MEAN CURVATURE IN MINKOWSKI SPACES *

ESTHER CABEZAS–RIVAS †

Departamento de Geometria y Topologia, Universidad de Valencia, 46100-Busjassot (Valencia), SPAIN

E-mail: Esther.Cabezas@uv.es

VICENTE MIQUEL ‡

Departamento de Geometria y Topologia, Universidad de Valencia, 46100-Busjassot (Valencia), SPAIN

E-mail: miquel@uv.es

Mean curvature of submanifolds in the Euclidean Space is a well established concept, an old and active research field. When we face to the same concept in a finite dimensional normed space, we have many choices for the definition of mean curvature of a submanifold. Here we shall concentrate on normed spaces with smooth and strictly convex unit sphere, also called Minkowski spaces. On such a space there are different natural definitions of volume. Then, if we consider the mean curvature as the first variation of the area, we face to different notions of mean curvature. In this very partial survey we shall focus on some recent work on the mean curvature showing the advantages and disadvantages of the definitions of mean curvature arising from the Hausdorff and Holmes-Thompson notions of volume.

1. Introduction

In Euclidean Geometry there are two equivalent approaches from which the notion of mean curvature of a submanifold arises. One starts with the definition of the second fundamental form as the orthogonal component of

* *MSC 2000:* 53C60, 51B20, 53C42.

Keywords: mean curvature, Holmes-Thompson volume, Busemann volume, Finsler Geometry, Minkowski Geometry.

† Work supported by the Beca del Programa Nacional de Formacin del Profesorado Universitario ref: AP2003-3344.

‡ Work partially supported by the AVCIT GRUPOS 03/169. Both authors were partially supported by the DGI(Espaa) and FEDER Project MTM No *2004-06015-C02-01 and by Generalitat Valenciana Grant ACOMP06/166.*

the directional derivative of a tangent vector field to the submanifold, and the mean curvature appears as the "trace" of the second fundamental form. The other one considers the volume functional defined on the submanifolds of the same dimension and the mean curvature appears as the gradient of this functional.

On a normed space, these two approaches give different concept of mean curvature. Our choice in this short survey is to give a definition of mean curvature based on the variational approach, because we think that it captures the essential geometric meaning of mean curvature. However, it has another difficulty impossible to overcome: there are more that one natural notion of volume in a normed space, then *we have as many definitions of mean curvature as definitions of volume we have.* Then, when we face to some problem on mean curvature in a normed space, we have to decide first which theory (definition of volume) we are developing in order to obtain a theorem.

In this paper we dedicate the first three sections to the basic notions: *Minkowski spaces* (normed spaces with not necessarily symmetric norm, but on which we can still use differential tools),*volume* (the axioms for a good definition and different definitions satisfying the axioms), and *mean curvature* associated to a definition of volume. It follows from this approach that mean curvature can be defined for "densities" not necessarily coming from a normed space. In fact, the definitions and computations given here are inspired in [3], who works on this more general context. More details for sections 2 and 3 can be found in [2], [11] and [14] . In sections 5 to 8 we give a rapid account of some results of J.C. Alvarez-Paiva, G. Berck, D. Burago, S. Ivanov, Q. She, Y.B.Shen, M. Souza, J. Spruck and K. Tennenblat on minimal submanifolds for the mean curvature associated to the two known natural definitions of volume: the Hausdorff volume (cf. Definition 3.1) and the symplectic volume (cf. Definition 3.4).

After looking at all these results, it is apparent that the symplectic volume gives a lot much more positive results than the Hausdorff one. This made some people to think that symplectic is the right definition of volume in a Minkowski space or, more general, in a Finsler manifold. However, the fact that all Randers norms with the same associated Euclidean metric have the volume form of this metric (Proposition 5.1) tells us that, sometimes, the symplectic volume is not sharp enough to distinguish between Minkowski and Euclidean Geometry. May be the symplectic volume gives better results just because it forgets many of the specific facts of Minkowski Geometry not present in Euclidean Geometry. That is: may be the symplectic volume gives not better results, but just results more similar to the Euclidean ones.

The authors would like to thank M. C. Domingo-Juan the stimulating talks we had about the subject of this survey.

2. The Minkowski Space

2.1. *Definitions*

Definition 2.1 A (non-symmetric in general) normed space of dimension n: is a pair (V, F), where V is a n-dimensional real vector space and $F : V \longrightarrow \mathbb{R}$ is a map which satisfies

i) $F(v) \geq 0$ for every $v \in V$, and $F(v) = 0$ iff $v = 0$,

ii) $F(\lambda v) = \lambda\ F(v)$ for every $\lambda \in \mathbb{R}$, $\lambda \geq 0$, and

iii) $F(v + w) \leq F(v) + F(w)$ for every $v, w \in V$.

If condition ii) holds under the form $F(\lambda v) = |\lambda|\ F(v)$ for every $\lambda \in \mathbb{R}$, we have the usual definition of (symmetric) normed space.

Definition 2.2 The unit ball B_F and the unit sphere S_F of (V, F) are defined by $B_F = \{v \in V;\ F(v) \leq 1\}$, and $S_F = \{v \in V;\ F(v) = 1\}$

From Def 2.1.iii) it follows that B_F and S_F are a convex. It is also clear that F is a (symmetric) norm if and only if B_F and S_F are centrally symmetric.

