THE k–STEIN CONDITION ON DAMEK–RICCI SPACES *

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A Riemannian manifold M with associated curvature tensor R and Jacobi operators R_X , X in TM, is said to be k-stein, $k \geq 1$, if there exists a function μ_k on M such that

$$\operatorname{tr}(R_X^k) = \mu_k \, |X|^{2k} \text{ for all } X \text{ in } TM.$$

We study the k-stein condition on Damek-Ricci spaces: these spaces are Einstein and 2-stein, since they are harmonic . We show that Damek-Ricci spaces are not k-stein for any $k \geq 3$, unless they are symmetric.

Let M be a Riemannian manifold, R its curvature tensor and R_X the Jacobi operator defined by $R_X Y = R(Y, X)X$, X a unit tangent vector in TM. For any natural $k \geq 1$, M is said to be k-stein (equivalently, M satisfies the k-stein condition) if there exist a real-valued function μ_k on M such that

$$\operatorname{tr}(R_X^k) = \mu_k(p) \, |X|^{2k} \text{ for all } X \text{ in } T_pM.$$

Note the difference between this definition and the one given in [5]. The k-stein conditions are related to the Osserman property as follows: A Riemannian manifold M is Osserman if and only if M is k-stein for all $k = 1, \ldots,$ $\dim M - 1$. A detailed proof is given in [4, Proposition 2.1] (see also [5, Proposition 2.1]).

It is immediate that irreducible symmetric spaces of rank one are k-stein for all $k \geq 1$ (the eingenvalues of the Jacobi operators R_X are constant for $X \in TM$, $|X| = 1$). The first examples of non-symmetric spaces we know that are k-stein for some $k \geq 2$ are the Damek-Ricci spaces; in this case for $k = 2$

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(also $k = 1$), since they are harmonic. Damek-Ricci spaces have sectional curvature $K \leq 0$ and they are the first examples of noncompact harmonic spaces which are not symmetric, in case that the sectional curvature is not strictly negative (see [3]).

We remark that in a locally symmetric space M the k-stein condition coincides with the so called k^{th}-Ledger conditions for all $k \geq 1$, satisfied for harmonic spaces. The first of these is that of being Einstein (or 1-stein if $\dim M \geq 3$) and the second one is the 2-stein condition (see [2]).

In this exposition we analyze the k-stein condition for $k \geq 3$ on Damek-Ricci spaces, which are a distinguished subclass in that of metric Lie groups S of Iwasawa type. They contain the symmetric spaces of noncompact type and rank one, and are defined as solvable extensions of codimension 1 of Heisenberg type groups. The rank one symmetric spaces of noncompact type are characterized among them as those whose sectional curvature is strictly negative. See [1] for details.

We show that if a Damek-Ricci space satisfies the k-stein condition for some $k \geq 3$ then it is a symmetric space of noncompact type and rank one. In this case it is k-stein for all $k \geq 1$.

We refer to [4] where is proved that: If S is a Carnot space that is k-stein for some $k \geq 2$, then S is a Damek-Ricci space (Theorem 4.1).

1. Preliminaries

A Lie group S of Iwasawa type and rank one is a simply connected Lie group with left invariant metric associated to a metric Lie algebra $\mathfrak{s}$ of Iwasawa type and rank one. That is, $\mathfrak{s}$ is a solvable Lie algebra with inner product $\langle , \rangle$ satisfying the conditions:

(i) $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $H \perp \mathfrak{n}$, $|H| = 1$.

(ii) $\mathrm{ad}_H|_{\mathfrak{n}}$ is symmetric and has all positive eigenvalues.

The Levi Civita connection ∇ and the curvature tensor R associated to the left invariant metric on S, can be computed by

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle \\ R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,, \text{ for any } X, Y, Z \in \mathfrak{s}. \end{aligned}$$

In this case $\mathfrak{n}$ decomposes $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and $\mathfrak{v}$ is the orthogonal complement of $\mathfrak{z}$ with respect to the metric $\langle , \rangle$ restricted to $\mathfrak{n}$. Moreover, $\mathfrak{z}$ and $\mathfrak{v}$ are invariant under ad_H.

For any $Z \in \mathfrak{z}$, the skew-symmetric operator $j_Z : \mathfrak{v} \to \mathfrak{v}$ is defined by

$$\langle j_Z X, Y \rangle = \langle [X, Y], Z \rangle \quad \text{for all } X,\ Y \in \mathfrak{v},$$

and plays an important role for describing the geometry of $\mathfrak{n}$ or $\mathfrak{s}$.

