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Cyclic Symmetry in Geometrical Nonlinear Analysis of Structures

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1 Introduction

A possible challenge in the work of a research or designer is when a structure with cyclic period geometry appears. This type of symmetry remains an up-to-date subject in the theoretical and practicable field of engineering. The cyclically symmetry is present in many civil engineering structures (domes, cooling towers, chimneys, etc.) or in mechanical engineering (milling cutters, turbine bladed disks, gears, fan or pump impellers, etc.).

Such a structure may be considered as a domain composed by identical, coupled subdomains positioned symmetrically with respect to an axis. Analysis of one of the subdomains, named fundamental, and its high degree of repetition represents the key in obtaining major savings in calculus.

The symmetry of a state of a structural system is an intrinsic property that is independent from the external loading or of the analysis type: linear or nonlinear, static or dynamic, etc. Specific methods have been developed and implemented in FEM programs (MSC NASTRAN, ANSYS, PAFEC, etc.) **only** in the linear elastic and modal analysis. The published literature on cyclic symmetry presented the advantages of these specific methods (e.g. an important reduction of the computational effort comparative with the common approach). However, the aim of this paper is to survey the main directions possible to use the initial cyclic symmetrical configuration in a geometrical nonlinear analysis of structures and to present an example on a Schwedler dome.

2 Cyclical Symmetry

The cyclical symmetry was used in a FEM application for the first time by Richard Courant in his historical paper [Cou43] to reduce the effort of his "numerical treatment of the plane torsion problem for multiple-connected domains" [Vas94]. The mathematician Hermann Weyl presented in his essay [Wey52] the symmetry in nature

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(from the cell in biology to the mineral crystal) and art closely related to some aspects of the mathematical theory of groups and subgroups. This group theory, especially the representation theory, is penetrating now in the FEM domain.

It is relevant to present the overall stiffness matrix **K** for a cyclically symmetric structure. It is essential to the development of the theory that the reference system of coordinates should itself be cyclically symmetric: the axes attached to a subdomain are carried into those of the next similar subdomain when the whole domain (the structure) is rotated through an angle of $2\pi/N$. N is the number of identical subdomains and it represents the order of cyclic symmetry. In this way the overall stiffness matrix has a special form: it is quasicirculant or block circulant. The theory of circulant and quasicirculant matrices shows that **K** is similar to a matrix with diagonal blocks [Dav79]. This is also justified for the mass matrix **M**, damping matrix **C**, etc. of full structure.

The immediate consequence is that the initial problem, for any form of matrix analysis, is multiplied N times, but each of them is drastically reduced in size. This reduction is the main advantage to be considered in the following because the cost of solving a problem rises much faster than its size. For instance, the mathematical representation of the model in terms of physical quantities leads to the system of simultaneous linear equations in the static analysis

$$\mathbf{KU} = \mathbf{P} \tag{2.1}$$

where **U** is the displacement vector and **P** is the nodal force vector. This problem can be split into N decoupled problems. Hence, there are preferable to solve many small sets of linear algebraic equations, quickly and involving less computer memory, using common matrix manipulation capabilities. For the decoupled eigenvalue problems there is another advantage: the condition number of each matrix from reduced eigenvalue problem is less or equal to the condition number of the overall matrix from uncoupled problem, respectively. Other advantage is that the analyst will be required to input data for the fundamental domain (geometry, material properties, connections, boundary conditions, etc.). Transformation of mathematical representation of the model from physical components to cyclic components is possible taking into account that the deflections of a subdomain j are related to the adjacent subdomains j+1 by a complex constant [Tho79].

