ON OSSERMAN MANIFOLDS OF DIMENSION 16 *

YURI NIKOLAYEVSKY †

Department of Mathematics, La Trobe University,
Bundoora, 3083 Victoria, Australia

E-mail: Y.Nikolayevsky@latrobe.edu.au

For a Riemannian manifold M^n with the curvature tensor R, the Jacobi operator R_X is defined by $R_X Y = R(X,Y)X$. The manifold M^n is called *pointwise Osserman* if, for every $p \in M^n$, the eigenvalues of the Jacobi operator R_X do not depend on the choice of a unit vector $X \in T_pM^n$, and is called *globally Osserman* if they do not depend of the point p either. R. Osserman conjectured that globally Osserman manifolds are flat or rank-one symmetric. This Conjecture was proved in all the cases, except for manifolds of dimension $n = 16$ whose Jacobi operator has an eigenvalue of multiplicity $m \in \{7, 8, 9\}$. Here we give the proof in the case $m = 9$.

1. Introduction

An *algebraic curvature tensor* R in a Euclidean space $\mathbb{R}^n$ is a $(3,1)$ tensor having the same symmetries as the curvature tensor of a Riemannian manifold. For $X \in \mathbb{R}^n$, the *Jacobi operator* $R_X : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $R_X Y = R(X,Y)X$. The Jacobi operator is symmetric and $R_X X = 0$ for all $X \in \mathbb{R}^n$. Throughout the paper, "eigenvalues of the Jacobi operator" refers to eigenvalues of the restriction of R_X, with X a unit vector, to the subspace $X^\perp$.

Definition 1.1 An algebraic curvature tensor R is called *Osserman* if the eigenvalues of the Jacobi operator R_X do not depend of the choice of a unit vector $X \in \mathbb{R}^n$.

Definition 1.2 A Riemannian manifold M^n is called *pointwise Osserman* if its curvature tensor is Osserman. If, in addition, the eigenvalues of the

* *MSC 2000:* 53C25.
Keywords: Osserman manifold, Jacobi operator, Clifford structure.
† Work supported by the ARC Discovery grant DP0342758.

Jacobi operator are constant on M^n, the manifold M^n is called *globally Osserman.*

Flat and rank-one symmetric spaces are globally Osserman, since the isometry group of each of them acts transitively on its unit tangent bundle. Osserman [9] conjectured that the converse is also true:

Conjecture 1.1 *A globally Osserman manifold is flat or locally rank-one symmetric.*

Except for a few cases in dimension 16, the answer to the Osserman Conjecture is affirmative, as well as to its "pointwise" version (see the Corollary below).

In this paper, we treat one of the remaining cases and prove the following:

Theorem 1.1 *The Jacobi operator of a pointwise Osserman manifold of dimension sixteen cannot have an eigenvalue of multiplicity* 9*.*

Combining this with Theorems 1 and 2 of [7], and with the result of [8] we get:

Corollary 1.1 *Let M^n be a Riemannian globally Osserman manifold of dimension $n \geq 2$, or a pointwise Osserman manifold of dimension $n \neq 2, 4$. Then M^n is flat or locally rank-one symmetric except, possibly, in the following case: $n = 16$ and the Jacobi operator has an eigenvalue of multiplicity* 7 *or* 8*.*

In the cases covered by the Corollary, there is not much difference between globally and pointwise Osserman conditions, except in dimension 2, where any Riemannian manifold is pointwise Osserman, and in dimension 4, where any globally Osserman manifold is flat or locally rank-one symmetric [1], but there exist pointwise Osserman manifolds that are not even locally symmetric ("generalized complex space forms", Corollary 2.7 of [3]).

We would also like to announce the following theorem, the proof of which is to appear elsewhere:

Theorem 1.2 *A pointwise Osserman manifold of dimension sixteen whose Jacobi operator has two eigenvalues, of multiplicity* 7 *and* 8*, respectively, is locally isometric to the Cayley projective plane or to its hyperbolic dual.*

The paper is organized as follows.

In section 2, we introduce algebraic curvature tensors with a Clifford structure and show that Theorem 1.1 follows from Proposition 2.1 saying that an Osserman algebraic curvature tensor in $\mathbb{R}^{16}$ whose Jacobi operator has an eigenvalue of multiplicity 9 has a Clifford structure. Proposition 2.1 is then proved in section 3.

2. Manifolds with Clifford structure. Proof of Theorem 1.1

We will follow the two-step approach to the Osserman Conjecture suggested in [3]:

(i) find all Osserman algebraic curvature tensors;

(ii) classify Riemannian manifolds having curvature tensor as in (i).

The standard tool for (ii) is the second Bianchi identity. The difficult part is (i), but thanks to the remarkable construction of [2,3], we know the right candidate for (i), a typical Osserman algebraic curvature tensor:

Definition 2.1 An algebraic curvature tensor R in $\mathbb{R}^n$ has a Cliff(ν)-*structure* ($\nu \geq 0$), if there exist anticommuting skew-symmetric orthogonal operators $J_1, \dots, J_\nu$, and the numbers $\lambda_0, \mu_1, \dots \mu_\nu$, with $\mu_s \neq \lambda_0$, such that

$$\begin{aligned} R(X,Y)Z &= \lambda_0(\langle X,Z\rangle Y - \langle Y,Z\rangle X) \\ &\quad + \sum_{s=1}^{\nu} \tfrac{1}{3}(\mu_s - \lambda_0)(2\langle J_sX,Y\rangle J_sZ + \langle J_sZ,Y\rangle J_sX - \langle J_sZ,X\rangle J_sY). \end{aligned} \quad (1)$$

A Riemannian manifold M^n has a Cliff(ν)-*structure* if its curvature tensor at every point does.

The fact that skew-symmetric operators J_s are orthogonal and anticommute is equivalent to each of the following sets of equations: $\langle J_sX, J_qX\rangle = \delta_{sq}\|X\|^2$ and $J_sJ_q + J_qJ_s = -2\delta_{sq}I_n$, for all $s,q = 1,\dots,\nu$ and all $X \in \mathbb{R}^n$.

The Jacobi operator of the algebraic curvature tensor R with the Clifford structure given by (1) has the form

$$R_XY = \lambda_0(\|X\|^2Y - \langle Y,X\rangle X) + \sum_{s=1}^{\nu}(\mu_s - \lambda_0)\langle J_sX,Y\rangle J_sX, \quad (2)$$

and the tensor R can be reconstructed from (2) using polarization and the first Bianchi identity.

A Cliff(ν) algebraic curvature tensor (manifold) is Osserman (pointwise Osserman, respectively). For any unit vector X, the Jacobi operator R_X given by (2) has constant eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$, where $\lambda_1, \dots, \lambda_{k-1}$

are the μ_s's without repetitions. The eigenspace corresponding to the eigenvalue λ_α, $\alpha \neq 0$, is $E_{\lambda_\alpha}(X) = \mathrm{Span}_{s:\mu_s=\lambda_\alpha}(J_s X)$, and the λ_0-eigenspace is $E_{\lambda_0}(X) = (\mathrm{Span}(X, J_1 X, \dots, J_\nu X))^\perp$, provided $\nu < n-1$.

Theorem 1.1 follows from

Proposition 2.1 *An Osserman algebraic curvature tensor in* $\mathbb{R}^{16}$ *whose Jacobi operator has an eigenvalue of multiplicity* 9 *has a Clifford structure* Cliff(6).

By Theorem 1.2 of [5], a Riemannian manifold M^{16} with a Cliff(6)-structure is locally rank-one symmetric. However, no rank-one symmetric space of dimension 16 has the Jacobi operator with an eigenvalue of multiplicity 9.

3. Proof of Proposition 2.1

Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^{16}$ whose Jacobi operator has an eigenvalue of multiplicity 9. Shifting R by a multiple of the curvature tensor of a unit sphere (which preserves both the Osserman property and the property to have a Clifford structure) we can assume this eigenvalue to be 0.

Define an *eigenvalue structure for* R to be the list of multiplicities of its Jacobi operator, in the nondecreasing order. There are eleven possible eigenvalue structures:

$$(9,6), \tag{3}$$

$$(9,3,3), \tag{4}$$

$$(9,5,1), (9,4,2), (9,4,1,1), (9,3,2,1), (9,3,1,1,1), \tag{5}$$

$$(9,2,2,2), (9,2,2,1,1), (9,2,1,1,1,1), (9,1,1,1,1,1,1). \tag{6}$$

For the eigenvalue structure $(9,6)$ the claim follows from Theorem 2 of [6]. We will separately consider the case of the eigenvalue structure $(9,3,3)$, in subsection 3.1, and all the other cases (when the Jacobi operator has an eigenvalue of multiplicity 1 or 2), in subsection 3.2. In the proof, we will use some facts from commutative algebra collected in subsection 3.3.

