ELEMENTARY GRADES STUDENTS' CAPACITY FOR FUNCTIONAL THINKING[i]

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This is a study of how urban elementary grades students develop and express functions. Data were analyzed according to the forms of representations students used, the progression in students' mathematical language and the operations they employed, and how they attended to one or more varying quantities. Findings indicate that students are capable of functional thinking at grades earlier than perhaps thought. In particular, data suggest that students can engage in co-variational thinking as early as kindergarten and are able to describe how quantities correspond as early as 1st-grade. Although pattern finding in single variable data sets is common in elementary curricula, we conclude that elementary grades mathematics should extend further to include functional thinking as well.

BACKGROUND FOR THE STUDY

Research increasingly documents the ability of elementary grades (PreK-5) students from diverse socioeconomic and educational backgrounds to engage in algebraic reasoning[ii] in ways that dispel developmental constraints previously imposed on them (e.g., Bastable & Schifter, 2003; Blanton & Kaput, 2003; Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, & Brizuela, in press; Dougherty, 2003; Kaput & Blanton, in press; Schifter, 1999; Schliemann, Lara-Roth, & Goodrow, 2001). One of the forms algebraic reasoning takes involves *functional thinking*, which Smith (2003) describes as "representational thinking that focuses on the relationship between two (or more) varying quantities" and for which *functions* denote the "representational systems invented or appropriated by children to represent a generalization of a relationship among quantities". As reported earlier (Blanton & Kaput, 2002), our interest in the development of algebraic reasoning in elementary school mathematics led us to identify design aspects of tasks that might be used to exploit algebraic ideas, particularly in tasks where algebraic reasoning occurred through generalizing from numerical patterns to develop functional relationships. This study extends that work and builds on the emerging research base in early algebraic thinking by examining how students in elementary grades are able to develop and express functional relationships.

METHODOLOGY

The data for this study were taken from GEAAR, a 6-year, teacher professional development program in an urban school district designed to help teachers transform their instructional resources and teaching practices to build on classroom opportunities for algebraic reasoning. We base the particular findings reported here on PreK-5 student responses from one of the district's schools to the task "Eyes and Tails". The task involves developing a functional relationship between an arbitrary amount of dogs and the corresponding total number of eyes or the total number of eyes and tails:

Eyes and Tails:

Suppose you were at a dog shelter and you wanted to count all the dog eyes you saw. If there was one dog, how many eyes would there be? What if there were two dogs? Three dogs? 100 dogs? Do you see a relationship between the number of dogs and the total number of eyes? How would you describe this relationship? How do you know this works?

Suppose you wanted to find out how many eyes and tails there were all together. How many eyes and tails are there for one dog? Two dogs? Three dogs? 100 dogs? How would you describe the relationship between the number of dogs and the total number of eyes and tails? How do you know this works?

"Eyes and Tails" was selected because its accessibility across the grades allowed us to look for longitudinal trends in students' functional thinking. Student responses were collected from written work and teacher interviews and were analyzed by grade according to the types of representations students used at different grades, the progression of mathematical language in students' descriptions of functional relationships, how students tracked and organized data, the mathematical operations they employed to interpret functional relationships (i.e., additive vs. multiplicative), and how they expressed variation among quantities.

RESULTS

Pre-kindergarten (Ages 3–5)

The teacher and students spent time with paper cutouts of dogs, counting their eyes and tails. Students described the amounts as "even" or "odd". With the teacher's guidance, the whole class used a t-chart to organize their data. As a class, they recorded that one dog had 2 eyes and 1 tail, or a total of 3. They also determined that 2 dogs had 4 eyes and 2 tails, or 6 total. As the children offered the number of eyes, the teacher wrote that number in the appropriate box in the t-chart and put a corresponding number of dots below the box. She recorded the number of tails for a given number of dogs in a similar manner.