Remark 2.1 It is also possible to define a norm F on a vector space V by giving a convex body B with $0 \in B$ and defining the norm F by $F(v) = \lambda$ iff $\lambda\ v_0 = v$ and v_0 is the intersection point of the boundary of B with the half-line $\{t\ v; t \geq 0\}$. In this case $B = B_F$.

In the Euclidean case, there is a natural isomorphism $\flat$ between V and V^* induced by the metric. The generalization to normed spaces is a bijection $\mathcal{L}$ called Legendre transformation.

Definition 2.3 The Legendre Transformation from V to V^* is defined on S_F by

$$\begin{aligned} \mathcal{L} : S_F &\longrightarrow V^* \\ v &\longmapsto \mathcal{L}(v) : V \longrightarrow \mathbb{R} \\ &\qquad\qquad w \longmapsto 0 \text{ if } w \in T_v S_F, \text{ or } 1 \text{ if } w = v. \end{aligned}$$

This is equivalent to say that $\mathcal{L}(v)$ is the linear map satisfying

$T_vS_F = \mathrm{Ker}\mathcal{L}(v)$[a] and $\mathcal{L}(v)(v) = 1$. Now, we extend $\mathcal{L}$ to all V by

$$\mathcal{L}(\lambda v) = \lambda \mathcal{L}(v) \ \text{ for } \ \lambda \geq 0 \ \text{ and } \ v \in S_F.$$

It follows that $\mathcal{L}$ is positively homogeneous of degree 1.

The above definition is univoque if T_vS_F is well defined (unique) for every $v \in S_F$. This happens if S_F is C^1.

Lines are geodesics in a normed space. If we want the geodesic between two points to be unique always, we need that S_F be strictly convex. The same condition is necessary if we want that the geodesic realizing the distance between a point and an hyperplane be unique.

Let us denote $L = \frac{1}{2}F^2$. If F is smooth,

$$\mathcal{L}(v) = dL(v) = F(v)\ dF(v/F(v)) \text{ for every } v \in V.$$

Definition 2.4 A Minkowski space is a finite dimensional normed space (V, F), with F smooth on $V - \{0\}$ and strongly convex. By "F strongly convex" we mean that, for every $u \in V - \{0\}$

$$D^2\left(L\right)_u(v, v) > 0 \text{ for every } v \in V - \{0\}. \qquad (1)$$

Let us remark that condition (1) implies the properties Def 2.1.i) and iii), then we can give a more economical definition of Minkowski space as a pair(V, F) where F is a smooth function on $V - \{0\}$, strongly convex, and satisfying Def 2.1 ii).

From condition (1), F defines a Riemannian metric on the unit bundle $S_F \times V$ of (V, F) or, equivalently, on S_F, by

$$g_{(v,x)} := g_v := D^2\left(L\right)_v.$$

From the homogeneity of L it follows that g_v has the following properties:

a) $g_v(v, v) = F(v)^2$,

b) $g_v(v, w) = 0$ if $v \in S_F$ and $w \in T_vS_F$.

c) $g_{\lambda v} = \ g_v$ if $\lambda > 0$.

Condition (1) is not just a trick in order to apply Riemannian Geometry. There are two good reasons to introduce this condition in the definition of Minkowski Space. These are given by the next propositions 2.2 and 2.4.

[a]We consider T_vS_F as a vector subspace. The definition looks different (but it is the same) in the books where T_vS_F is considered as an affine subspace.

2.2. *Banach-Mazur distance*

If $T : (V, F) \longrightarrow (W, G)$ is a linear map between symmetric normed spaces, the norm of T is defined by

$$||T|| := \sup\{G(T(x));\ x \in S_F\}. \tag{2}$$

Proposition 2.1 *Given any symmetric norm $||\cdot||$ on a finite dimensional vector space V, there is a symmetric Minkowski norm F on V satisfying*

$$F \leq ||\cdot|| \leq (1+\varepsilon)F.$$

The Banach - Mazur distance between two n-dimensional Banach spaces (V, F) and (W, G) is the infimum of the numbers $\ln(||T||||T^{-1}||)$ where T ranges over all the invertible linear maps from V to W. Then, it follows from Proposition 2.1 that

Proposition 2.2 *The set of symmetric Minkowski norms on a given n-dimensional vector space is dense (in the topology given by the Banach-Mazur distance) in the set of symmetric norms on the same spaces.*

2.3. *Dual unit sphere*

Definition 2.5 The dual or polar set D^* of a subset $D \subset V$ is $D^* = \mathcal{L}(D)$.