3.1. *Jacobi operator has an eigenvalue of multiplicity* 1 *or* 2

In this subsection, we show that an Osserman algebraic curvature tensor whose Jacobi operator has an eigenvalue of multiplicity 9 and an eigenvalue of multiplicity at most 2 has a Clifford structure. Our starting point is the following two Lemmas:

Lemma 3.1 (1.) *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^n$ whose Jacobi operator has k distinct eigenvalues, one of which is zero, and let $\lambda \neq 0$ be a simple eigenvalue. Then for every $X \neq 0$, the eigenspace $E_{\lambda\|X\|^2}(X)$ of R_X is spanned by a vector $P(X)$ all of whose nonzero components are odd homogeneous polynomials of degree $2m+1 \leq k-1$, and for all unit vectors X,*

$$\langle P(X), X\rangle = 0, \qquad \|P(X)\|^2 = 1, \qquad P(P(X)) = -X. \tag{7}$$

(2.) *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^n$ whose Jacobi operator has k distinct eigenvalues, one of which is zero, and let $\lambda \neq 0$ be an eigenvalue of multiplicity 2. Then for every $X \neq 0$, the eigenspace $E_{\lambda\|X\|^2}(X)$ of R_X is spanned by vectors $U(X), V(X)$ all of whose nonzero components are odd homogeneous polynomials of degree $2m+1 \leq k-1$, and for all unit vectors X,*

$$\langle U(X), X\rangle = \langle V(X), X\rangle = \langle U(X), V(X)\rangle = 0, \quad \|U(X)\|^2 = \|V(X)\|^2 = 1.$$

The proof is given in Lemma 2 of [8].

Lemma 3.2 (1.) *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^{16}$, and let J be an orthogonal skew-symmetric operator and $C \neq 0$ a real constant. Suppose that an algebraic curvature tensor $\tilde{R}$ defined by $\tilde{R}_X Y = R_X Y - C\langle JX, Y\rangle JX$ is Osserman (this condition is automatically satisfied, if JX is an eigenvector of R_X, for all $X \in \mathbb{R}^{16}$) and has a Clifford structure. Then R also has a Clifford structure.*

(2.) *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^{16}$ having an eigenvalue 0 of multiplicity 9 and an eigenvalue $\lambda \neq 0$ of multiplicity 1 or 2. If the polynomial eigenvector $P(X)$ (respectively, at least one of the polynomial eigenvector $U(X), V(X)$) constructed as in Lemma 3.1 has the form $\|X\|^{2m} JX$ for some linear operator J, then R has a Clifford structure* Cliff(6)*.*

Proof. (1.) We assume (shifting by a constant curvature tensor) that the eigenvalue of the Jacobi operator of $\tilde{R}$ with the biggest multiplicity is 0.

Let $J_1, \ldots, J_\nu$ be a Clifford structure for $\tilde{R}$, so that

$$\tilde{R}_X Y = \sum_{\alpha=1}^{\nu} \mu_\alpha \langle J_\alpha X, Y\rangle J_\alpha X,$$

where $\mu_\alpha \neq 0$, although some of the μ_α's can be equal. Let $\lambda_1, \ldots, \lambda_p$ be the set of nonzero eigenvalues (the μ_α's without repetitions), and let

$m_1, \dots, m_p$ be the corresponding multiplicities. Relabeling the J_α's accordingly we get $\tilde{R}_X Y = \sum_{i=1}^p \lambda_i (\sum_{j=1}^{m_i} \langle J_j^{(i)} X, Y \rangle J_j^{(i)} X)$. For a unit vector X, the eigenspaces of $\tilde{R}_X$ are $E_{\lambda_i}(X) = \mathrm{Span}(J_1^{(i)} X, \dots, J_{m_i}^{(i)} X)$ and $E_0(X) = \mathrm{Ker} \tilde{R}_X = (\mathrm{Span}_\alpha (J_\alpha X))^\perp$. For R to be Osserman, it is necessary and sufficient that every projection $\pi_i(JX)$ of JX to $E_{\lambda_i}(X)$ has a constant length $c_i \geq 0$, for any unit vector X. For an arbitrary X we have $\pi_i(JX) = \sum_{j=1}^{m_i} (\langle J_j^{(i)} X, JX \rangle \|X\|^{-2}) J_j^{(i)} X$, and so $\|\pi_i(JX)\|^2 = \sum_{j=1}^{m_i} \langle J_j^{(i)} X, JX \rangle^2 \|X\|^{-2}$. This must be equal to $c_i \|X\|^2$, so $\sum_{j=1}^{m_i} \langle J_j^{(i)} X, JX \rangle^2 = \|X\|^4$. The left-hand side is a sum of squares of $m_i \leq \nu \leq 8$ polynomials, which is divisible by $\|X\|^2$. According to Lemma 3.11, each of them must be divisible by $\|X\|^2$, so that each $\langle J_j^{(i)} X, JX \rangle$ is a constant multiple of $\|X\|^2$, This implies that $\pi_i \circ J$ is a linear operator, which is a constant linear combination of $J_1^{(i)}, \dots, J_{m_i}^{(i)}$. Replacing $J_1^{(i)}, \dots, J_{m_i}^{(i)}$ by appropriate orthonormal linear combinations we can assume that $\pi_i(JX) = c_i J_1^{(i)} X$. Then the map $X \to J_{\nu+1} X$, the projection of JX to $E_0(X)$ is also linear, and the operator $J_{\nu+1}$ is skew-symmetric, is orthogonal times a constant $c_0 = \sqrt{1 - \sum_i c_i^2}$ (which can be zero), and anticommutes with all the J_α's (as $J_{\nu+1} X \perp J_\alpha X$, for $\alpha = 1, \dots, \nu$). So $J = \sum_{i=1}^p c_i J_1^{(i)} + c_0 (c_0^{-1} J_{\nu+1})$ (if $c_0 = 0$, we simply omit the last term). Substituting this to $R_X Y = \tilde{R}_X Y + C \langle JX, Y \rangle JX$ and replacing the operators $J_1^{(1)}, \dots, J_1^{(p)}, J_{\nu+1}$ by their orthonormal linear combinations (to diagonalize the symmetric matrix $\mathrm{diag}\{\lambda_1, \dots, \lambda_p, 0\} + C(c_1, \dots, c_p, c_0)^t (c_1, \dots, c_p, c_0))$ we get a Clifford structure for R.

(2.) As it follows from Lemma 3.1, the operator J is orthogonal and skew-symmetric. What is more, for any $X \in \mathbb{R}^{16}$, JX is an eigenvector of R_X, with the eigenvalue λ. Then an algebraic curvature tensor $\tilde{R}$ defined by $\tilde{R}_X Y = R_X Y - \lambda \langle JX, Y \rangle JX$ is again Osserman, with the Jacobi operator having an eigenvalue 0 of multiplicity 10. By Proposition 1 of [7], such an algebraic curvature tensor has a Clifford structure Cliff(5). By assertion 1, R has a Clifford structure (which is Cliff(6)).

One of the applications of Lemma 3.2 is immediate: for the eigenvalue structures $(9, 5, 1)$ and $(9, 4, 2)$, the degree of all the nonzero components of $P(X)$ (of $U(X), V(X)$, respectively) is 1, according to Lemma 3.1. Hence in the both cases, R has a Clifford structure Cliff(6).

We next consider the cases (6), when all the multiplicities, other than 9, are 1 or 2.

Lemma 3.3 *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^{16}$. Suppose that the Jacobi operator of R has the eigenvalue* 0*, with multiplicity* 9*, and that all the other eigenvalues have multiplicity* 1 *or* 2*. Then R has a Clifford structure* $\mathrm{Cliff}(6)$.

Proof. Let $\lambda_1, \dots, \lambda_p$ be the nonzero eigenvalues, each of multiplicity $m_j \le 2$. Let $\mu_1, \dots, \mu_6$ be the list of the λ_j's counting multiplicities (each λ_j appears m_j times in that list), and let $P_1(X), \dots, P_6(X)$ be the list of the corresponding polynomial eigenvectors of R_X (take $P(X)$ for each simple eigenvalue and $U(X), V(X)$ for each eigenvalue of multiplicity 2, as constructed in Lemma 3.1). For every i, all the nonzero components of $P_i(X)$ are homogeneous polynomials of the same odd degree d_i. Denote $d = \max_i d_i$ and replace every P_i's with $d_i < d$ by $P_i \|X\|^{d-d_i}$ (note that $d - d_i$ is even). We now have six vectors $P_i(X)$, all whose nonzero components are odd homogeneous polynomials of the same degree d, and at least one of the components of some vector $P_i(X)$ is not divisible by $\|X\|^2$. What is more, they still span the corresponding eigenspaces of R_X, and $\langle P_i(X), X\rangle = 0$, $\langle P_i(X), P_j(X)\rangle = \delta_{ij}\|X\|^{2d}$. Choose an orthonormal basis for $\mathbb{R}^{16}$ and denote $P(X)$ a 16×6 matrix whose columns are $P_1(X), \dots, P_6(X)$. Note that $P(X)^t P(X) = \|X\|^{2d} I_6$.