S is a Carnot space if its metric Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ satisfies

$$\mathrm{ad}_H|_{\mathfrak{z}} = \mathrm{Id},\ \mathrm{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\mathrm{Id}.$$

Moreover, S is said to be a Damek-Ricci space if S is Carnot and

$$j_Z^2 = -\left|Z\right|^2 \mathrm{Id} \text{ for all } Z \in \mathfrak{z}$$

holds, whenever $\mathfrak{v} \neq 0$. If $\mathfrak{v} = 0$, S corresponds to the real hyperbolic space.

Recall that Damek-Ricci spaces S contain the symmetric spaces of noncompact type and rank one, which are those satisfying $\nabla R = 0$. Indeed, they are given by either $\mathfrak{v} = 0$ or $\dim \mathfrak{z} = 1, 3$ and 7; in these cases $\mathfrak{s}$ corresponds to the solvable part of the Iwasawa decomposition of the Lie algebra of the isometry group of the real hyperbolic space $\mathbf{R}H^{n+1}$, the complex hyperbolic space $\mathbf{C}H^{n+1}$, the quaternionic hyperbolic space $\mathbf{Q}H^{n+1}$ and the Cayley hyperbolic plane $\mathbf{Cay}H^2$, respectively (see [1] for details).

In what follows we assume that S is a Damek-Ricci space.

1.1. *The k-stein condition*

We say that S is k-stein, or $\mathfrak{s}$ satisfies the k-stein condition, if for some constant μ_k

$$\mathrm{tr}(R_X^k) = \mu_k \left|X\right|^{2k} \text{ for all } X \in \mathfrak{s}.$$

Note that for $k = 1$, it means that S is Einstein.

Let $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ be unit vectors and set $n = \dim \mathfrak{z}$, $m = \dim \mathfrak{v}$. Recall that $\mathfrak{v}$ decomposes as an orthogonal direct sum

$$\mathfrak{v} = \ker \mathrm{ad}_X|_{\mathfrak{v}} \oplus j_{\mathfrak{z}} X.$$

We express $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}^* \oplus \mathfrak{v}^*$ where $\mathfrak{s}_0$, $\mathfrak{s}^*$ and $\mathfrak{v}^*$ are defined by

$$\begin{aligned} \mathfrak{s}_0 &= \mathrm{span}\{Z, X, j_Z X, H\},\ \mathfrak{s}^* = \mathfrak{z} \cap Z^{\perp} \oplus j_{\mathfrak{z} \cap Z^{\perp}} X \\ \mathfrak{v}^* &= \ker \mathrm{ad}_X|_{\mathfrak{v}} \cap X^{\perp}, \text{ respectively.} \end{aligned}$$

Note that $\mathfrak{s}_0$ and $\mathfrak{z} \oplus \mathbf{R}H$ are totally geodesic subalgebras of $\mathfrak{s}$; that is $\nabla_U V \in \mathfrak{s}_0$ ($\mathfrak{z} \oplus \mathbf{R}H$) whenever U, $V \in \mathfrak{s}_0$ ($\mathfrak{z} \oplus \mathbf{R}H$). Moreover, $\mathfrak{s}_0$ is the metric Lie algebra of Iwasawa type associated to the symmetric space $\mathbf{C}H^2$

($n = 1$, $m = 2$) and $\mathfrak{z} \oplus \mathbf{R}H$, as subalgebra of $\mathfrak{s}$, has associated Lie group that corresponds to the real hyperbolic space $\mathbf{R}H^{n+1}$.

We remark that any symmetric space of noncompact type and rank one is k-stein for all $k \geq 1$ (see [4, Section 2]). In particular, the Lie algebras $\mathfrak{z} \oplus \mathbf{R}H$ and $\mathfrak{s}_0$, as defined above, satisfy the k-stein condition. Consequently, for any unit vectors $Z \in \mathfrak{z}$, $X \in \mathfrak{v}$ and real numbers r, s, with $r^2 + s^2 = 1$ we have that for all $k \geq 1$,

$$\operatorname{tr}\left(R^k_{rZ+sH}\big|_{\mathfrak{z}\oplus\mathbf{R}H} \right) = \operatorname{tr}(\, -\operatorname{ad}^2_H\big|_{\mathfrak{z}\oplus\mathbf{R}H})^k$$

and

$$\operatorname{tr}\left(R^k_{rZ+sX}\big|_{\mathfrak{s}_0} \right) = \operatorname{tr}(\, -\operatorname{ad}^2_H\big|_{\mathfrak{s}_0})^k.$$