An interesting property of this definition is the following

Proposition 2.3 *Given a normed space (V, F), there is an induced norm F^* on V^* defined by*

$$F^*(w) = \sup\{w(v);\ v \in S_F\}, \tag{3}$$

then one has:

$$S_{F^*} = S_F^*. \tag{4}$$

In fact, for every $w \in S_F^*$, there is a $u \in S_F$ such that $w = \mathcal{L}(u)$. On the other hand, for every $v \in S_F$ not parallel to T_uS_F, there is a $\lambda \in \mathbb{R}$, $\lambda > 1$ or $\lambda < 0$, such that $\lambda v \in T_uS_F$, and $w(v) = (1/\lambda)\mathcal{L}(u)(\lambda v) = 1/\lambda \leq 1$, with the equality when $u = v$. Then $F^*(w) = \sup\{w(v);\ v \in S_F\} = 1$, and $S_F^* \subset S_{F^*}$. The other inclusion follows from the bijectivity of $\mathcal{L}$. □

In a normed space with F smooth and B_F strictly convex, it is clear that S_F is smooth. If we want to apply differential geometry in the study of (V, F), it is natural to require that also S_F^* be smooth. Then, the following result gives a justification of condition (1) in the definition of Minkowski space.

Proposition 2.4 *Let* (V, F) *be a normed space with* F *smooth and* B_F *strictly convex. Then* S_F^* *is smooth if and only if* F *satisfies* (1)

Related with Proposition 2.3 is the following result, very useful for some computations where we need $\mathcal{L}^{-1}$.

Proposition 2.5 *Let* (V, F) *be a Minkowski space,* F^* *the dual norm on* V^**. After the natural identification of* V *with* V^{**}*,* F *becomes the norm induced by* F^* *on* V *by* (3)*. If* $\mathcal{L}_{F^*} : V^* \longrightarrow V$ *and* $\mathcal{L}_F : V \longrightarrow V^*$ *are the Legendre transformations associated to* F^* *and* F *respectively, then* $\mathcal{L}_{F^*} = \mathcal{L}_F^{-1}$*.*

2.4. *Orthogonality in normed spaces*

Definition 2.6 In (V, F), we say that u is orthogonal to v $(u \dashv v)$ if $v \in T_{\frac{u}{F(u)}} S_F$. This is equivalent to say that the minimizing line from the origin to a line of the form $p + t\ v$ is in the direction of u.

This definition of orthogonal is not symmetric: $u \dashv v$ does not imply $v \dashv u$. But we can consider the reciprocal notion of orthogonality:

Definition 2.7 In (V, F), we say that u has v as orthogonal $(u \vdash v)$ if $v \dashv u$.

There is also a third definition of orthogonality,

Definition 2.8 We say that two vectors $v, w \in V$ are u-orthogonal if they are orthogonal in the metric g_u

In this context, the following relations are useful:

(1) $u \dashv v$ iff $g_u(u, v) = 0$
(2) $g_u(u, v) = \mathcal{L}(u)(v) = F(u)dF(u)(v) = dL(u)(v)$.
(3) If $\gamma(t)$ is a curve parametrized by arc-length, then $\gamma'(t) \dashv \gamma''(t)$.

3. Volume in Minkowski spaces

3.1. *Axioms and definitions*

Before giving a definition of volume in a Minkowski space, it is natural to consider a series of natural axioms that such a definition should satisfy. We follow [2] to state the axioms. As usual, the volume of a body Ω in V will be given as the integral along Ω of a density μ_V on V. A density μ_V on a normed space V defines a "symmetric" norm on $\Lambda^n V$ (where $n = dim(V)$). We require this symmetric norm to satisfy the following axioms.

(A1) If $T : X \longrightarrow Y$ is a contractive linear map, $n = dim(X) = dim(Y)$, then the induced linear map $T_* : (\Lambda^n X, \mu_X) \longrightarrow (\Lambda^n Y, \mu_Y)$ is contractive.

(A2) The map $(X, F) \mapsto (\Lambda^n X, \mu_X)$ is continuous with respect the Banach-Mazur distance.

(A3) If (X, F) is Euclidean, then μ_X is the standard Euclidean volume on X.

(A4) Every k-subspace has an induced norm, which induces a volume on it. In this way, each definition of volume induces a k-density on V, that is, a continuous function, homogeneous and of degree 1, on $\Lambda^k_s V$, the set of simple (or decomposable) exterior k-vectors on V. When $k = n-1$, then $\Lambda^{n-1}_s V = \Lambda^{n-1} V$, and we impose:
The induced density σ on each $n-1$-dimensional subspace gives a norm on $\Lambda^{n-1} V$.

Condition (A4) is natural because it is equivalent to the following one: *If P is a closed polyhedron in V, then the area of any of its facets is less than or equal to the sum of the areas of the remaining facets.*

From condition (A1), volume is invariant by isometries, then it is invariant by translations, and every volume has to be a multiple of the Lebesgue measure. The freedom is in the choosing of the constant. This is not relevant when we consider n-volumes in V, but it gives dramatic differences when we consider k-volumes of k-submanifolds, since the norms induced on each k-dimensional subspace are not equivalent in general. There are three well known definitions of volume satisfying conditions (A1) to (A4):

Definition 3.1 The Busemann or Hausdorff volume is the multiple of the Lebesgue measure for which the volume of the unit ball equals the volume ϵ_n of the Euclidean unit ball of dimension n. In other words, the density μ^b associated to this volume is

$$\mu^b(x_1 \wedge ... \wedge x_n) = \frac{\epsilon_n}{vol(B_F : x_1 \wedge ... \wedge x_n)},$$

where $vol(B_F : x_1 \wedge ... \wedge x_n)$ indicates the volume of B_F in the Lebesgue measure determined by the basis $x_1, ..., x_n$.