Let $\Lambda = \mathrm{diag}\{\mu_1, \dots, \mu_6\}$. For every $X \ne 0$, the 16×16 matrices $\|X\|^{-2d} P(X)\Lambda P(X)^t$ and $\|X\|^{-2} R_X$ have the same eigenvalues and the same eigenvectors. It follows that $P(X)\Lambda P(X)^t = \|X\|^{2d-2} R_X$. Squaring both sides we get

$$P(X)\Lambda^2 P(X)^t = \|X\|^{2d-4} R_X^2, \quad \text{as} \quad P(X)^t P(X) = \|X\|^{2d} I_6.$$

If $d \ge 3$, then every entry of the polynomial matrix $P(X)\Lambda^2 P(X)^t$ is divisible by $\|X\|^2$. The diagonal (α, α) entry of this matrix ($\alpha = 1, \dots, 16$) is $\sum_{i=1}^6 (\mu_i (P_i(X))_\alpha)^2$, which is a sum of squares of six polynomials in the variables $x_1, \dots, x_{16}$, the coordinates of X. By Lemma 3.11 of subsection 3.3, it follows that for every $i = 1, \dots, 6$ and every $\alpha = 1, \dots, 16$, the polynomial $(P_i(X))_\alpha$ is divisible by $\|X\|^2$. This contradicts the construction of the P_i's.

So $d = 1$, hence all the $P_i(X)$'s are linear, which implies that R has a Clifford structure.

We have three remaining cases for the eigenvalue structure to consider: $(9,4,1,1), (9,3,2,1)$ and $(9,3,1,1,1)$.

This is done in the following Lemma:

Lemma 3.4 *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^{16}$ whose eigenvalue structure is one of the following:* $(9,4,1,1)$, $(9,3,2,1)$, $(9,3,1,1,1)$, *with* 0 *being the eigenvalue of multiplicity* 9. *Then R has a Clifford structure* $\mathrm{Cliff}(6)$.

Proof. We start as in the proof of Lemma 3.3. Let $\lambda, \lambda_1, \dots, \lambda_p$ be the nonzero eigenvalues of the Jacobi operator, with λ the eigenvalue of multiplicity bigger than 2, and each of the λ_j of multiplicity 1 or 2. Let $\mu_1, \dots, \mu_s$ be the list of the λ_j's counting multiplicities (s here can be 2 or 3), and let $P_1(X), \dots, P_s(X)$ be the list of the corresponding polynomial eigenvectors of R_X, as constructed in Lemma 3.1. We can assume that all the nonzero components of all the P_i's are homogeneous polynomials of the same odd degree d, and at least one of the components of some of them is not divisible by $\|X\|^2$. Choose an orthonormal basis for $\mathbb{R}^{16}$ and consider a $16 \times s$ matrix $P(X)$ whose columns are $P_1(X), \dots, P_s(X)$. We have $P(X)^t P(X) = \|X\|^{2d} I_s$.

Let $\Lambda = \mathrm{diag}\{\mu_1(\mu_1 - \lambda), \dots, \mu_s(\mu_s - \lambda)\}$ (all these numbers are nonzero). For every $X \neq 0$, the 16×16 matrices $\|X\|^{-2d} P(X) \Lambda P(X)^t$ and $\|X\|^{-4} R_X(R_X - \lambda \|X\|^2 I_{16})$ have the same eigenvalues and the same eigenvectors. So

$$P(X)\Lambda P(X)^t = \|X\|^{2d-4} R_X(R_X - \lambda \|X\|^2 I_{16}). \tag{8}$$

Squaring both sides of equation (8) we get

$$P(X)\Lambda^2 P(X)^t = \|X\|^{2d-8} (R_X(R_X - \lambda \|X\|^2 I_{16}))^2.$$

Suppose $d \geq 5$. Then the diagonal entries of the polynomial matrix $P(X)\Lambda^2 P(X)^t$ are divisible by $\|X\|^2$. The diagonal (α, α) entry is $\sum_{i=1}^s (\mu_i(\mu_i - \lambda)(P_i(X))_\alpha)^2$, for every $\alpha = 1, \dots, 16$, which is a sum of squares of no more than three polynomials in $x_1, \dots, x_{16}$, hence, by Lemma 3.11 of subsection 3.3, every $(P_i(X))_\alpha$ is divisible by $\|X\|^2$. This contradicts the construction of the P_i's.

Hence, as d is odd, it can only be 1 or 3. If $d = 1$, all the $P_i(X)$'s are linear and R has a Clifford structure, by assertion 2 of Lemma 3.2.

Let $d = 3$. Then every entry of the matrix $P(X)\Lambda P(X)^t$ is divisible by $\|X\|^2$, by (8).

First, suppose that all the numbers $\mu_i(\mu_i - \lambda)$, the diagonal entries of Λ, have the same sign. The diagonal (α, α) entry of $P(X)\Lambda P(X)^t$ is then $\pm \sum_{i=1}^s (|\mu_i(\mu_i - \lambda)|^{-1/2} (P_i(X))_\alpha)^2$, for every $\alpha = 1, \dots, 16$, which is a sum of squares of no more than three polynomials in $x_1, \dots, x_{16}$. This again implies that all the $(P_i(X))_\alpha$'s are divisible by $\|X\|^2$, which leads to a contradiction.

Now, assume that from among the $\mu_i(\mu_i-\lambda)$'s there are numbers of both signs. Here we consider two possibilities, $s=2$ and $s=3$, separately.

Let $s=2$ (this corresponds to the eigenvalue structure $(9,4,1,1)$). Without loss of generality, let $\mu_1(\mu_1-\lambda)=a^2>0,\ \mu_2(\mu_2-\lambda)=-b^2<0$. The fact that all the entries of the matrix $P(X)\Lambda P(X)^t=a^2P_1(X)P_1(X)^t-b^2P_2(X)P_2(X)^t$ are divisible by $\|X\|^2$ implies that all the components of at least one of the $aP_1(X)\pm bP_2(X)$ are also divisible by $\|X\|^2$. Let say $aP_1(X)-bP_2(X)=(a^2+b^2)^{1/2}\|X\|^2JX$, for some linear operator J. Then, from (7) we have $0=\langle aP_1(X)-bP_2(X),X\rangle=(a^2+b^2)^{1/2}\|X\|^2\langle JX,X\rangle$, so J is skew-symmetric, and $(a^2+b^2)\|X\|^6=\|aP_1(X)-bP_2(X)\|^2=(a^2+b^2)\|X\|^4\|JX\|^2$, so J is orthogonal (since $P_1(X)\perp P_2(X)$, as they are eigenvectors of R_X corresponding to different eigenvalues). Moreover, although JX is not an eigenvector of R_X, an algebraic curvature tensor $\tilde R$ defined by $\tilde R_XY=R_XY-C\langle JX,Y\rangle JX$ is Osserman, for any real C. Indeed, for any unit vector X, the projections of JX to the eigenspaces $E_{\mu_1}(X)$ and $E_{\mu_2}(X)$ have the same length (and JX is orthogonal to all the other eigenspaces of R_X). In fact, for a unit vector X, the eigenvalues of $\tilde R_X$ are the same as those of R_X, except that instead of the eigenvalues μ_1,μ_2, $\tilde R_X$ has two eigenvalues, which are the roots of the equation $x^2-x(\mu_1+\mu_2-C)+\mu_1\mu_2(1-C/(\mu_1+\mu_2-\lambda))=0$ (note that $\mu_1+\mu_2-\lambda\neq 0$, as $\mu_1(\mu_1-\lambda)$ and $\mu_2(\mu_2-\lambda)$ have opposite signs). If we take $C=\mu_1+\mu_2-\lambda$, these roots are 0 and λ, so the Osserman algebraic curvature tensor $\tilde R$ has the eigenvalue structure $(10,5)$, with the corresponding eigenvalues 0 and λ. Such a $\tilde R$ has a Clifford structure $\mathrm{Cliff}(5)$. Then R has a Clifford structure $\mathrm{Cliff}(6)$ by assertion 1 of Lemma 3.2.

Let now $s=3$. Again, without loss of generality, we can assume that $\mu_1(\mu_1-\lambda)=a_1^2>0,\ \mu_2(\mu_2-\lambda)=a_2^2>0,\ \mu_3(\mu_3-\lambda)=-b^2<0$.