1.2. *The curvature formulas*

For all unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$, using the curvature formulas, we get

$$\begin{aligned} &R_Z|_{\mathfrak{z}\oplus\mathbf{R}H\cap Z^\perp} = -\mathrm{Id}, \quad R_Z|_{\mathfrak{v}} = -\frac{1}{4}\,\mathrm{Id}, \qquad R_H = -\operatorname{ad}^2_H,\\ &R_X|_{\mathfrak{z}\oplus\mathbf{R}H} = -\frac{1}{4}\,\mathrm{Id}, \quad R_X|_{\ker \operatorname{ad}_X|_{\mathfrak{v}}\cap X^\perp} = -\frac{1}{4}\,\mathrm{Id}, \quad R_X|_{j_{\mathfrak{z}}X} = -\mathrm{Id}, \end{aligned}$$

for all unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$.

1.3. *Properties of the operator* $j_{(.)}$.

For all X, $Y \in \mathfrak{v}$ and Z, $Z^* \in \mathfrak{z}$

$$\begin{aligned} &j_Z^2 = -|Z|^2\,\mathrm{Id}, && [Y, j_{Z^*}Y] = |Y|^2\, Z^*,\\ &\langle j_Z Y, j_{Z^*}Y\rangle = |Y|^2 \langle Z, Z^*\rangle, && \langle j_Z X, Y\rangle + \langle X, j_Z Y\rangle = 0,\\ &j_Z \circ j_{Z^*} + j_{Z^*} \circ j_Z = -2\langle Z, Z^*\rangle\, \mathrm{Id}_{\mathfrak{v}}. \end{aligned}$$

Recall that the symmetric spaces of noncompact type and rank one are characterized in the class of Damek-Ricci spaces, as those satisfying the so called J^2-condition; that is,

$$j_{Z^*} j_Z X \in j_{\mathfrak{z}} X \quad \text{for all } X \in \mathfrak{v} \text{ and } Z \perp Z^* \text{ in } \mathfrak{z}$$

unit vectors. See [1, Chapter 4] for details.

2. Damek-Ricci spaces and the k-stein condition

Next we show in Theorem 2.1 that a Damek-Ricci space S is not k-stein for any $k \geq 3$, unless S is symmetric. Let S be a Damek-Ricci space with metric Lie algebra $\mathfrak{s} = \mathfrak{z} \oplus \mathfrak{v} \oplus \mathbf{R}H$, $|H| = 1$. Let $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ be unit vectors and set $n = \dim \mathfrak{z}$, $m = \dim \mathfrak{v}$. We use the same notation as in Preliminaries.

In what follows we fix unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ and take r, s real numbers such that $r^2 + s^2 = 1$. We express

$$R_{rZ+sX} = r^2 R_Z + s^2 R_X + rsT,$$

where T is the symmetric operator on $\mathfrak{s}$ defined by $T(\cdot) = R(\cdot, Z)X + R(\cdot, X)Z$.

We compute

$$T(Z^*) = \frac{3}{4} j_{Z^*} j_Z X, \qquad T(Y) = \frac{3}{4}[j_Z X, Y],$$

for any $Z^* \perp Z$ in $\mathfrak{z}$ and $Y \perp X$ in $\mathfrak{v} \cap (j_Z X)^{\perp}$. It is a direct computation to see that

$$T^2\big|_{\mathfrak{z}\cap Z^{\perp}} = \frac{9}{16}\mathrm{Id} \text{ and } T^2\big|_{j_{\mathfrak{z}\cap Z^{\perp}}(j_Z X)} = \frac{9}{16}\mathrm{Id}.$$

Some properties of T related with the Jacobi operators R_Z and R_X are given in the following lemma, which is very useful for the proof of Proposition 2.1. See the proofs in [4, Lemma 3.2 and Proposition 3.3].