Busemann proved that this volume coincides with the Hausdorff measure defined on the metric space (V, d_F).

Definition 3.2 The symplectic volume on $T^*V = V \times V^*$ is defined by the density $\Omega = \frac{1}{n!}\, \omega \wedge \cdots \wedge \omega$, where ω is the symplectic form on $V \times V^*$ defined by

$$\omega((x,\xi),(y,\zeta)) = \xi(y) - \zeta(x).$$

Definition 3.3 The Dazord-Holmes-Thompson or symplectic volume on V is the multiple of the Lebesgue measure for which the volume of the unit ball equals the symplectic volume of $B_F \times B_F^*$ divided by the volume of the Euclidean unit ball of dimension n. In other words, the density μ^s associated to this volume is

$$\mu^s(x_1 \wedge ... \wedge x_n) = \frac{1}{\epsilon_n}\, vol(B_F^* : \xi_1 \wedge ... \wedge \xi_n),$$

where $\xi_1, \ldots, \xi_n$ is the dual basis of $x_1, \ldots, x_n$ and $vol(B_F^* : \xi_1 \wedge ... \wedge \xi_n)$ indicates the volume of B_F^* in the Lebesgue measure determined by the basis $\xi_1, ..., \xi_n$.

Definition 3.4 The Benson's volume or Gromov mass* is the multiple of the Lebesgue measure for which the volume of the minimal parallelotope circumscribed to the unit ball equals 2^n. In other words, the density μ^* associated to this volume is

$$\mu^*(x_1 \wedge ... \wedge x_n) = \frac{1}{\mu^m_{V^*}(\xi_1 \wedge ... \wedge \xi_n)},$$

where $\mu^m_{V^*}(\xi_1 \wedge ... \wedge \xi_n) = \inf\{F^*(\zeta_1) \ldots F^*(\zeta_n);\ \zeta_1 \wedge ... \wedge \zeta_n = \lambda\, \xi_1 \wedge ... \wedge \xi_n \text{ for some } \lambda\}$.

Remark 3.1 A *Finsler* metric on a *manifold* is a continuous function on its tangent bundle that is smooth away from the zero section and such that its restriction to each tangent space is a Minkowski norm.

Given a definition of volume of normed spaces, we have a definition of volume on Finsler manifolds: the volume density on an n-dimensional Finsler manifold M assigns to each parallelotope formed by the tangent vectors $v_1, ..., v_n \in T_xM$ its volume in the normed space T_xM. The condition that the volume density be smooth is satisfied by both the Busemann and Dazord-Holmes-Thompson definitions, but not by mass*. Then, only two, among the known definitions of volume in a Minkowski space remain natural.

3.2. *The volumes induced on k-submanifolds*

The k-densities induced on V by the above definitions of volume are: for every $a \in \Lambda^k_s V$,

$$\sigma^b(a) = \frac{\epsilon_k}{vol(B_F \cap < a >; a)}, \quad \sigma^s(a) = \epsilon_k^{-1} \int_{\pi(B_F^*)} |a|.$$

Where $\pi : X^* \longrightarrow < a >^*$ is the dual projection of the inclusion $i :< a > \longrightarrow X$ and a is considered as a k-form on $< a >^*$ (then $|a|$ is a k-density).

3.3. *Minkowski contents, isoperimetrix and Gauss map.*

An important concept related with the $(n-1)$-volume of a hypersurface M of V is the

Definition 3.5 The $(n-1)$-Minkowski density associated to a convex body I of V and a Lebesgue measure λ on V is the symmetric norm σ_I defined on $\Lambda^{n-1}V$ by

$$\sigma_I(x_1 \wedge ... \wedge x_{n-1}) := \frac{1}{n} \lim_{t \to 0^+} \frac{\lambda([x_1, ..., x_{n-1}] + t\ I)}{t},$$

where $[x_1, ..., x_{n-1}]$ denotes the parallelotope generated by the vectors $x_1, ..., x_{n-1}$, and $A + B = \{a + b;\ a \in A,\ b \in B\}$.

The corresponding $(n-1)$-Minkowski contents of a hypersurface M of V is defined by

$$\mu^{mi}(M) = \int_M \sigma_I.$$

In the opposite sense, given a symmetric norm σ on $\Lambda^{n-1}V$ and a Lebesgue measure λ on V, we can ask: Is there a convex set $I \subset V$ such that $\sigma = \sigma_I$? Before giving the answer, we introduce some machinery. Given a volume form Ω on V, we define

$$i_\Omega : \Lambda^{n-1}V \longrightarrow V^*;\ i_\Omega(x_1 \wedge \cdots \wedge x_{n-1})(x) = \Omega(x_1, ..., x_{n-1}, x). \qquad (5)$$

It is easy to check that i_Ω is an isomorphism. Then it induces a norm σ^* on V^* by $\sigma^*(\ i_\Omega(x_1 \wedge \cdots \wedge x_{n-1})) = \sigma(x_1 \wedge \cdots \wedge x_{n-1})$. Obviously, if B_σ is the unit ball of σ, then $i_\Omega(B_\sigma)$ is the unit ball of σ^*.