Every entry of the 16×16 matrix

$$(a_1P_1(X))(a_1P_1(X))^t+(a_2P_2(X))(a_2P_2(X))^t-(bP_3(X))(bP_3(X))^t$$

is divisible by $\|X\|^2$. Let $\mathbf{K}$ be the ring $\mathbb{R}[x_1,\dots,x_{16}]/\left(x_1^2+\dots+x_{16}^2\right)$, with $\pi:\mathbb{R}[x_1,\dots,x_{16}]\to\mathbf{K}$ the natural projection. Denote by $p_i=\pi(P_i)$ the corresponding elements of the free module $\mathbf{K}^{16}$, and by $p_{i\alpha}$ their components ($\alpha=1,\dots,16,\ i=1,2,3$). Then for any $1\le\alpha\neq\beta\le 16$ we have

$$a_1^2p_{1\alpha}^2+a_2^2p_{2\alpha}^2=b^2p_{3\alpha}^2,\qquad a_1^2p_{1\alpha}p_{1\beta}+a_2^2p_{2\alpha}p_{2\beta}=b^2p_{3\alpha}p_{3\beta},\tag{9}$$

and so $a_1^2a_2^2(p_{1\alpha}p_{2\beta}-p_{2\alpha}p_{1\beta})^2=0$. As $a_1,a_2\neq 0$ and $\mathbf{K}$ is an integral domain, this implies that $p_{1\alpha}p_{2\beta}-p_{2\alpha}p_{1\beta}=0$.

Then the 16×2 matrix over $\mathbf{K}$ with entries $p_{i\alpha}$ has the rank at most one (every 2×2 minor vanishes). As $\mathbf{K}$ is a UFD (Nagata Theorem 3.1), there exist $u_i, v_\alpha \in \mathbf{K}$ such that $p_{i\alpha} = u_i v_\alpha$, for all $i = 1, 2,\ \alpha = 1, \dots, 16$.

If all the v_α's vanish, or if at least one of the u_i's is zero (say $u_1 = 0$), then $p_1 = 0$, so $P_1(X)$ is divisible by $\|X\|^2$. Hence there exists a linear operator J such that $P_1(X) = \|X\|^2 JX$ and the claim follows from assertion 2 of Lemma 3.2.

We assume therefore, that both u_1, u_2 and at least one of the v_α's is nonzero.

Substituting $p_{i\alpha} = u_i v_\alpha,\ i = 1, 2$, to the equations (9) we find that $(a_1^2 u_1^2 + a_2^2 u_2^2) v_\alpha v_\beta = b^2 p_{3\alpha} p_{3\beta}$ for all $\alpha, \beta = 1, \dots, 16$. As $\mathbf{K}$ is a UFD, this implies that $a_1^2 u_1^2 + a_2^2 u_2^2$ is a square: there exists $u_3 \in \mathbf{K}$ such that $a_1^2 u_1^2 + a_2^2 u_2^2 = (b u_3)^2$. Then $u_3^2 v_\alpha v_\beta = p_{3\alpha} p_{3\beta}$ for all $\alpha, \beta = 1, \dots, 16$ and so $p_{3\alpha} = \pm u_3 v_\alpha$, with the same choice of the sign for all $1 \leq \alpha \leq 16$. Replacing u_3 by $-u_3$, if necessary, we can assume that $p_3 = u_3 v_\alpha$. Again, we can assume that $u_3 \neq 0$, as otherwise $P_3(X)$ is divisible by $\|X\|^2$ and the assertion 2 of Lemma 3.2 applies.

If u_3 is not a unit in $\mathbf{K}$, then $\gcd_{\mathbf{K}}(p_{3,1}, \dots, p_{3,16}) \neq 1$. Also, the eigenvalue μ_3 of the Jacobi operator is simple, so $P_3(X)$ satisfies (7). The claim is then follows from the Lemma below (its proof is given at the end of the subsection) and assertion 2 of Lemma 3.2.

Lemma 3.5 *Let $P(X)$ be a 16-dimensional vector whose components $P_\alpha(X)\,(\alpha = 1, \dots, 16)$ are homogeneous cubic polynomials satisfying* (7) *such that for all X, $P(X)$ is an eigenvector of R_X. Let $p_\alpha = \pi(P_\alpha(X))$, where $\pi : \mathbb{R}[x_1, \dots, x_{16}] \to \mathbf{K} = \mathbb{R}[x_1, \dots, x_{16}] / \left(x_1^2 + \dots + x_{16}^2\right)$ is the natural projection. If $\gcd_{\mathbf{K}}(p_1, \dots, p_{16}) \neq 1$, then $p_\alpha = 0$, that is, $P(X) = \|X\|^2 JX$ for an orthogonal skew-symmetric operator J.*

Assume now that u_3 is a unit in $\mathbf{K}$. Then, as $p_{i\alpha} = u_i v_\alpha$, for all $i = 1, 2, 3,\ \alpha = 1, \dots, 16$, we have $p_1 = (u_1 u_3^{-1}) p_3,\ p_2 = (u_2 u_3^{-1}) p_3$. Lifting up to $\mathbb{R}[x_1, \dots, x_{16}]$ we obtain $P_1(X) = f_1(X) P_3(X) + \|X\|^2 Q_1(X)$, $P_2(X) = f_2(X) P_3(X) + \|X\|^2 Q_2(X)$ for some polynomials $f_i(X) \in \pi^{-1}(u_i u_3^{-1})$ and some 16-dimensional polynomial vectors $Q_1(X), Q_2(X)$. As all the nonzero components of P_1, P_2 and P_3 are homogeneous cubic polynomials, we can take f_1, f_2 to be constants (which are nonzero, since $\pi(f_i) = u_i u_3^{-1} \neq 0$).

Take $c_1 = -f_2 (f_1^2 + f_2^2)^{-1/2},\ c_2 = f_1 (f_1^2 + f_2^2)^{-1/2}$. Then $c_1^2 + c_2^2 = 1$, and $c_1 P_1 + c_2 P_2$ is divisible by $\|X\|^2$, so $c_1 P_1(X) + c_2 P_2(X) = \|X\|^2 JX$, with J a skew-symmetric orthogonal operator. Then an algebraic curvature tensor $\tilde{R}$

with $\tilde{R}_X Y = R_X Y - C\langle JX, Y\rangle JX$ is Osserman, for any C. For a unit vector X, the eigenvalues of $\tilde{R}_X$ are the same as those of R_X, except that instead of the eigenvalues μ_1, μ_2, $\tilde{R}_X$ has two eigenvalues, which are the roots of the equation $(\mu_1 - x)(\mu_2 - x) - C(\mu_1 c_2^2 + \mu_2 c_1^2 - x) = 0$. The coefficient of C must be nonzero for at least one of two values of $x:\ x = 0$ or $x = \lambda$. Then, for the corresponding C, $\tilde{R}$ is the Osserman algebraic curvature tensor with the eigenvalue structure $(10, \dots)$ (or $(9, 4, \dots)$, respectively). In the both cases, $\tilde{R}$ has a Clifford structure, which implies (assertion 1 of Lemma 3.2) that R has a Clifford structure (which must be Cliff(6)).

We now give a proof of Lemma 3.5.

Proof. Let $f = \gcd_{\mathbf{K}}(p_1, \dots, p_{16}) \neq 1$. Lifting up to $\mathbb{R}[x_1, \dots, x_{16}]$ we get $P(X) = F(X)Q(X) + \|X\|^2 C(X)$ for some polynomial vectors Q and C and a polynomial $F \in \pi^{-1}(f)$. As the nonzero components of P are homogeneous cubic polynomials, we can take C to be a linear operator, F to be a homogeneous polynomial of degree $r = 1, 2, 3$ coprime with $\|X\|^2$, and all the nonzero components of Q to be homogeneous, of degree $3 - r$. It is sufficient to prove that all the components of Q are divisible by $\|X\|^2$.

If $r = 3$, the equation $\langle P(X), X\rangle = 0$ from (7) can only be satisfied, when $Q = 0$ (as F and $\|X\|^2$ are coprime).

Assume that $r = 2$, so that Q is a linear operator. From the second equation of (7) it follows that $\|QX\|^2$ is divisible by $\|X\|^2$, so that Q is a constant, say c_0, times an orthogonal operator. From the third equation of (7) we get $P(P(X)) = -\|X\|^8 X$, so all the components of $F(F(X)QX)Q(F(X)QX) = F(X)^3 F(QX) Q^2 X$ are divisible by $\|X\|^2$. As $F(X)$ is coprime with $\|X\|^2$, this is only possible when $F(QX)$ is divisible by $\|X\|^2$. But if $c_0 \neq 0$, the operator Q is invertible and so $F(QX)$ is coprime with $\|QX\|^2 = c_0^{-2}\|X\|^2$. Hence $c_0 = 0$, that is, $Q = 0$.