We fix $\{Z_i^* : i = 1, ..., n-1\}$ an orthonormal basis of $\mathfrak{z} \cap Z^{\perp}$ and set

$$\mathrm{tr}\left((-R_X)^i\Big|_{j_{\mathfrak{z}\cap Z^{\perp}}(j_Z X)} \right) = \sum_{l=1}^{n-1} \left\langle (-R_X)^i \left(j_{Z_l^*} j_Z X\right), j_{Z_l^*} j_Z X \right\rangle \text{ for any } i \geq 1.$$

Lemma 2.1 *If $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ are unit vectors of the Lie algebra $\mathfrak{s}$ with $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ then,*

(*i*) *for all odd $k \geq 1$ and $j, l \geq 1$,*

$$tr\left((r^2 R_Z + s^2 R_X)^j\, T^k \left(r^2 R_Z + s^2 R_X\right)^l \right) = 0.$$

(*ii*)

$$\left| tr\left(R_X^j T R_X^i T\Big|_{\mathfrak{s}^* \oplus \mathfrak{v}^*} \right) \right| = \frac{9}{16}\frac{1}{4^{i+j}} \left(4^i\, tr\left((-R_X)^i\big|_{j_{\mathfrak{z}\cap Z^{\perp}}(j_Z X)} \right) + 4^j\, tr\left((-R_X)^j\big|_{j_{\mathfrak{z}\cap Z^{\perp}}(j_Z X)} \right) \right)$$

$$\leq \frac{9}{16}\,(n-1)\left(\frac{4^i + 4^j}{4^{i+j}} \right).$$

Proposition 2.1 *Let S be a Damek-Ricci space that is k-stein for some $k \geq 3$. Then, for any unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ the operators R_Z, R_X and the associated T defined by $T(\cdot) = R(\cdot, Z)X + R(\cdot, X)Z$, are related by the following condition*

$$0 = k tr \left(R_Z R_X^{k-1} - \left(-ad_H^2\right)^k \right)\Big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*} + k tr \left(R_X^{k-2}T^2\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}$$
$$+ \sum_{l=1}^{k-3}\sum_{i=0}^{k-3-l} tr\big(R_X^{i+l} T R_X^{k-2-l-i} T\big)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*} + \sum_{i=1}^{k-3} tr\big(R_X^{i} T R_X^{k-2-i} T\big)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}.$$

Theorem 2.1 *Let S be a Damek-Ricci space. If S is k-stein for some $k \geq 3$, then S is a symmetric space of noncompact type and rank one.*

Proof. Assume that S is k-stein for some $k \geq 3$ and we will prove that S is symmetric. To that end we will show that the J^2-condition (see 1.3) is satisfied. Let $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ be unit vectors; we use the terminology given at the beginning of this section.

Next, we express the condition given by Proposition 2.1. For this purpose we compute:

(i) $$\mathrm{tr}\left(R_X^{k-1} R_Z - \left(-\mathrm{ad}_H^2\right)^k \right)\Big|_{\mathfrak{v}^*} = 0,$$

which is immediate since $R_Z|_{\mathfrak{v}^*} = R_X|_{\mathfrak{v}^*} = -\frac{1}{4}\mathrm{Id} = -\mathrm{ad}_H^2\big|_{\mathfrak{v}^*}$.

(ii) $$\mathrm{tr}\left(R_X^{k-1} R_Z - \left(-\mathrm{ad}_H^2\right)^k \right)\Big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*} = 3(n-1)(-1)^k \left(\frac{1-4^{k-1}}{4^k}\right).$$

In fact, since

$$\begin{aligned}
\mathrm{tr}\left(R_X^{k-1}R_Z\right)\big|_{\mathfrak{s}^*} &= \mathrm{tr}\left(R_X^{k-1}R_Z\right)\big|_{\mathfrak{z}\cap Z^\perp} + \mathrm{tr}\left(R_X^{k-1}R_Z\right)\big|_{j_{\mathfrak{z}\cap Z^\perp}(X)} \\
&= -\mathrm{tr}\left(R_X^{k-1}\big|_{\mathfrak{z}\cap Z^\perp}\right) + (-\frac{1}{4})\,\mathrm{tr}\left(R_X^{k-1}\big|_{j_{\mathfrak{z}\cap Z^\perp}(X)}\right) \\
&= -(n-1)\left(\left(-\frac{1}{4}\right)^{k-1} + \frac{1}{4}(-1)^{k-1}\right) \\
&= (n-1)(-1)^k\left(\frac{1+4^{k-2}}{4^{k-1}}\right)
\end{aligned}$$