In finding the total number of eyes and tails for a given number of dogs, the teacher pointed to each of the dots as the class simultaneously counted. When the teacher asked about 3 dogs and 4 dogs, the children counted the number of eyes and tails using dog pictures on the floor in front of them. No predictions were made at this grade and answers were determined by counting visible objects. Moreover, there was no indication from the data that students looked for patterns. However, we maintain that a significant mathematical event for these students was not only the development of correspondence between numeral and object, but also the introduction of a function table (t-chart) as a means to organize quantities that co-vary. The latter event reflects the early development of representational infrastructure to support algebraic reasoning.

Kindergarten

In one class, students recorded data by making a dot for each eye and a long mark for each tail. Dots were grouped in pairs or in 4-dot, 2x2 arrays. Dots (eyes) and marks (tails) were recorded under the number sentences that represented the total number of eyes and tails for a given amount of dogs. Data were aggregated by groups of dogs, which were drawn and painted by students, and the corresponding number sentences for total eyes or total eyes and tails, as well as dots and marks representing eyes and tails, were recorded and encircled by students (see Figure 1). Data were calculated for up to 10 dogs. T-charts were used in some kindergarten classes (with data recorded on the charts by the teacher) and some students identified the pattern in the amount of eyes as "counting by 2s", "more and more", and "every time we add one more dog, we get two eyes".

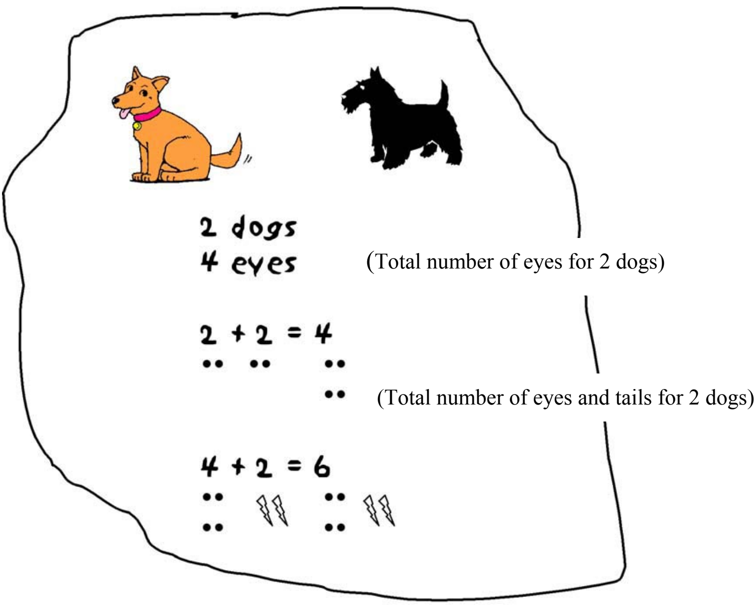

Figure 1. Kindergarten students' representation for 2 dogs.

In one class, after the teacher and students built a t-chart that recorded the number of eyes on 1, 2, 3 and 4 dogs (the teacher recorded the data), the following exchange occurred in which the teacher called on students to identify a pattern based on parity in the data:

> Teacher: What if [we have] 5 dogs? Odd or even?
>
> Student: Even.
>
> Teacher: Why?
>
> Student: We're skipping all the odd numbers.

We include this particular transcript here because it illustrates an important point. By asking students to analyze the data in terms of parity (even or odd) and not just quantity, the teacher required a further abstraction in student thinking. We find it mathematically significant that *kindergarten* students were not only able to recognize even and odd numbers (a concept we had observed as difficult for some *third-grade* students during the early stages of GEAAR), but were also able to articulate a pattern, albeit primitive, about parity in the data.

First grade

First-grade teachers noted that students had used t-charts previous to "Eyes and Tails". Moreover, students, rather than the teacher, recorded data on t-charts. They described patterns in the case of counting eyes and tails as "we are counting by 3s". Literacy activity was integrated into the problem in one class, where students made rhyming words and constructed poems in conjunction with the pattern "counting by 2s". With the teacher, students in this class tried to predict the number of eyes 7 dogs would have and used skip counting to find the answer. Students saw that the pattern would "double" (for total eyes), and then "triple" (for total eyes and tails).