Let $r = 1$, so that $F(X)$ is a nonzero linear form, and all the nonzero components of Q are quadratic forms. Then from the first equation of (7), $F(X)\langle Q(X), X\rangle = -\|X\|^2 \langle CX, X\rangle$, so $F(X)$ divides the quadratic form $\langle CX, X\rangle$: $\langle CX, X\rangle = F(X)\langle a, X\rangle$, with some vector $a \in \mathbb{R}^{16}$. Define a linear operator $\tilde{C}$ and a vector of quadratic forms $\tilde{Q}$ by $\tilde{C}X = CX - F(X)a$, $\tilde{Q}(X) = Q(X) + \|X\|^2 a$, respectively. Then P is still given by $F(X)\tilde{Q}(X) + \|X\|^2 \tilde{C}X$, but now with a skew-symmetric $\tilde{C}$. From the second equation of (7) it follows that $F(X)$ divides the quadratic form $\|X\|^2 - \|\tilde{C}X\|^2$, which is only possible (for a skew-symmetric $\tilde{C}$) when $\tilde{C}$ is orthogonal. From the third equation of (7) we get $P(P(X)) = -\|X\|^8 X$, which simplifies to $F(P(X))\tilde{Q}(P(X)) = -\|X\|^6 F(X)\tilde{C}\tilde{Q}(X)$. On the kernel of F, we have $P(X) = \|X\|^2\tilde{C}X$, so $(F(\tilde{C}X)\tilde{Q}(\tilde{C}X))_{|\mathrm{Ker}F} = 0$. As $\tilde{C}$ is an orthogonal operator, $F(\tilde{C}X)$ cannot be identically zero on Ker$F$, so $\tilde{Q}(\tilde{C}X)_{|\mathrm{Ker}F} = 0$. It follows that $F(X)$ divides $\tilde{Q}(\tilde{C}X)$, hence $F(\tilde{C}X)$ divides $\tilde{Q}(X)$, as $\tilde{C}$ is orthogonal and skew-symmetric, which reduces this case to the case $r = 2$.

3.2. *Eigenvalue structure* $(9, 3, 3)$

We follow the plan of proof of Proposition 1 of [7]. Let $\lambda_1 \neq \lambda_2$ be two nonzero eigenvalues of the Jacobi operator, each of multiplicity 3. In a Euclidean space $\mathbb{R}^6$, choose an orthonormal basis $e_1, \dots e_6$ and define an operator $\Lambda : \mathbb{R}^6 \to \mathbb{R}^6$ by $\Lambda\, e_s = \mu_s e_s$, $s = 1, \dots, 6$. The matrix of Λ is then $\mathrm{diag}\{\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2, \}$.

The key ingredient of the proof is the following Lemma similar to Proposition 3 of [7]:

Lemma 3.6 *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^{16}$ whose Jacobi operator has three different eigenvalues, $0, \lambda_1, \lambda_2$, with the multiplicities $9, 3, 3$, respectively.*

There exists a linear map $M : \mathbb{R}^{16} \to \mathrm{Hom}(\mathbb{R}^6, \mathbb{R}^{16})$, $X \to M_X$ such that

$$R_X = M_X\, \Lambda\, M_X^t. \tag{10}$$

The map $X \to M_X$ is determined uniquely up to a precomposition $X \to M_X N$ with an element N from the group $\mathrm{O}_\Lambda = \{N : N\Lambda\, N^t = \Lambda\}$.

Assuming Lemma 3.6, the claim (the existence of a Clifford structure for R) follows from the Lemma below (Proposition 4 of [7]):

Lemma 3.7 *Let R be an Osserman algebraic curvature tensor in $\mathbb{R}^n$, with the Jacobi operator having the form* (10). *Assume that $n > \frac{(\nu+1)^2}{4}$, where ν is the sum of multiplicities of all the nonzero eigenvalues of the Jacobi operator. Then R has a* $\mathrm{Cliff}(\nu)$*-structure.*

In our case, $n = 16, \nu = 6$, so the condition of Lemma 3.7 is satisfied. Before proving Lemma 3.6 we need some preparations.

First of all, we will use the Rakič duality principle [10], for the eigenvalue 0, in a slightly modified form: for any two vectors $X, Y \in \mathbb{R}^{16}$,

$$Y \in \mathrm{Ker}R_X \quad \text{iff} \quad X \in \mathrm{Ker}R_Y. \tag{11}$$

One immediate consequence of (11) is the following fact: if $\{X_\alpha\}$ and $\{Y_\beta\}$ are two sets of vectors with the same span, then $\cap_\alpha \mathrm{Ker}R_{X_\alpha} = \cap_\beta \mathrm{Ker}R_{Y_\beta}$, so that $\cap_\alpha \mathrm{Ker}R_{X_\alpha}$ depends only on the $\mathrm{Span}_\alpha(X_\alpha)$.

For every nonzero $X \in \mathbb{R}^{16}$, $\dim \mathrm{Ker} R_X = 10$ and $\dim \mathrm{Im} R_X = 6$. One can expect that for a generic pairs of vectors, the intersection of the images of the corresponding Jacobi operators is zero, and the set of such pairs is large enough. This is formalized in Lemma 3.8 below.

Denote V_2 the Stiefel manifold of pairs of orthonormal vectors in $\mathbb{R}^{16}$, and V_3 the Stiefel manifold of triples of orthonormal vectors in $\mathbb{R}^{16}$. Define $\tilde{V_3}$ to be the set of triples (X, Y, Z) of unit vectors in $\mathbb{R}^{16}$ such that $Y \perp Z$, and $X \notin \mathrm{Span}(Y, Z)$. $\tilde{V_3}$ is an open manifold (with the topology induced from $S^{15} \times V_2$) and is a fibration over V_2 with a fiber homeomorphic to $S^{15} \setminus S^1$.

Let $S_2 \subset V_2$ be the set of pairs of orthonormal vectors (Y, Z) with the property $\mathrm{Im} R_Y \cap \mathrm{Im} R_Z = 0$. By assertion 1 of Lemma 2 of [7], the subset S_2 is open and dense in V_2.

Lemma 3.8 *There exists an open and dense subset $S_3 \subset V_3$ such that for every triple $(X, Y, Z) \in S_3$, the following conditions are satisfied:*

(a) *for any ϕ,* $\dim(\mathrm{Im} R_X \cap \mathrm{Im} R_{\cos\phi Y + \sin\phi Z}) \le 1$*;*

(b) $\mathrm{Im} R_Y \cap \mathrm{Im} R_Z = \mathrm{Im} R_X \cap \mathrm{Im} R_Y = \mathrm{Im} R_X \cap \mathrm{Im} R_Z = 0$*, that is,* $(X, Y), (Y, Z), (Z, X) \in S_2$*;*

(c) $\mathrm{Ker} R_X \cap \mathrm{Ker} R_Y \cap \mathrm{Ker} R_Z = 0$.

Proof. First we prove the following fact: there exists an open and dense subset $\tilde{S_3} \subset \tilde{V_3}$ such that for any triple $(X, Y, Z) \in \tilde{S_3}$, condition (a) of the Lemma is satisfied.

Let (Y, Z) be an arbitrary pair of vectors from the set S_2, and let $C = \{Y(\phi) = \cos\phi Y + \sin\phi Z,\ \phi \in [0, 2\pi)\}$ be the unit circle passing through Y and Z. The sphere bundle $S_1 C$ with a fiber over a point $Y(\phi) \in C$ being a unit 9-dimensional sphere in the space $\mathrm{Ker} R_{Y(\phi)}$ is a ten-dimensional compact submanifold of TS^{15}. Let $S_2 C$ be a sphere bundle over $S_1 C$, with the fiber over a point $(Y(\phi), U) \in S_1 C$ being the 9-dimensional unit sphere in the space $\mathrm{Ker} R_U$. The space $S_2 C$ is a compact analytic manifold of dimension 19 (as $\mathrm{Ker} R_X$ depends analytically on X, for X nonzero).

Define the projection $\pi : S_2 C \to S^{15}$ by $\pi(((Y(\phi), U), X)) = X$. Then $\pi^{-1}(X) = \cup_\phi \{((Y(\phi), U), X) \in S_2 C : U \in \mathrm{Ker} R_X \cap \mathrm{Ker} R_{Y(\phi)}\}$, for any $X \in S^{15}$, by the duality principle.

Since for every ϕ, $\mathrm{Ker} R_X \cap \mathrm{Ker} R_{Y(\phi)}$ is a linear space of dimension at least 4, the map π is surjective and $\pi_1 \circ \pi^{-1}(X) \supset C$, where $\pi_1 : S_2 C \to C$ sends $((Y(\phi), U), X)$ to $Y(\phi)$.

The set A_{YZ} of the regular values of π is open and dense in S^{15}. Moreover, for each regular value $X \in A_{YZ}$ we must have $\mathrm{rk}\, d\pi_{((Y(\phi), U), X)} = 15$, so

$\dim(\mathrm{Ker}R_X \cap \mathrm{Ker}R_{Y(\phi)}) \leq 5$, for every ϕ. It follows that $A_{YZ} \cap C = \varnothing$ and that $\dim(\mathrm{Im}R_X + \mathrm{Im}R_{Y(\phi)}) \geq 11$, which is equivalent to condition (a), $\dim(\mathrm{Im}R_X \cap \mathrm{Im}R_{Y(\phi)}) \leq 1$, as both spaces are six-dimensional.