and

$$\mathrm{tr}\left((-\mathrm{ad}_H^2)^k\big|_{\mathfrak{s}^*}\right) = (n-1)\left((-1)^k + \left(-\frac{1}{4}\right)^k\right) = (n-1)(-1)^k\left(\frac{4^k+1}{4^k}\right),$$

it follows that

$$\operatorname{tr}\left(R_X^{k-1}R_Z - \left(-\mathrm{ad}_H^2\right)^k\right)\Big|_{\mathfrak{s}^*} = (n-1)(-1)^k\left(\frac{1+4^{k-2}}{4^{k-1}} - \frac{4^k+1}{4^k}\right)$$
$$= 3(n-1)(-1)^k\left(\frac{1-4^{k-1}}{4^k}\right).$$

The assertion follows from (i) above.

(iii) $$\operatorname{tr}\left(R_X^{k-2}T^2\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*} = (-1)^k\left(\frac{9}{16}(n-1)\left(\frac{1+4^{k-2}}{4^{k-2}}\right) + \left(\frac{1-4^{k-2}}{4^{k-2}}\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right)\right).$$

Expressing

$$\operatorname{tr}\left(R_X^{k-2}T^2\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*} = \operatorname{tr}\left(R_X^{k-2}T^2\right)\big|_{\mathfrak{z}\cap Z^\perp} + \operatorname{tr}\left(R_X^{k-2}T^2\right)\big|_{j_{\mathfrak{z}\cap Z^\perp}(X)\oplus\mathfrak{v}^*},$$

we compute these two terms separately, by taking orthonormal bases $\{Z_i^* : i = 1, ..., n-1\}$ of $\mathfrak{z}\cap Z^\perp$ and $\{Y_j : j = 1, ..., m-n-1\}$ of $\mathfrak{v}^*$. We set

$$\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right) = \sum_{j=1}^{m-n-1}\left\langle T^2(Y_j), Y_j\right\rangle = \sum_{j=1}^{m-n-1}\left|T(Y_j)\right|^2$$

Thus,

$$\operatorname{tr}\left(R_X^{k-2}T^2\right)\big|_{\mathfrak{z}\cap Z^\perp} = \sum_{l=1}^{n-1}\left\langle R_X^{k-2}T^2(Z_l^*), Z_l^*\right\rangle = \frac{9}{16}\sum_{l=1}^{n-1}\left\langle R_X^{k-2}(Z_l^*), Z_l^*\right\rangle$$
$$= \frac{9}{16}(n-1)\left(-\frac{1}{4}\right)^{k-2} = \frac{9}{16}(n-1)(-1)^k\frac{1}{4^{k-2}}$$

and

$$\operatorname{tr}\left(R_X^{k-2}T^2\right)\big|_{j_{\mathfrak{z}\cap Z^\perp}X\oplus\mathfrak{v}^*} =$$
$$= \sum_{l=1}^{n-1}\left\langle T^2R_X^{k-2}(j_{Z_l^*}X), j_{Z_l^*}X\right\rangle + \sum_{j=1}^{m-n-1}\left\langle T^2R_X^{k-2}(Y_j), Y_j\right\rangle,$$

which gives

$$
\begin{aligned}
&= (-1)^{k-2}\sum_{l=1}^{n-1}\left\langle T^2(j_{Z_l^*}X), j_{Z_l^*}X\right\rangle + \left(-\frac{1}{4}\right)^{k-2}\sum_{j=1}^{m-n-1}\left\langle T^2(Y_j), Y_j\right\rangle \\
&= (-1)^k\left(\operatorname{tr} T^2\big|_{j_{\mathfrak{z}\cap Z^\perp}(X)\oplus\mathfrak{v}^*} + \left(\frac{1}{4^{k-2}}-1\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right)\right) \\
&= (-1)^k\left(\operatorname{tr} T^2\big|_{\mathfrak{s}_0^\perp\cap\mathfrak{v}} + \left(\frac{1}{4^{k-2}}-1\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right)\right) \\
&= (-1)^k\left(\frac{9}{16}(n-1) + \left(\frac{1-4^{k-2}}{4^{k-2}}\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right)\right),
\end{aligned}
$$

since $\operatorname{tr}\Big(T^2\big|_{\mathfrak{s}_0^\perp\cap\mathfrak{v}}\Big) = \operatorname{tr}\Big(T^2\big|_{j_{\mathfrak{z}\cap Z^\perp}(j_Z X)}\Big) = \frac{9}{16}(n-1)$.