Second grade

Students in one 2^{nd}-grade class recorded their data on a t-chart for 1 to 10 dogs and were able to give a multiplicative relationship using natural language ("You have to double the number of dogs to get the number of eyes"). They then used this to predict the number of eyes for 100 dogs without counting the eyes. They constructed a similar t-chart for counting eyes and tails and used this, based on the information recorded in their t-charts, to predict that the total number of eyes and tails for 100 dogs would be 300.

Third, fourth, and fifth grades

Students in 3^{rd}-grade classes used t-charts fluently, were able to express the rule multiplicatively in words and symbols, and could predict the number of eyes or eyes and tails for 100 dogs using their rule. In counting the number of eyes, students noted that "It doesn't matter how many dogs you have, you can just multiply it by 2". Students were able to describe this relationship as '$n \times 2$' and '$2 \times n$'. One 3^{rd}-grade class graphed their results comparing the number of eyes with dogs (see Figure 2) and comparing the number of eyes and tails with the number of dogs. Fourth- and 5^{th}-grade student work was similar to that in 3^{rd}-grade, with the only noticeable difference being that students in later grades needed less data (only up to 3 dogs) to develop a function.

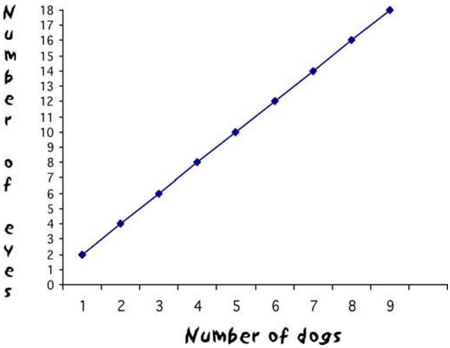

Figure 2. Third-grade students' graphical representation of the total number of eyes versus number of dogs.

DISCUSSION

These data indicate that very young learners are capable of functional thinking and suggest how that thinking might progress over grades Pre-K–5. Particularly, shifts occurred in how students were able to (1) use representational forms such as t-charts, (2) articulate and symbolize patterns, from natural language descriptions of additive relationships to symbolic representations of multiplicative relationships, and (3) account for co-varying quantities. The following discussion details that progression.

The development of representational infrastructure and students' symbol sense

Across the grades, students used tables, graphs, pictures, words and symbols to make sense of the task and to express mathematical relationships. Regarding the scaffolding of these representational forms, teachers were typically the recorders for t-charts in earliest grades, although by first grade students began to assume responsibility for this. In one kindergarten class, students did record the data on a class chart, but the teacher played a large role in organizing the data. By 2^{nd}- and 3^{rd}-grades, students seemed to use this representational tool fluently.

In grades Pre-K through 1, students relied on counting visible objects, keeping track of their counting in various ways through t-charts or making dots and marks for eyes and tails (see Figure 1). In early grades, t-charts became opportunities to re-represent marks with numerals as children worked on the correspondence between quantity and numeral representation. T-charts were the most common way, especially from 1^{st}-grade through 5^{th}-grade, that students organized and tracked data.

We observed that, by 3rd-grade, students were able to symbolize varying quantities with letters, and they seemed to have an emergent understanding of what these symbols represented. (We did note some confusion as to when a variable represented the number of dogs versus the number of eyes (or eyes and tails).) Moreover, third-grade students could express relationships in symbolized form (e.g., "number of eyes is *2n"*), although they did not fully symbolize the relationship in a form such as '$f(n)=2n$'. In 4th grade, some students wrote '$\square \times 3 = n$' after constructing a t-chart. Although students primarily used words and symbols to describe the function, one 3rd-grade class did construct a line graph representing the number of dogs versus the total number of eyes (see Figure 2).