Now let $\tilde{S}_3 \subset \tilde{V}_3$ be the set of triples (X, Y, Z) such that $(Y, Z) \in S_2$, $X \in A_{YZ}$. As $\tilde{V}_3$ fibers over V_2, S_2 is open and dense in V_2, and A_{YZ} is open and dense in the fiber of $\tilde{V}_3$ for every $(Y, Z) \in S_2$, the subset $\tilde{S}_3$ is open and dense in $\tilde{V}_3$.

Using the set $\tilde{S}_3$, we will now construct an open and dense subset of V_3, every element of which satisfies condition (a).

The manifold $\tilde{V}_3$ is a fibre bundle over the Stiefel manifold V_3, with the projection p defined via the Gramm-Schmidt orthogonalization. Namely, $p(X, Y, Z) = (X, Y', Z')$, where $Y' = (Y - \langle X, Y\rangle X)(1 - \langle X, Y\rangle^2)^{-1/2}$, $Z' = (Z - \langle Y', Z\rangle Y' - \langle X, Z\rangle X)(1 - \langle Y', Z\rangle^2 - \langle X, Z\rangle^2)^{-1/2}$ (this is well defined, as $\mathrm{rk}\,\{X, Y, Z\} = 3$.

The set $\tilde{S}_3$ projects under p onto an open and dense subset $B_1 \subset V_3$. For every triple $(X, Y', Z') \in B_1$, condition (a) is still satisfied, since $\mathrm{Span}(X, \cos\phi Y' + \sin\phi Z') = \mathrm{Span}(X, \cos\psi Y + \sin\psi Z)$ for some ψ, hence $\mathrm{Ker}R_X \cap \mathrm{Ker}R_{\cos\phi Y' + \sin\phi Z'} = \mathrm{Ker}R_X \cap \mathrm{Ker}R_{\cos\psi Y + \sin\psi Z}$, which implies that $\mathrm{Im}R_X + \mathrm{Im}R_{\cos\phi Y' + \sin\phi Z'} = \mathrm{Im}R_X + \mathrm{Im}R_{\cos\psi Y + \sin\psi Z}$, and so $\dim(\mathrm{Im}R_X \cap \mathrm{Im}R_{\cos\phi Y' + \sin\phi Z'}) = \dim(\mathrm{Im}R_X \cap \mathrm{Im}R_{\cos\psi Y + \sin\psi Z})$.

Next, there is an open and dense subset $B_2 \subset V_3$, for each element of which condition (b) is satisfied. To see that, consider three projections, $p_1, p_2, p_3 : V_3 \to V_2$ defined by $p_1(X, Y, Z) = (X, Y)$, $p_2(X, Y, Z) = (Y, Z)$, $p_3(X, Y, Z) = (Z, X)$. The set S_2 is open and dense in V_2, hence each of the $p_i^{-1}(S_2)$ is open and dense in V_3, therefore a set $B_2 = p_1^{-1}(S_2) \cap p_2^{-1}(S_2) \cap p_3^{-1}(S_2)$ is also open and dense in V_3.

Finally, there exists an open and dense subset $B_3 \subset V_3$, for each element of which condition (c) is satisfied. This follows from the dimension count. Indeed, $\dim V_3 = 42$. If condition (c) is violated, then there exists a unit vector $U \in \mathbb{R}^{16}$ such that $U \in \mathrm{Ker}R_X \cap \mathrm{Ker}R_Y \cap \mathrm{Ker}R_Z$. By the duality principle, $X, Y, Z \in \mathrm{Ker}R_U$. Consider a Stiefel fiber bundle F over S^{15} whose fiber over $U \in S^{15}$ is the set of triples (X, Y, Z) of orthonormal vectors from $\mathrm{Ker}R_U$. Then F is a compact manifold of dimension 39, hence the image of its projection to V_3 defined by $(U, (X, Y, Z)) \to (X, Y, Z)$ is closed and nowhere dense. For all the triples in the complement B_3 of that image, condition (c) is satisfied.

It now follows that for every triple (X, Y, Z) in an open and dense subset $S_3 = B_1 \cap B_2 \cap B_3$ of V_3, all three conditions (a), (b), (c) are satisfied.

Combining the results of Lemma 1 and Lemma 3 of [7] we get:

Lemma 3.9 (1.) *For every $X \in \mathbb{R}^{16}$ there exists a linear operator $M_X : \mathbb{R}^6 \to \mathbb{R}^{16}$ such that $R_X = M_X \Lambda M_X^t$. The operator M_X is uniquely defined, up to a precomposition with an element $N \in \mathrm{O}_\Lambda = \{N : N\Lambda N^t = \Lambda\}$, and can be chosen in such a way that $M_X^t M_X = \|X\|^2 \mathrm{id}_{\mathbb{R}^6}$. Moreover, $\mathrm{Im} R_X = \mathrm{Im} M_X$.*

(2.) *For every pair of orthonormal vectors $(X, Y) \in S_2$, there exist linear operators $M_1, M_2 : \mathbb{R}^6 \to \mathbb{R}^{16}$ such that*

$$R_{xX+yY} = (M_1 x + M_2 y)\, \Lambda\, (M_1 x + M_2 y)^t$$

for all $x, y \in \mathbb{R}$. The operators M_1, M_2 are determined uniquely up to a precomposition $M_1 N$, $M_2 N$ with an element $N \in \mathrm{O}_\Lambda$.

We claim that a factorization of the Jacobi operator similar to that of Lemma 3.9 exists for any triple from the open and dense set $S_3 \subset V_3$ constructed in Lemma 3.8:

Lemma 3.10 *For any triple $(X, Y, Z) \in S_3$, there exist operators $M_1, M_2, M_3 : \mathbb{R}^6 \to \mathbb{R}^{16}$ such that for all $x, y, z \in \mathbb{R}$,*

$$R_{xX+yY+zZ} = (M_1 x + M_2 y + M_3 z)\, \Lambda\, (M_1 x + M_2 y + M_3 z)^t.$$

Proof. Choose a triple (X, Y, Z) is in S_3. Denote $Y(\phi) := \cos\phi Y + \sin\phi Z$. Then for any ϕ, $\dim(\mathrm{Im} R_X \cap \mathrm{Im} R_{Y(\phi)}) \le 1$ and $\mathrm{Im} R_X \cap \mathrm{Im} R_{Y(0)} = \mathrm{Im} R_X \cap \mathrm{Im} R_{Y(\pi/2)} = 0$.

The fact that $\dim(\mathrm{Im} R_X \cap \mathrm{Im} R_{Y(\phi)}) > 0$ is equivalent to the following: a linear operator $\Phi(\phi) : \mathbb{R}^{16} \to R^{32}$ defined for every ϕ by $\Phi(\phi) U = (R_{Y(\phi)} U, R_X U)$ has rank less than 12. This condition can be expressed as a set of polynomial equations for $\sin\phi, \cos\phi$. Since $\mathrm{rk}\, \Phi(0) = 12$ (as $\mathrm{Im} R_X \cap \mathrm{Im} R_{Y(0)} = 0$), the condition $\mathrm{Im} R_X \cap \mathrm{Im} R_{Y(\phi)} = 0$ holds for all, but a finite number of $\phi \in [0, 2\pi)$.

It follows that $(X, Y(\phi)) \in S_2$ for all $\phi \in S$, where S is a subset of $[0, 2\pi)$ with a finite complement containing 0 and $\pi/2$. For every $\phi \in S$, there exist linear operators $M_1(\phi), \tilde{M}_2(\phi) : \mathbb{R}^6 \to \mathbb{R}^{16}$ such that for all $x, y \in \mathbb{R}$, $R_{xX+yY(\phi)} = (M_1(\phi) x + \tilde{M}_2(\phi) y)\, \Lambda\, (M_1(\phi) x + \tilde{M}_2(\phi) y)^t$. By Lemma 3.9 we can take $M_1(\phi)$ to be the same for all $\phi \in S$ and to satisfy $M_1^t M_1 = \mathrm{id}_{\mathbb{R}^6}$. Then

$$R_{xX+y(\cos\phi Y+\sin\phi Z)} = (M_1 x + \tilde{M}_2(\phi) y)\, \Lambda\, (M_1 x + \tilde{M}_2(\phi) y)^t, \qquad (12)$$

for all $\phi \in S$, $x, y \in \mathbb{R}$, with the operator $\tilde{M}_2(\phi) : \mathbb{R}^6 \to \mathbb{R}^{16}$ being uniquely determined for every $\phi \in S$. In particular, by uniqueness, $\tilde{M}_2(\phi + \pi) = -\tilde{M}_2(\phi)$, whenever ϕ and $\phi + \pi \mod (2\pi)$ are in S.