Theorem 3.1 *Let σ be a symmetric norm on $\Lambda^{n-1}V$, B_σ its unit ball, λ a Lebesgue measure on V, Ω a volume form on V such that $\lambda = |\Omega|$. By the natural identification of V with the dual of V^*, we can consider the polar set $I = (i_\Omega(B_\sigma))^*$ with respect to σ^*. Then $\sigma_I = \sigma$.*

I (or, sometimes, its boundary ∂I) is called the isoperimetrix or the Wulf shape of σ.

Let us consider σ, λ and Ω, with the meaning they have in the above theorem, fixed on V. The $(n-1)$-density σ on V allows us to define the $(n-1)$-volume of any hypersurface M of V by $\int_M \sigma$. The name of isoperimetrix for I comes from the following result:

Theorem 3.2 *Among all convex bodies in V with a fixed $(n-1)$-volume of their boundary, the one that encloses the largest λ-volume is, up to translation, a dilate of $I = (i_\Omega(B_\sigma))^*$.*

Then I gives, for any definition of volume in a Minkowski space, the solution of the isoperimetric problem among convex bodies.

There are more ways for describing the isoperimetric I determined by σ:

Definition 3.6 Given $0 \neq a \in \Lambda^{n-1}V$, we say that $v_I \equiv v_\sigma \in V$ is a unit vector normal to a with respect to I (or σ) if $v_\sigma \in \partial I$, $T_{v_\sigma}\partial I =< a >$, and $\Omega(a \wedge v_\sigma) > 0$.

Notice that v_σ is constructed in such a way that

$$\Omega(a \wedge v_\sigma) = \sup\{|\Omega(a \wedge x)|;\ x \in I\}$$

and $v_\sigma \dashv< a >$ in the norm defined by I on V, that is, in the norm on V dual of σ^* induced on V^* by σ and i_Ω as we indicated above. Then, we have the following result, which is the normed version of the fact, in the the Euclidean space, that the volume form a hypersurface is the contraction with the unit normal vector of the volume form of the ambient space.

Proposition 3.1 *If $v_\sigma \in V$ is normal to $< a >$ ($a \in \Lambda^{n-1}V$) with respect to $I = (i_\Omega(B_\sigma))^*$, then*

$$\sigma(a) = \Omega(a \wedge v_\sigma).$$

When ∂I is smooth, v_σ is uniquely defined by $< a >$, then, given a, v_σ is the unique vector satisfying the above equation, and also the vector in I maximizing $|\Omega(a \wedge x)|$. All these considerations allows to say that, when ∂I is smooth, there is a well defined map

$$v_\sigma : \Lambda^{n-1}V - \{0\} \longrightarrow \partial I/\ a \mapsto v_\sigma(a);\ \sigma(a) = \Omega(a \wedge v_\sigma(a)).$$

which gives a diffeomorphism between S_σ and ∂I.

Let M be an oriented hypersurface of (V,F), and let σ be the $(n-1)$-density induced by F through some definition of volume. The σ-unit normal ν_σ to M is defined by

$$\nu_\sigma(x) = v_\sigma(e_1 \wedge \cdots \wedge e_{n-1}),$$

for a positively oriented basis $\{e_1, ..., e_{n-1}\}$ of T_xM. By analogy with the Euclidean case, we call the map $\nu_\sigma : M \longrightarrow \partial I$ the *σ-Gauss map*. We also define the *σ-Weingarten map* $W_{\sigma x} : T_xM \longrightarrow T_xM$ by $W_{\sigma x}(X) = -d\nu_{\sigma x}(X)$ after identification of T_xM with $T_{v_\sigma}\partial I$ (because they are parallel by the definition of ν_σ).

4. Mean curvature

4.1. *Generic mean curvature*

Let M be a hypersurface of (V,F), and let σ be the $(n-1)$-density induced by F through a definition of volume. After the above definition for the Weingarten map of M, it is natural to define the *σ-mean curvature $H_{\sigma x}$ of M at x* by

$$H_{\sigma x} = -\mathrm{tr}\ d\nu_{\sigma x}. \tag{6}$$

In this section we shall show that this definition is in agreement with the concept of mean curvature arising in a variational problem on the $(n-1)$-volume of a hypersurface.

Given an immersion $x : M \longrightarrow V$, let $X : M \times I \longrightarrow V$ be a variation of x, $x_t(u) := X(u,t)$, with variation vector field $Y = \left.\frac{\partial X}{\partial t}\right|_{t=0}$. If (U,u) is a coordinate system of M, the σ-volume form in this coordinate system for the immersion x_t can be written as $\omega_t = \sigma(\overline{x}_t)\,du$, where $\overline{x}_t := \frac{\partial x_t}{\partial u^1} \wedge \cdots \wedge \frac{\partial x_t}{\partial u^{n-1}}$ and $du := du^1 \wedge ... \wedge du^{n-1}$. For the volume $vol_\sigma(x_t(U)$ of $x_t(U)$ we have

$$\left.\frac{dvol_\sigma(x_t(U))}{dt}\right|_{t=0} = \left.\frac{d}{dt}\right|_{t=0} \int_{u(U)} \sigma(\overline{x}_t)\ du = \int_{u(U)} \left.\frac{\partial}{\partial t}\right|_{t=0} \sigma(\overline{x}_t)\ du$$ and

