ with the product para-Kähler structure is of quasi-constant P-sectional curvature (which is not constant if $k \neq 0$).

6. Curvature properties

On a para-Kähler manifold (M, P, g, ξ) of a dimension m, the Ricci tensor field Ric, the Ricci operator Q, the identity operator I, the scalar curvature

r and the Bochner tensor field are defined respectively by:

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\, R(-,X,Y,-); g(QX,Y) = \mathrm{Ric}(X,Y);$$

$$r = \mathrm{Trace}(\mathrm{Ric})$$

$$\begin{aligned} B(X,Y,Z) = R(X,Y)Z + \frac{1}{m+4}\{&\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y + \\ &+ g(Y,Z)QX - g(X,Z)QY + \mathrm{Ric}(PX,Z)PY - \\ &- \mathrm{Ric}(PY,Z)PX + g(PX,Z)Q(PY) - \\ &- g(PY,Z)Q(PX) + 2\,\mathrm{Ric}(PX,Y)PZ + \\ &+ 2g(PX,Y)Q(PZ)\} + \frac{r}{(m+2)(m+4)}\,\rho_0(X,Y,Z), \\ &\qquad \forall X,Y,Z \in \Gamma(TM), \end{aligned}$$

where $R_0(X,Y,Z,V) = (g(\rho_0(X,Y,Z),V)/4$.

By analogy with the contact metric manifolds [5, pp. 105], it is natural to introduce the following

Definition 6.1 A para-Kähler manifold (M,P,g,ξ) is called η-Einstein provided the Ricci tensor field is of the form

$$\mathrm{Ric} = ag + b[\eta \otimes \eta + (\eta \circ J) \otimes (\eta \circ J)],$$

where a, b are functions on M.

By a straightforward calculation as in [4], we obtain

Theorem 6.2 *If a para-Kähler manifold is η-Einstein and Bochner flat, then M is of quasi-constant P-sectional curvature.*

Conversely, we obtain

Theorem 6.3 *Let (M,P,g,ξ) be a para-Kähler manifold of quasi-constant P-sectional curvature. Then: (i) M is η-Einstein; (ii) M is Bochner flat iff $c_2 = 0$.*

Remark 6.4 The manifold constructed in Example 1, §5, is not Bochner flat.

Let recall the following

Theorem 6.4 [1] *If (M,P,g) is a Bochner flat para-Kähler manifold of constant Ricci scalar curvature, then the Pontrjagin classes of M can be expressed only with the fundamental 2-form $\Omega(-,-) = g(P-,-)$ and with the first Pontrjagin closed form.*

Corollary 6.5 *If (M, P, g) is a para-Kähler manifold of quasi-constant P-sectional curvature with constant Ricci scalar curvature, then its Pontrjagin classes are expressed by the fundamental 2-form Ω and the first Pontrjagin closed form only.*

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