Let $y_1 = y\cos\phi$, $y_2 = y\sin\phi$. The equation (12) then takes the form

$$R_{xX+y_1Y+y_2Z} = (M_1x + M_2(y_1,y_2))\,\Lambda\,(M_1x + M_2(y_1,y_2))^t, \tag{13}$$

where $M_2(y\cos\phi, y\sin\phi) = yM_2(\phi)$ is well defined outside a finite union of lines on the plane (y_1, y_2).

At this point, it would be more convenient to switch to the matrix language. Choose an orthonormal basis $\{e_1,\dots,e_{16}\}$ for $\mathbb{R}^{16}$ such that the matrix of M_1 is $\left(\begin{smallmatrix} I_6 \\ 0\end{smallmatrix}\right)$, where I_6 is the 6×6 identity matrix (this is always possible, as $M_1^tM_1 = I_6$). Then $\mathrm{Im}R_X = \mathrm{Span}(e_1,\dots,e_6)$, $\mathrm{Ker}R_X = \mathrm{Span}(e_7,\dots,e_{16})$. With respect to this basis, let

$$M_2(y_1,y_2) = \begin{pmatrix} F(y_1,y_2) \\ P(y_1,y_2)\end{pmatrix},$$

where $F(y_1,y_2)$ is a 6×6 matrix, and $P(y_1,y_2)$ is a 10×6 matrix.

From the terms of (13) linear in x we get

$$\begin{pmatrix} F(y_1,y_2)\Lambda + \Lambda F(y_1,y_2)^t & \Lambda P(y_1,y_2)^t \\ P(y_1,y_2)\Lambda & 0\end{pmatrix} = 2y_1R_{XY} + 2y_2R_{XZ}$$

(where $R_{XY}:\mathbb{R}^{16}\to\mathbb{R}^{16}$ is defined by $2R_{XY}U = R(X,U)Y + R(Y,U)X$). As all the entries of the matrix on the right-hand side are linear in y_1, y_2 and $\det\Lambda \neq 0$, we get that all the entries of both $F(y_1,y_2)\Lambda + \Lambda F(y_1,y_2)^t$ and $P(y_1,y_2)$ are linear in y_1,y_2. It follows that

$$P(y_1,y_2) = y_1P_1 + y_2P_2, \quad F(y_1,y_2) = y_1F_1 + y_2F_2 + K(y_1,y_2)\Lambda^{-1},$$

where P_i and F_i are constant matrices, and $K(y_1,y_2)$ is a skew-symmetric 6×6 matrix

The terms of (13) not containing x give

$$\begin{pmatrix} F(y_1,y_2)\Lambda F(y_1,y_2)^t & F(y_1,y_2)\Lambda P(y_1,y_2)^t \\ P(y_1,y_2)\Lambda F(y_1,y_2)^t & P(y_1,y_2)\Lambda P(y_1,y_2)^t\end{pmatrix} = y_1^2R_Y + 2y_1y_2R_{YZ} + y_2^2R_Z, \tag{14}$$

which implies that all the entries of the matrix on the left-hand side are quadratic forms in y_1, y_2 (whenever they are defined). It follows that all the entries of both

$$F(y_1,y_2)\Lambda F(y_1,y_2)^t \quad \text{and} \quad K(y_1,y_2)(y_1P_1 + y_2P_2)^t \tag{15}$$

must be quadratic forms.

By the choice of the triple (X,Y,Z), the conditions (b) and (c) of Lemma 3.8 must be satisfied. In particular, the fact that $\mathrm{Im}R_X \cap \mathrm{Im}R_Y = 0$ means (Lemma 3.9) that $M_2(1,0)$ is defined and that $\mathrm{Im}R_Y = \mathrm{Im}M_2(1,0)$, which implies $\operatorname{rk} P_1 = 6$. Similarly, the fact that $(X, Z) \in S_2$ means that $M_2(0,1)$ is defined and $\operatorname{rk} P_2 = 6$. Condition (c) gives that no nonzero vector from $\mathbb{R}^{16}$ is orthogonal to the column spaces of all three matrices $M_2(1,0), M_2(0,1)$ and M_3 simultaneously, which gives

$$\operatorname{rk}(P_1 \mid P_2) = 10. \tag{16}$$

As all the entries of the matrix $K(y_1, y_2)(y_1 P_1 + y_2 P_2)^t =: Q(y_1, y_2)$ are quadratic forms in y_1, y_2 and $\operatorname{rk}(y_1 P_1 + y_2 P_2) = 6$ for generic pairs (y_1, y_2), $K(y_1, y_2) = Q(y_1, y_2)(y_1 P_1 + y_2 P_2)((y_1 P_1 + y_2 P_2)^t (y_1 P_1 + y_2 P_2))^{-1}$, hence the entries of $K(y_1, y_2)$ are rational functions, which are the ratios of homogeneous polynomials in y_1, y_2 of homogeneity 1 (that is, with the numerator of each nonzero entry is of degree one bigger then that of the denominator). Let $q(y_1, y_2)$ be the lowest common multiple of the denominators of the nonzero entries of $K(y_1, y_2)$ (after cancelation in every entry). The polynomial q is homogeneous and its zero locus is a subset of $\{(y_1, y_2) : \operatorname{rk}(y_1 P_1 + y_2 P_2) < 6\}$. In particular, both $q(1,0)$ and $q(0,1)$ are nonzero, hence q and $y_1 y_2$ are coprime.

Introduce a 6×6 skew-symmetric matrix $\hat{K}(y_1, y_2) := K(y_1, y_2) - y_1 K(1,0) - y_2 K(0,1)$. As $\hat{K}(1,0) = \hat{K}(0,1) = 0$ all the numerators of the entries of $\hat{K}(y_1, y_2)$ are divisible by $y_1 y_2$. Define a 6×6 skew-symmetric matrix $G(y_1, y_2)$ by $\hat{K}(1,0) = y_1 y_2 q^{-1} G$. The entries of G are homogeneous polynomials in y_1, y_2, with the degree of every nonzero entry one smaller than the degree of q, and with the greatest common divisor of the entries being coprime with q (otherwise we can cancel it out).

It follows now from (15) that for some matrices $Q_1(y_1, y_2), Q_2(y_1, y_2)$ whose nonzero entries are quadratic forms in y_1, y_2,

$$y_1 y_2 q (G (y_1 \tilde{F}_1 + y_2 \tilde{F}_2)^t - (y_1 \tilde{F}_1 + y_2 \tilde{F}_2) G) + y_1^2 y_2^2 G \Lambda^{-1} G^t = q^2 Q_1 \tag{17}$$

$$y_1 y_2 G (y_1 P_1 + y_2 P_2)^t = q Q_2, \tag{18}$$

where $\tilde{F}_1 = F_1 + K(1,0)$, $\tilde{F}_2 = F_2 + K(0,1)$ are constant matrices. As q and $y_1 y_2$ are coprime, equation (17) implies that all the entries of $G \Lambda^{-1} G^t$ are divisible by q^2. As the degree of every nonzero entry of G is one smaller than the degree of q, this is only possible when $G \Lambda^{-1} G^t = 0$. Since $\Lambda^{-1} = \operatorname{diag}\{\lambda_1^{-1}, \lambda_1^{-1}, \lambda_1^{-1}, \lambda_2^{-1}, \lambda_2^{-1}, \lambda_2^{-1}\}$, this implies that $G = 0$, if $\lambda_1 \lambda_2 > 0$, or $\operatorname{rk} G \leq 3$, if $\lambda_1 \lambda_2 < 0$. As G is skew-symmetric, we have $\operatorname{rk} G \leq 2$. We want to show that the condition $\operatorname{rk} G(y_1, y_2) = 2$ for at least one point (y_1, y_2) leads to a contradiction, so that G must vanish.