Finally, the ways students labelled their t-charts reflected increasingly sophisticated language. In grades PreK-1, t-chart headings were described in words ('dogs'; 'eyes'); in 2nd-grade, t-charts were labelled as "number of dogs" and "number of eyes". By 3rd-grade, symbols such as 'D' and 'E' were used for the number of dogs and eyes.

All of this suggests that teachers were able to scaffold students' thinking from a very early age so that diverse representational and linguistic tools became an increasing part of students' repertoire of doing mathematics.

How students accounted for varying quantities

Although finding patterns and predicting future values seemed understandably tentative in grades Pre-K–1, there were notable instances of this, such as the protocol recorded earlier in which kindergarten students found an "even" pattern in their data. When using a t-chart, 1st-grade students noticed patterns in how the number of eyes varied, and they described patterns in everyday language using both additive relationships ("we are counting by 3's) and multiplicative relationships ("double" and "triple"). Skip counting seemed to be the most common process for finding unknown values, and additive relationships were more common than multiplicative ones. By 2nd-grade, students were able to articulate a multiplicative relationship using everyday language ("You have to double the number of dogs to get the number of eyes") and use this to predict the number of eyes for 100 dogs without counting the eyes. In later grades, students needed increasingly fewer data values to determine a functional relationship and make predictions.

What we found particularly compelling in the data was how early students began to think about how quantities co-varied. One kindergarten class described an additive relationship between the number of eyes and dogs as "every time we add one more dog we get two more eyes", indicating that they were attending to both the number of dogs and eyes simultaneously and were able to describe how these quantities co-varied. In 1st-grade, students identified a multiplicative relationship of "doubles" and "triples" to describe the number of eyes and the number of eyes and tails, respectively, for an arbitrary number of dogs. In 2nd-grade, students also saw a multiplicative relationship ("doubles"; "If you double the number of dogs you get the number of eyes"). The observation that the pattern "doubles" or "triples" indicates that students were attending to how quantities corresponded. That is, some quantity needed to be doubled to get the total amount of eyes or eyes and tails. Since data in the 'output' column (e.g., total number of eyes; 2, 4, 6, 8…) were *not* doubled (i.e., $4x2 \neq 6$; $6x2 \neq 8$), this suggests that students were not looking 'down' the column of eye data (which would have resulted in a pattern of "add 2 every time" or "count by 2's"), but 'across'. By 3rd-grade and beyond, students seemed fairly sophisticated in their ability to attend to how two quantities varied simultaneously and to symbolize this relationship as a functional correspondence (e.g., "the number of eyes = $2n$").

Although elementary grades mathematics has in more recent years included notions of patterning, it has not traditionally attended to functional thinking, especially in grades Pre-K–2. Yet, from our analysis, we found that students could engage in co-variational thinking as early as kindergarten and were subsequently able to describe how quantities corresponded as early as 1st-grade. More abstract symbolizing using letters as variables occurred as early as 3rd-grade. We conjecture that the typical emphasis on pattern finding in single variable data sets in early elementary grades might impede an emphasis on functional thinking in later elementary grades and beyond. In particular, there was evidence that when 1st-grade students engaged in functional thinking, they were sometimes redirected to an analysis of a single variable (e.g., finding a pattern in the total number of eyes). This focus could be a habit of mind engendered in teachers by existing curricula. Ultimately, pattern finding in single variable data has less predictive capacity and is less powerful mathematically than functional thinking. There is a fundamental conceptual shift that must occur in teachers' thinking in order to move from analyses of single variable data to those attending to two or more quantities simultaneously. As a result, we suggest that curricula for grades PreK-5 should attend to how two or more quantities vary simultaneously, not just simple patterning. This study supports the claim that young students have the capacity for this type of functional thinking.

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[ii] While algebraic reasoning can take on various mathematical forms in the classroom (Kaput, 1998), we see it broadly as a habit of mind that permeates all of mathematics and that involves students' capacity to build, justify, and express conjectures about mathematical structure and relationship.