$$\mathbf{U}_j = \mathbf{U}_{j+1} e^{i\mu} \tag{2.2}$$

where $\mu = 2\pi n/N$ is propagation constant, $n = 0,1,2,...\mathrm{Int}\left(\frac{N}{2}\right)$. The force acting in the k-th degree of freedom (d.o.f.) of the j-th subdomain may be expressed by discrete Fourier transform [BSR91] as:

$$p_{jk} = \sum_{n=1}^{N} f_{nk} e^{-i(j-1)\mu} \tag{2.3}$$

The cyclic component of force for the k-th d.o.f. is obtained by inverse transform:

$$f_{nk} = \frac{1}{N}\sum_{j=1}^{N} p_{jk} e^{i(j-1)\mu} \tag{2.4}$$

Since all forces acting on adjacent subdomains are related by the same phase constant, $e^{-i\mu}$, the deflection on adjacent subdomains should be connected in the same way [Tho79]:

$$u_{jk} = \sum_{n=1}^{N} d_{nk} e^{-i(j-1)\mu} \tag{2.5}$$

Using these equations in first equation, the decoupled matrix equation for various harmonics is given by

$$\mathbf{K_n D_n} = \mathbf{F_n} \tag{2.6}$$

The application of complex constraints produces a fully populated complex stiffness matrix $\mathbf{K}_n$ and a set of simultaneous complex equation is to be solved for each harmonic, n.

After the solution phase of the problem, it is necessary to transform the results from cyclic components back into physical components. The obtained displacement vector $\mathbf{D}_n$ may be transform to cover displacements on the entire domain. The displacement of the k-th d.o.f. of the j-th subdomain is calculate [BSR91] for N odd:

$$u_{jk} = d_{0k} + 2\sum_{n=1}^{(N-1)/2} [Re(d_{nk})\cos\{(j-1)\mu\} + Im(d_{nk})\sin\{(j-1)\}\mu]\,; \tag{2.7}$$

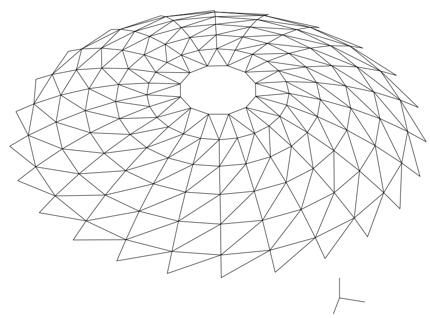
for N even:

$$u_{jk} = d_{0k} + (-1)^{(j+1)} d_{\frac{N}{2}k} + 2\sum_{n=1}^{\frac{N}{2}-1} [Re(d_{nk})\cos\{(j-1)\mu\} + Im(d_{nk})\sin\{(j-1)\}\mu] \tag{2.8}$$

Similar relations may be used for other physical components like stress, temperature, etc.

Any more savings in calculus are possible to obtain if the loading is also cyclical symmetric of the same order N. Then each subdomain is assumed to be loaded identically. It follows that the corresponding nodal displacements from the two boundaries of the fundamental domain are identically in the radial and circumferential direction, respectively [ZS72]. Since the constraints between two boundaries are of real type, will be necessary to solve only one system of simultaneous linear equations written for the fundamental domain [Vas95].

Figure 1



3 Geometric Nonlinear Analysis

In linear structural analysis, it is assumed that the joint displacements of the structure under the applied loads are negligible with respect to the original joint coordinates. Thus, the geometric changes in the structure can be ignored and the overall stiffness of the structure in the deformed shape can be assumed to equal the stiffness of the undeformed structure. However, in the space truss structures like domes, significant changes in the initial geometry can occur. In such a case, the stiffness of the dome in the deformed shape should be computed from the new geometry of the structure. That means to formulate the condition of equilibrium in the deformed configuration, since the truss members of the dome are assumed to have linear stress-strain relationships.

The problem to be solved in static geometric nonlinear structural system is the determination of the displacements, $\mathbf{U}$, corresponding to some load, $\mathbf{P}$, since the stiffness matrix, $\mathbf{K}$, in the equilibrium equation is a function of the joint displacements $\mathbf{U}$, which are as yet unknown. The analysis will typically proceed in two phases: the first is a solution which may be termed the incremental phase; the second is a corrective procedure applied to the first in an attempt to obtain the solution close to the equilibrium path.

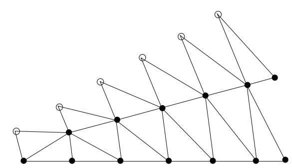
The way to preserve the advantage induced from the initial cyclic symmetry of the structure is to choose the modified Newton-Raphson method (MNR), in which the same matrix $\mathbf{K}_{(0)}$ is used for all the iterations. $\mathbf{K}_{(0)}$ is the stiffness matrix from the linear analysis, named also the tangent stiffness matrix. This matrix is computed at the beginning, and has a circulant or quasicirculant form.