(iv) $$\left|\operatorname{tr}\left(R_X^{i+l}TR_X^{k-2-l-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right| \le \frac{9}{16}(n-1)\frac{(4^{i+l}+4^{k-2-l-i})}{4^{k-2}}$$

This is a direct application of Lemma 2.1 (ii). Moreover, we remark that

$$
\begin{aligned}
&\operatorname{tr}\left(R_X^{i+l}TR_X^{k-2-l-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*} = \\
&\quad= \frac{9}{16}(-1)^k\Big(\frac{1}{4^{i+l}}\operatorname{tr}\,(-R_X)^{k-2-l-i})\big|_{j_{\mathfrak{z}\cap Z^\perp}(j_Z X)} \\
&\quad+ \frac{1}{4^{k-2-l-i}}\operatorname{tr}\,(-R_X)^{i+l})\big|_{j_{\mathfrak{z}\cap Z^\perp}(j_Z X)}\Big) \\
&\quad= (-1)^k\left|\operatorname{tr}\left(R_X^{i+l}TR_X^{k-2-l-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right|,\ \text{since } (-1)^k = (-1)^{k-2}.
\end{aligned}
$$

Now, taking into account the above remark and substituting the equalities given by (ii), (iii) in the condition given by Proposition 2.1, we obtain

$$
\begin{aligned}
0 &= 3k(n-1)(-1)^k\left(\frac{1-4^{k-1}}{4^k}\right) \\
&+ (-1)^k k\left(\frac{9}{16}(n-1)\left(\frac{1+4^{k-2}}{4^{k-2}}\right) + \left(\frac{1-4^{k-2}}{4^{k-2}}\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right)\right) \\
&+ (-1)^k\sum_{l=1}^{k-3}\sum_{i=0}^{k-3-l}\left|\operatorname{tr}\left(R_X^{i+l}TR_X^{k-2-l-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right| \\
&+ (-1)^k\sum_{i=1}^{k-3}\left|\operatorname{tr}\left(R_X^{i}TR_X^{k-2-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right|,
\end{aligned}
$$

which in turn is equivalent to

$$\begin{aligned} 0 = k\left(\frac{1-4^{k-2}}{4^{k-2}}\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right) + 3k(n-1)\left(\frac{1-4^{k-1}}{4^k}\right) \\ + \frac{9}{16}(n-1)k\left(\frac{1+4^{k-2}}{4^{k-2}}\right) \\ + \sum_{l=1}^{k-3}\sum_{i=0}^{k-3-l}\left|\operatorname{tr}\left((R_X)^{i+l}T(R_X)^{k-2-l-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right| \\ + \sum_{i=1}^{k-3}\left|\operatorname{tr}\left((R_X)^{i}T(R_X)^{k-2-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right|. \end{aligned} \tag{1}$$

By using the inequality obtained in (iv) we have that

$$\begin{aligned} &\sum_{l=1}^{k-3}\sum_{i=0}^{k-3-l}\left|\operatorname{tr}\left(R_X^{i+l}TR_X^{k-2-l-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right| + \sum_{i=1}^{k-3}\left|\operatorname{tr}\left(R_X^{i}TR_X^{k-2-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right| \\ &\leq \frac{9}{16}(n-1)\frac{1}{4^{k-2}}\left(\sum_{l=1}^{k-3}\sum_{i=0}^{k-3-l}\left(4^{i+l}+4^{k-2-l-i}\right) + \sum_{i=1}^{k-3}\left(4^{i}+4^{k-2-i}\right)\right). \end{aligned}$$

Therefore, since $k\left(\frac{1+4^{k-2}}{4^{k-2}}\right)$ is exactly the sum of the k terms equal to $\frac{1+4^{k-2}}{4^{k-2}}$ in the sum

$$\frac{1}{4^{k-2}}\sum_{l=0}^{k-2}\sum_{i=0}^{k-2-l}\left(4^{i+l}+4^{k-2-l-i}\right),$$

corresponding to the values $l=0$, $i=0$, $i=k-2$ and for each $1\leq l\leq k-2$, $i=k-2-l$, we have