$$\frac{\partial}{\partial t}\,\sigma\left(\frac{\partial x_t}{\partial u^1} \wedge \cdots \wedge \frac{\partial x_t}{\partial u^{n-1}}\right)\Bigg|_{t=0} = d\sigma_{\overline{x}_0}\left(\frac{\partial}{\partial t}\left(\frac{\partial x_t}{\partial u^1} \wedge \cdots \wedge \frac{\partial x_t}{\partial u^{n-1}}\right)\Bigg|_{t=0}\right)$$
$$= \sum_{j=1}^{n-1} d\sigma_{\overline{x}_0}\left(\frac{\partial x_0}{\partial u^1} \wedge \cdots \wedge \left.\frac{\partial^2 x_t}{\partial t \partial u^j}\right|_{t=0} \wedge \cdots \wedge \frac{\partial x_0}{\partial u^{n-1}}\right).$$

But $\left.\dfrac{\partial^2 x_t}{\partial t \partial u^j}\right|_{t=0} = \left.\dfrac{\partial^2 x_t}{\partial u^j \partial t}\right|_{t=0} = \dfrac{\partial Y}{\partial u^j} = Y_j^\nu + Y_j^T$,
where Y_j^ν and Y_j^T are the components of $\dfrac{\partial Y}{\partial u^j}$ under the decomposition $V =< \nu_\sigma(x_0(u)) > \oplus T_{x_0(u)}M$. Then

$$\left.\frac{\partial}{\partial t}\sigma\left(\overline{x}_t\right)\right|_{t=0} = \left(\sum_{j=1}^{n-1} du^j (Y_j^T)\right) \, d\sigma_{\overline{x}_0}\left(\overline{x}_0\right)$$
$$+ \sum_{j=1}^{n-1} d\sigma_{\overline{x}_0}\left(\frac{\partial x_0}{\partial u^1} \wedge \cdots \wedge Y_j^\nu \wedge \cdots \wedge \frac{\partial x_0}{\partial u^{n-1}}\right).$$

By the homogeneity of σ, $d\sigma_{\overline{x}_0}\left(\overline{x}_0\right) = \sigma\left(\overline{x}_0\right)$, and a direct computation using Proposition 3.1 gives $d\sigma_{\overline{x}_0}\left(\dfrac{\partial x_t}{\partial u^1} \wedge \cdots \wedge Y_j^\nu \wedge \cdots \wedge \dfrac{\partial x_t}{\partial u^{n-1}}\right) = 0$. From all this, we obtain the formula of the first variation for $vol(x_t(U))$:

$$\left.\frac{dvol_\sigma(x_t(U))}{dt}\right|_{t=0} = \int_U \left(\sum_{j=1}^{n-1} du^j (Y_j^T)\right) \sigma(\overline{x}_0) du. \tag{7}$$

Any variation vector field Y can be written under the form $Y = h\ \nu_\sigma + y^\top$, where $y^\top$ is tangent to M, and

$$\sum_{j=1}^{n-1} du^j (Y_j^T) = \sum_{j=1}^{n-1} du^j \left(\frac{\partial (h\ \nu_\sigma + y^\top)}{\partial u^j}\right)^T = \sum_{j=1}^{n-1} du^j \left(h \frac{\partial \nu_\sigma}{\partial u^j} + \frac{\partial y^\top}{\partial u^j}\right)^T$$
$$= h \ \mathrm{tr}\ d\nu_\sigma + \sum_{j=1}^{n-1} du^j \left(\frac{\partial y^\top}{\partial u^j}\right)^T. \tag{8}$$

Now, let us give an interpretation of the second adding term in the last formula. The $(n-1)$-density σ on V restricted to TM defines a $(n-1)$-density σ on M, and, on the domain U of the chart (U, u), there is a differential $(n-1)$-form $\overline{\sigma}$ such that $\sigma = |\overline{\sigma}|$. Then, associated to $\overline{\sigma}$, we have a divergence div_σ defined, as usual, by

$$\mathrm{div}_\sigma Z \ \overline{\sigma} = \mathfrak{L}_Z \overline{\sigma} \quad \text{for every } Z \in \mathfrak{X}(M), \tag{9}$$

where $\mathfrak{L}_Z$ denotes the Lie derivative respect to Z. Since $\overline{\sigma}$ is a form of maximal degree we have $\mathfrak{L}_Z \overline{\sigma} = d\iota_Z \overline{\sigma}$, and a standard computation gives

$$d\iota_Z \overline{\sigma} = \sum_{j=1}^{n-1} \left(\frac{\partial Z^j}{\partial u^j} + Z^j \sum_{i=1}^{n-1} \left(\frac{\partial^2 x}{\partial u^i \partial u^j}\right)^T\right) \overline{\sigma} = \sum_{j=1}^{n-1} du^j \left(\frac{\partial Z}{\partial u^j}\right)^T,$$

and the second adding term in (8) is

$$\sum_{j=1}^{n-1} du^j \left(\frac{\partial y^\top}{\partial u^j}\right) = d\iota_{y^\top}\overline{\sigma} = \mathrm{div}_\sigma y^\top \ \overline{\sigma}. \tag{10}$$