4 Solution Algorithm

The advantages of the cyclic symmetry techniques are embedded in solving the linear equations' system. In the common geometrical nonlinear analysis, the global stiffness matrix $\mathbf{K}_{(0)}$ can be constructed only once, from the stiffness matrices of the individual members of the structure by the general assembly procedure.

In particular, the cyclic symmetry capabilities enable to solve the linear equations' system without assembling $\mathbf{K}$ - only the stiffness matrices of an appropiate fundamental domain are required. The algorithm for this method is summarized in the following:

Figure 2



Implementation

1. Initialization and input parameters such as geometric and material properties, connectivity, boundary conditions, and reference loads.

2. Initial step: solve $\mathbf{K}_{(0)}\mathbf{U}_{(0)} = \mathbf{P} \Rightarrow$ get $\mathbf{U}_{(0)}$.

3. Compute initial member end forces in global coordinates.

4. For each iteration (n)

4.1. For each element, form its stiffness matrix in local and then, in global coordinates $\mathbf{k}_{(n)}=\mathbf{k}(\mathbf{U}_{(n-1)})$.

4.2. Compute unbalanced joint loads in global coordinate system $\mathbf{P}_{(n)} = \mathbf{P}$ - $\mathbf{K}_{(n)}\mathbf{U}_{(n-1)}$.

4.3. Solve $\mathbf{K}_{(0)}\triangle\mathbf{U} = \mathbf{P}_{(n)} \Rightarrow$ get $\triangle\mathbf{U}$.

4.4. Update joint coordinates $\mathbf{U}_{(n)} = \mathbf{U}_{(n-1)} + \triangle\mathbf{U}$

4.5. Check convergence by considering force norm. If no convergence and $n <$ max. number of iterations GO TO Step 4.1.

5. Print on files final results.

6. Stop.

Practical Considerations

The type of the conservative structure considered in this approach is a cyclically symmetric space truss. In the iterative procedure, the stiffness matrices of the truss elements are updated corresponding to the new joint coordinates. The updating of initial length for each element is used to calculate the rotation matrix. This improves the convergence of the iterative process.

Apparently, we must assemble the overall tangent stiffness matrix of the structure to calculate in each iteration the unbalanced forces (see Step 4.2). This would compromise the present method as it is extremely expensive in the memory allocation and CPU time. In fact, we need to process only the new element stiffness in local and then in global coordinates. The unbalanced forces in global coordinate system are established by contribution of each element and is no need to assemble the overall tangent stiffness matrix of the structure.

Schwedler Dome Example

A computer program has been developed in FORTRAN 77 language to perform geometrical nonlinear analysis on cyclically symmetrical space truss structures. The structure analyzed was a spherical dome (25.6 m radius) of Schwedler type. The order of cyclic symmetry is 12. It can be observed that no plan of symmetry exists. Some different constructive solutions were studied to cover a cylindrical oil tank (32.0 m diameter). The constructive details (geometry, bar sections, loading, etc.) are from [SR85]. Since the whole structure (see Figure 1) is made up of 408 trusses connected in 166 nodes (396 active d.o.f.), the fundamental subdomain to be analyzed (see Figure 2) contains 34 trusses pinned in 20 nodes. In this case, the size of the problem is given by the 33 active d.o.f. on the subdomain after the boundary constraints were applied. We need only 6 iterations to obtain the specified convergence (0.01) of unbalanced force norm.

5 Conclusions

A solution strategy for the geometrical nonlinear analysis of elastic cyclically symmetric structures like dome have been presented. The formulation has been adopted the modified Newton-Raphson method (MNR) to take full advantage of cyclic symmetry of the structure. We remember that in nonlinear analysis solution algorithms are problem dependent. It is well known, however, that the convergence of the MNR method is linear and may run into serious difficulties if the loading level produces large changes in displacements. A simple method to prevent this is to scale back the load. At this stage, it is necessary to underline the limit of the MNR method in the cyclically symmetric context: it works on mild geometrical nonlinear structures like domes.

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