$$\begin{aligned} &\frac{9}{16}(n-1)k\left(\frac{1+4^{k-2}}{4^{k-2}}\right) + \sum_{l=1}^{k-3}\sum_{i=0}^{k-3-l}\left|\operatorname{tr}\left(R_X^{i+l}TR_X^{k-2-l-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right| \\ &+\sum_{i=1}^{k-3}\left|\operatorname{tr}\left(R_X^{i}TR_X^{k-2-i}T\right)\big|_{\mathfrak{s}^*\oplus\mathfrak{v}^*}\right| \\ &\leq \frac{9}{16}(n-1)\frac{1}{4^{k-2}}\sum_{l=0}^{k-2}\sum_{i=0}^{k-2-l}\left(4^{i+l}+4^{k-2-l-i}\right) = 3k(n-1)\left(\frac{4^{k-1}-1}{4^k}\right), \end{aligned} \tag{2}$$

provided that

$$\sum_{l=0}^{k-2}\sum_{i=0}^{k-2-l}\frac{1}{4^{k-2}}\left(4^{k-2-l-i}+4^{i+l}\right) = \frac{1}{3}\,k\,\frac{4^{k-1}-1}{4^{k-2}}.$$

This formula was showed in [4, Section 3].

Therefore, condition (1) implies that

$$0 = k\left(\frac{1-4^{k-2}}{4^{k-2}}\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right) + 3k(n-1)\left(\frac{1-4^{k-1}}{4^k}\right) + A,$$

where

$$0 \le A \le 3k(n-1)\left(\frac{4^{k-1}-1}{4^k}\right).$$

Thus, equality occurs in (2) and also

$$k\left(\frac{4^{k-2}-1}{4^{k-2}}\right)\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right) = 0.$$

Hence, since $k \ge 3$ we have that

$$\operatorname{tr}\left(T^2\big|_{\mathfrak{v}^*}\right) = 0.$$

From this condition, it follows that for any unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ the associated operator T satisfies $T(Y) = \frac{3}{4}[j_Z X, Y] = 0$ for all $Y \in \mathfrak{v}^* = \ker \operatorname{ad}_X|_{\mathfrak{v}} \cap X^\perp$. Thus, for all $Z^* \perp Z$ in $\mathfrak{z}$ and $Y \in \ker \operatorname{ad}_X|_{\mathfrak{v}}$, $\langle [j_Z X, Y], Z^*\rangle = 0$ or equivalently,

$$\langle j_{Z^*} j_Z X, Y\rangle = 0 \text{ for all } Y \in \ker \operatorname{ad}_X|_{\mathfrak{v}}.$$

Hence, $j_{Z^*} j_Z X \in \left(\ker \operatorname{ad}_X|_{\mathfrak{v}}\right)^\perp = j_{\mathfrak{z}} X$. This fact means that $\mathfrak{n}$ satisfies the J^2-condition, which in turn is equivalent for S to be a symmetric space. The assertion of the theorem is proved. □

Example 2.1 Any non symmetric Damek-Ricci space provides examples of homogeneous spaces S which are Einstein and 2-stein but are not k-stein for any other $k \ge 3$.

In particular, an example in dimension seven is obtained when we put $n = 2$, $m = 4$. Assume that $\{Z_1, Z_2\}$ and $\{X, Y, j_{Z_1}X, j_{Z_2}X\}$ are orthonormal bases of $\mathfrak{z}$ and $\mathfrak{v}$, respectively, with $Y = j_{Z_1} j_{Z_2} X$. Here, j_{Z_i} $i = 1, 2$, are skew-symmetric operators on $\mathfrak{v}$ satisfying $j_{Z_i}^2 = -\mathrm{Id}$ and $j_{Z_1} j_{Z_2} = -j_{Z_2} j_{Z_1}$. Let $\mathfrak{s}$ be the Lie algebra spanned by the orthonormal basis $\{Z_1, Z_2, X, Y, j_{Z_1}X, j_{Z_2}X, H\}$ with bracket

$$\begin{aligned}
&[Z_1, Z_2] = [X, Y] = 0, && [X, j_{Z_i}X] = Z_i,\ i = 1, 2,\\
&[Y, j_{Z_1}X] = -Z_2, && [Y, j_{Z_2}X] = Z_1,\\
&\operatorname{ad}_H|_{\mathfrak{z}} = \mathrm{Id}, && \operatorname{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\,\mathrm{Id}.
\end{aligned}$$

Then S, the simply connected Lie group associated to $\mathfrak{s}$, is a homogeneous space of dimension 7 satisfying the required properties.

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