If the variation X of the immersion x is normal ($y^\top = 0$) or keeps fixed the boundary of U, the substitution of (10) in (8) and (7) gives

$$\left.\frac{dvol_\sigma(x_t(U))}{dt}\right|_{t=0} = \int_U h \ \mathrm{tr}\ d\nu_\sigma \ \sigma. \tag{11}$$

From (11) we have the announced variational justification of the definition (6) for the mean curvature:

Proposition 4.1 *An immersion $x : I \longrightarrow M$ is a critical point for $vol(U)$ for normal variations or for variation preserving the boundary of U if and only if $H_\sigma = 0$*

An important fact is that mean curvature depends on the concept of volume that we are taking. In the definition of mean curvature of a hypersurface in a generic Minkowski space, the isoperimetrix plays the role that metric sphere played in the Euclidean space. Then there are two natural phenomena. i) the ties between metric properties and minimality are not necessarily as deepen as they are in the Euclidean case, ii) these ties will be different for different notions of volume.

In the next sections we shall examine some properties related to mean curvature both for the symplectic and the Hausdorff measure notions. In all cases, we obtain stronger theorems using symplectic volume. Does it mean that this is the right concept of measure to be used in Finsler Geometry, or does it gives more properties just because its definition corrects in some extent the differences with the Euclidean case and makes things more similar to Euclidean. We hope that future research will give the right answer to these questions.

5. Minimal surfaces in a 3-dimensional Minkowski space for the symplectic notion of volume

Perhaps the simplest non-euclidean and non-symmetric Minkowski norms are *Randers norms*, which are of the form $F = || \ || + b$, where $|| \ ||$ is the norm induced by an Euclidean metric a and b is a 1-form with $||b|| < 1$. We call a the Euclidean metric of F.

A friendly property of a Randers norm F on a vector space V is that it induces a Randers norm $\overline{F}$ on every vector hyperplane and its Euclidean metric is the restriction to this hyperplane of the Euclidean metric of F. As a consequence, if $x : M \longrightarrow V$ is an isometric immersion of a hypersurface M of V with a Randers norm F with metric a, then M has a Randers metric with Riemannian metric $\overline{a}$, and $x : (M, \overline{a}) \longrightarrow (V, a)$ is an isometric immersion.

An easy to prove but surprising property for the symplectic volume induced by a Randers norm is the following

Proposition 5.1 *(He-Shen [8]) The Dazord-Holmes-Thompson densities of a Randers vector space and any isometrically immersed hypersurface M are just those of their Riemannian metrics.*

As a consequence, in a Randers vector space the minimal hypersurfaces are the same that in the Euclidean vector space of the same dimension.

For general norms and the same notion of density, the study of minimal hypersurfaces not necessarily reduces to the Euclidean case, but we still have the following Bernstein's type theorem.

Theorem 5.1 *(He-Shen [8]): Any complete minimal graph in a 3-dimensional Minkowski space is a plane.*

6. Minimal surfaces in a 3-dimensional Minkowski space for the Hausdorff notion of volume

With the Hausdorff notion of volume, results on minimal submanifolds are much more complicated. As far as we know, apart from planes, there are only examples and theorems for Randers metrics.

First, the classification of revolution minimal surfaces

Theorem 6.1 *(Souza-Tennenblat [13]) Let $(V, || \; || + b)$ be a Randers Minkowski space. Let us take the coordinates of V in such a way that $b = \beta \, dz$. For each b, $0||b|| < 1$, there exists a unique, up to homothety, forward complete*[b]

[b]A Finsler manifold (M, F) is said to be forward complete if every forward Cauchy sequence converges in M. (cf. [4]) minimal surface of revolution around the axis z. The surface is embedded, symmetric with respect to a plane perpendicular to the rotation axis, and it is generated by a concave plane curve.

Moreover, when $1/\sqrt{3} < ||b|| < 1$, the slope of the tangent lines to the curve is bounded by $\pm\sqrt{1-b^2}/\sqrt{3\,b^2-1}$. In this case, besides the forward complete minimal surfaces of rotation, there are non complete ones which include explicit minimal cones

For the Bernstein problem, we have

Theorem 6.2 *(Souza-Spruck-Tennenblat [12]) A minimal surface in a 3-dimensional Randers Space $(V^3, || \; || + b)$, $0 \leq ||b|| < 1/\sqrt{3}$, which is the graph of a function defined over a plane, is a plane.*

This theorem is proved by using the ellipticity of a partial differential equation equivalent to the vanishing of the mean curvature. This equation is not elliptic any more for $||b|| \geq 1/\sqrt{3}$, which gives the restriction in the hypothesis of the theorem.

Theorem 5.1 is proved following the same technique that the above one, but in that case the equation is elliptic for any Minkowski norm.

7. About minimizers of the volume

From the definition (6) and Proposition 4.1 it is clear that hyperplanes are minimal hypersurfaces ($H_\sigma = 0$) or, equivalently, they are critical points for the $(n-1)$-volume functional defined on all the hypersurfaces of a given Minkowski space.