Assume $\operatorname{rk} G(y_1, y_2) = 2$ for at least one point (y_1, y_2) (and hence for an open and dense set on the plane (y_1, y_2)). From (18), as q and $y_1 y_2$ are coprime, it follows that $Q_2 = y_1 y_2 C_2$, with a constant 6×10 matrix C_2 whose rank is at most 2 (since $\operatorname{rk} G \le 2$). Then

$$G(y_1P_1 + y_2P_2)^t = qC_2, \tag{19}$$

which implies that a 10×10 matrix $(y_1P_1+y_2P_2)C_2$ is skew-symmetric and has rank at most 2, for all y_1, y_2. If $\operatorname{rk} C_2 < 2$, then $(y_1P_1+y_2P_2)C_2 = 0$ and so $G = 0$ (since $\operatorname{rk}(y_1P_1+y_2P_2) = 6$ for an open dense set of points (y_1, y_2)). Assume $\operatorname{rk} C_2 = 2$. The vectors $e_7, \dots, e_{16}$ of the orthonormal basis e_i are arbitrary orthonormal vectors spanning the kernel of R_X. We can specify their choice in such a way that the matrix C_2 has a form $(a\,|\,b\,|\,0)$, where $a \perp b$ are nonzero six-dimensional vector-columns, and 0 represents the 6×8 zero matrix. Then there exists a nonzero linear form $l = l(y_1, y_2)$ such that

$$(y_1P_1 + y_2P_2)a = (0, -l, 0, \dots, 0)^t,\ (y_1P_1 + y_2P_2)b = (l, 0, 0, \dots, 0)^t. \tag{20}$$

For any $c \in \mathbb{R}^6$ orthogonal to a and b we have $C_2^t c = 0$, so by (19) $Gc = 0$. It follows that the column space of the skew-symmetric matrix G is spanned by a and b, which implies that $G = f(ab^t - ba^t)$ for some polynomial $f = f(y_1, y_2)$. Substituting this to (19) and using (20) we get $fl = q$. As q is coprime with the greatest common divisor of the entries of G, we obtain

$$G = ab^t - ba^t, \qquad q = l(y_1, y_2),\ \text{a linear form}. \tag{21}$$

It follows from (21) and (19) that

$$\begin{aligned} F(y_1, y_2)\Lambda P(y_1, y_2)^t &= (y_1\tilde{F}_1 + y_2\tilde{F}_2 + y_1y_2 l^{-1} G L^{-1})\Lambda(y_1P_1 + y_2P_2)^t \\ &= (y_1\tilde{F}_1 + y_2\tilde{F}_2)(y_1P_1 + y_2P_2)^t + y_1y_2(a\,|\,b\,|\,0). \end{aligned} \tag{22}$$

From (17), as $G\Lambda^{-1}G^t = 0$, we have $\langle \Lambda^{-1}a, a\rangle = \langle \Lambda^{-1}a, b\rangle = \langle \Lambda^{-1}b, b\rangle = 0$, and also $G(y_1\tilde{F}_1 + y_2\tilde{F}_2)^t - (y_1\tilde{F}_1 + y_2\tilde{F}_2)G = l(y_1, y_2)C_1$ for a constant symmetric 6×6 matrix C_1. In particular, if $l(y_{01}, y_{02}) = 0$, the matrix $G(y_{01}\tilde{F}_1 + y_{02}\tilde{F}_2)^t$ is skew-symmetric, and it follows from (21) that for some constant μ,

$$(y_{01}\tilde{F}_1 + y_{02}\tilde{F}_2)a = \mu a, \quad (y_{01}\tilde{F}_1 + y_{02}\tilde{F}_2)b = \mu b. \tag{23}$$

Now take y_{01}, y_{02} such that $y_{01}^2 + y_{02}^2 = 1$ and $l(y_{01}, y_{02}) = 0$. Then the 10×6 matrix $P_0 := P(y_{01}, y_{02}) = y_{01}P_1 + y_{02}P_2$ has rank 4. Indeed, by (20) its rank is not bigger than 4. On the other hand, by (16), $\operatorname{rk}(P_1 \mid P(y_{01}, y_{02})) = \operatorname{rk}(P_1 \mid P_2) = 10$, so $\operatorname{rk} P(y_{01}, y_{02}) = 4$. Moreover, by (20), the column space of the 6×10 matrix P_0^t (which is 4-dimensional) is orthogonal to both a and b. As $a, b \perp \Lambda^{-1}a, \Lambda^{-1}b$, it follows that both

$\Lambda^{-1}a$ and $\Lambda^{-1}b$ lie in the column space of P_0^t, hence a and b belong to the column space of ΛP_0. Then the subspace $\mathcal{V} \subset \mathbb{R}^{10}$ of vectors v such that $P_0^t v \in \mathrm{Span}(a, b)$ has dimension 8.

Let $U = (0, 0, 0, 0, 0, 0, u)^t \in \mathrm{Ker} R_X$, where $u \in \mathbb{R}^{10}$. From (14) and (22) we obtain

$$\begin{aligned} R_{y_1 Y + y_2 Z} U &= \begin{pmatrix} F(y_1, y_2)\Lambda P(y_1, y_2)^t \\ P(y_1, y_2)\Lambda P(y_1, y_2)^t \end{pmatrix} u \\ &= \begin{pmatrix} (y_1 \tilde{F}_1 + y_2 \tilde{F}_2)\Lambda(y_1 P_1 + y_2 P_2)^t + y_1 y_2 (a \,|\, b \,|\, 0) \\ P(y_1, y_2)\Lambda P(y_1, y_2)^t \end{pmatrix} u \end{aligned}$$

Take $(y_1, y_2) = (y_{01}, y_{02})$ and $u \in \mathcal{V}$. Then $\Lambda P_0^t u = l_1(u)a + l_2(u)b$ for some linear functionals l_1, l_2 on $\mathcal{V}$, as $\Lambda P_0^t(\mathcal{V}) = \mathrm{Span}(a, b)$. We have: $P_0 \Lambda P_0^t u = 0$, as $P_0 a = P_0 b = 0$ by (20), and by (23), $(y_{01}\tilde{F}_1 + y_{02}\tilde{F}_2)\Lambda P_0^t + y_{01}y_{02}(a \,|\, b \,|\, 0)u = \mu(l_1(u)a + l_2(u)b) + y_{01}y_{02}(u_1 a + u_2 b) = (\mu l_1(u) + y_{01}y_{02}u_1)a + (\mu l_2(u) + y_{01}y_{02}u_2)b$, where u_1 and u_2 are the first two components of $u \in \mathcal{V} \subset \mathbb{R}^{10}$.

The common kernel $\mathcal{U}$ of the linear functionals $u \to \mu l_i(u) + y_{01}y_{02}u_i$, $i = 1, 2$ on $\mathcal{V}$ is at least six-dimensional. Then any vector $U = (0, 0, 0, 0, 0, 0, u)^t$, with $u \in \mathcal{U}$ belongs to both $\mathrm{Ker} R_X$ and $\mathrm{Ker} R_{y_{01} Y + y_{02} Z}$, which contradicts condition (a) of Lemma 3.8, as $\dim \mathcal{U} \geq 6$.

It follows that $G = 0$, and so

$$M_2(y_1, y_2) = \begin{pmatrix} F(y_1, y_2) \\ P(y_1, y_2) \end{pmatrix} = y_1 \begin{pmatrix} \tilde{F}_1 \\ P_1 \end{pmatrix} + y_2 \begin{pmatrix} \tilde{F}_2 \\ P_2 \end{pmatrix},$$

with $\tilde{F}_1, \tilde{F}_2, P_1, P_2$ constant matrices. Substituting this to (13) completes the proof of the Lemma.

With Lemma 3.10, the proof of Lemma 3.6 (and hence of the fact that R has a Clifford structure Cliff(6)) is word-by-word the same as that in section 4 of [7], after Lemma 4.

3.3. *Two facts from Commutative Algebra*

Let for $X = (x_1, \dots, x_{16})$, $(\|X\|^2)$ be the ideal of $\mathbb{R}[X]$ generated by $\|X\|^2$, and $\mathbf{K} = \mathbb{R}[X]/(\|X\|^2)$, with $\pi : \mathbb{R}[X] \to \mathbf{K}$ the natural projection.

We need the following two facts (the first one is Nagata's Theorem [4]).

Theorem 3.1 *The ring* $\mathbf{K}$ *is a unique factorization domain* (*UFD*).

Lemma 3.11 *Let $f_1, \dots, f_m$ be polynomials in $\mathbb{R}^{16}$ such that $\|X\|^2$ divides $\sum_i f_i^2$. If $m \le 8$, then $\|X\|^2$ divides each of f_i.*

Proof. In $\mathbf{K}$, we have $\sum_i \hat{f}_i^2 = 0$, where $\hat{f}_i = \pi(f_i)$. Assume that at least one of the $\hat{f}_i$ is nonzero, say $\hat{f}_m \neq 0$. Then in the field of fractions $\mathbb{K}$ of the ring $\mathbf{K}$, we get $\sum_{i=1}^{m-1}(\hat{f}_i \hat{f}_m^{-1})^2 = -1$, which means that $s(\mathbb{K})$, the level of the field $\mathbb{K}$ is at most $m-1$ [11]. The field $\mathbb{K}$ is isomorphic to the field $\mathbb{L}_{15} = \mathbb{R}(x_1, \dots, x_{15}, \sqrt{-d_{15}})$, where $d_{15} = \sum_{j=1}^{15} x_j^2$ (an isomorphism from $\mathbb{L}_{15}$ to $\mathbb{K}$ is induced by the map $(a + b\sqrt{-d_{15}})/c \to (a + bx_{16})/c$, with $a, b, c \in \mathbb{R}[x_1, \dots, x_{15}],\ c \neq 0$). Then by Theorem 3.1.4 of [11], $s(\mathbb{K}) = 8$, a contradiction.

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