The problem is quite different if, instead of looking for minimal hypersurfaces we look for "minimizers" . We say that a d-dimensional submanifold M of V is a minimizer if, for every $x \in M$ and every convex neighborhood U_x of x in M diffeomorphic to a Euclidean d-ball D^d and every embedding $f : D^d \longrightarrow V$ such that $f|_{\partial D}$ is a diffeomorphism onto ∂U_x, one has $vol(f(D)) \geq vol(U_x)$. For minimizers we have the following result for planes

Theorem 7.1 *(Burago-Ivanov [7] and Ivanov [9]) . For the symplectic notion of volume, a 2-dimensional affine subspace of a Minkowski space (V, F) is a minimizer.*

And the following for hyperplanes:

Theorem 7.2 *(Alvarez Paiva- Berck [1]) A hyperplane in a Minkowski space (V, F) is a minimizer with respect to the symplectic notion of volume.*

As far as we know, the corresponding results for the Hausdorff measure are unknown. Look into the next section to see that the known results in general Finsler manifolds are negative for the Hausdorff measure.

8. Higher codimension and general ambient spaces

The notion of mean curvature H_σ that we have presented in sections 3 and 4 can be generalized (following the idea that H_σ must appear in the integrand of the first variation formula for the $(n-1)$-volume and $H_\sigma = 0$ must be the condition for M to be a critical point for the volume) to higher codimensions and to Finsler manifolds.

This is done, for instance, by Z. Shen [10] for the Hausdorff measure and by G. Berck [6] for the symplectic one. Although their approaches look different than our presentation in section 4, they are essentially equivalent when restricted to hypersurfaces in normed spaces. The approach given by Bellettini [5] coincides with ours if we use the norm σ_I defined by the isoperimetrix as the starting norm ϕ in [5].

The next theorems state that results 7.1 and 7.2 can be extended to general Finsler ambient spaces.

Theorem 8.1 *(Ivanov [9]) A totally geodesic two-dimensional submanifold N of a Finsler manifold (M, F) is a minimizer with respect to the symplectic notion of volume.*

Theorem 8.2 *(Alvarez Paiva and Berck [1]) A totally geodesic hypersurface N of a Finsler manifold (M, F) is a minimizer with respect to the symplectic notion of volume.*

However, when we use the Hausdorff notion of volume, even the minimality of hyperplanes that we noted at the beginning of section 7 fails for general ambient Finsler spaces, as the following amazing result states:

Theorem 8.3 *(Alvarez Paiva and Berck [1]) For any value of the parameter λ, all geodesics of the Finsler metric*

$$F_\lambda(x, v) = \frac{(1 + \lambda^2 ||x||^2)||v||^2 + \lambda^2 \langle x, v\rangle^2}{||v||}$$

on $\mathbb{R}^3$ (where $\langle\ ,\ \rangle$ is the standard scalar product, and $||\ ||$ its associated norm) are straight lines. However, the only value of λ for which all planes are minimal submanifolds of the Hausdorff 2-area functional of F_λ is $\lambda = 0$.

And, for minimality in every dimension and codimension, we have that symplectic volume has a "good behavior":

Theorem 8.4 *(Berck [6]) Every totally geodesic submanifold of a Finsler manifold is minimal for the symplectic notion of volume.*

References

1. J. Alvarez-Paiva and G. Berck *What is wrong with Hausdorff measure in Finsler spaces*, arXiv : math. DG/0408413, (2004).
2. J.C. Álvarez Paiva and A. C. Thompson *Volumes on normed and Finsler spaces*, A Sampler of Riemann-Finsler Geometry (D. Bao, R. Bryant, S.S. Chern and Z. Shen, eds.), Cambridge U. P., M.S.R.I. Publications volume **50** (2004), 1–49.
3. B. Andrews *Volume-Preserving anisotropic mean curvature flow*, Indiana Univ. Math. J., **50**(2001), 783–827.
4. D. Bao, S.S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*. GTM 200, Springer-Verlag, NewYork, 2000.
5. G. Bellettini *Anisotropic and Crystalline Mean Curvature Flow*, A Sampler of Riemann-Finsler Geometry (D. Bao, R. Bryant, S.S. Chern and Z. Shen, eds.), Cambridge U. P., M.S.R.I. Publications volume **50** (2004), 49–82.
6. G. Berck *Minimalit des sous-varits totalement godsiques en Gomtrie Finslerienne*, arXiv : math. DG/0409320 v1, (2004).
7. D. Burago and S. Ivanov, *On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume*, Annals of Mathematics **156** (2002), 891-914.
8. Q. He and Y.B. Shen, *On Bernstein type theorems in Finsler spaces with the volume form induced from the projective sphere bundle*, Proceedings Amer. Math. Soc. **134** (2005), 871–880.
9. S. Ivanov, *On two-dimensional minimal fillings*, St. Petersburg Math. J. **13** (2002), 17–25.
10. Z. Shen, *On Finsler Geometry of submanifolds*, Math. Ann. **311** (1998), 549–576.
11. Z. Shen, *Lectures on Finsler Geometry*, World Sc., Singapore, 2001.
12. M. Souza, J. Spruck and K. Tennenblat, *A Bernstein type theorem on a Randers space*, Math. Ann. **329** (2004), 291–305.
13. M. Souza and K. Tennenblat, *Minimal surfaces of rotation in a Finsler space with a Randers metric*, Math. Ann. **325** (2003), 625–642.
14. A.C. Thompson, *Minkowski Geometry*, Cambridge U. P., New York, 1996.