MAKING SENSE OF IRRATIONAL NUMBERS: FOCUSING ON REPRESENTATION

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In our investigation of preservice secondary teachers' understanding of irrational numbers we focus on how different representations influence participants' responses with respect to irrationality. As a theoretical perspective we use the distinction between transparent and opaque representations, that is, representations that "show" some features of numbers while they "hide" others. The results suggest that often participants do not rely on the given transparent representation (i.e. 53/83) in determining whether a number is rational or irrational. Further, the results indicate participants' tendency to rely on a calculator and a preference towards decimal over the common fraction representation. As a general recommendation for teaching practice we suggest a tighter emphasis on representations and conclusions that can be derived from considering them.

This report is a part of ongoing research on understanding of irrational numbers. Specifically, we focus here on how irrational numbers can be (or cannot be) represented and how different representations influence participants' responses with respect to irrationality.

ON REPRESENTATIONS AND IRRATIONAL NUMBERS

There is an extensive body of research on representations in mathematics and their role in mathematical learning (Cuoco, 2001; Goldin & Janvier, 1998, to name just a few recent collections). The role of representations is recognized in manipulating mathematical objects, communicating ideas, and assisting in problem solving.

Researchers draw strong connections between the representations students use and their understanding (Lamon, 2001). Janvier (1987) describes understanding as a "cumulative process mainly based upon the capacity of dealing with an 'ever-enriching' set of representations" (p. 67). Furthermore, representations are considered as a means in the formation of conceptual understanding. The ability to move smoothly between various representations of the same concept is seen as an indication of conceptual understanding and also as a goal for instruction (Lesh, Behr and Post, 1987). Moreover, according to Kaput (1991), possessing an abstract mathematical concept "is better regarded as a notationally rich web of representations and applications" (p. 61). Only a small part of the work on representation addresses representation of numbers, and it focuses primarily on fractions and *rational* numbers (e.g. Lesh, Behr & Post, 1987; Lamon, 2001).

In contrast, research on irrational numbers is rather slim. Fischbein, Jehiam & Cohen (1994, 1995) are the only research reports we found that treat the issue explicitly. The main objective of these studies was to examine the knowledge of irrational numbers of high school students and preservice teachers. Based on historical and psychological grounds, Fischbein et. al. assumed that the concept of irrational number presented two major obstacles: incommensurability and nondenumerability. Contrary to the expectations, the studies found that these intuitive difficulties did not manifest in participants' reactions. Instead, they found that subjects at all levels were not able to define correctly rational and irrational numbers or place the given numbers as belonging to either of these sets. It has been concluded that the expected obstacles are not of primitive nature – they imply certain mathematical maturity that the subjects in these studies did not possess.

These findings call for a more comprehensive study examining the understanding of irrationality and attending to issues of concern that were identified. Definitions of rational and irrational numbers rely on number representations. There has been no study to date that investigated understanding of irrational numbers from the perspective of representations.

THEORETICAL PERSPECTIVE: TRANSPARENT AND OPAQUE

As a theoretical perspective we use the distinction between transparent and opaque representations, introduced by Lesh, Behr and Post (1987). According to these researchers, a transparent representation has no more and no less meaning than the represented idea(s) or structure(s). An opaque representation emphasizes some aspects of the ideas or structures and de-emphasizes others. Borrowing Lesh's et. al. terminology in drawing the distinction between transparent and opaque representations, Zazkis and Gadowsky (2001) focused on representations of numbers introducing the notion of relative transparency and opaqueness. Namely, they suggested that all representations of numbers are opaque in the sense that they always hide some of the features of a number, although they might reveal other, with respect to which they would be "transparent". For example, representing the number 784 as 28^2 emphasizes that it is a perfect square, but de-emphasizes that it is divisible by 98. Representing the same number as $13 \times 60 + 4$ makes it transparent that the remainder of 784 in division by 13 is 4, but de-emphasizes its property of being a perfect square. In general, we say that a representation is transparent with respect to a certain property, if the property can be "seen" or derived from considering the given representation.

The definition of rational number relies on the existence of certain representation: a rational number is a number that can be represented as *a/b*, where *a* is an integer and *b* is a nonzero integer. When a real number cannot be represented in this way, it is called irrational. Until the exposure to a formal construction of irrational number using, for instance, Dedekind cuts, this distinguishing representational feature is used as a working definition of irrational number. That is to say, irrational number is a number that cannot be represented as a ratio of integers. An equivalent definition of irrational number refers to the infinite non-repeating decimal representation.

Applying the notions of opaqueness and transparency we suggest that infinite non-repeating decimal representation (such as 0.010011000111…, for instance) is a transparent representation of an irrational number (that is, irrationality can be derived from this representation), while representation as a common fraction is a transparent representation of a rational number (that is, rationality is embedded in the representation).

RESEARCH SETTING

As part of a larger research on understanding of irrational numbers we examined how the availability of certain representations influenced participants' decisions with respect to irrationality. To investigate this we designed the following questions:

1. Consider the following number 0.12122122212… (there is an infinite number of digits where the number of 2's between the 1's keeps increasing by one). Is this a rational or irrational number? How do you know?
2. Consider 53/83. Let's call this number M. In performing this division, the calculator display shows 0.63855421687. Is M a rational or an irrational number? Explain.

(Note that the numbers in Question 2 are carefully chosen so that the repeating is "opaque" on a calculator display. The length of the period in this case is 41 digits.)

These questions were presented to a group of 46 preservice secondary school mathematics teachers as part of a written questionnaire. These participants were in their final course in the teacher education program and had at least two calculus courses in their background. Upon completion of the questionnaire, 16 volunteers from the group participated in a clinical interview, where they had the opportunity to clarify and extend upon their responses. Participants' responses were analyzed with specific attention to the role that representation of a number played in their decision, and their reliance or non-reliance on the given representation.

RESULTS AND ANALYSIS

We first present quantitative summary of written responses. We then focus on the detail of one particular interview. Further, we present some common erroneous beliefs of participants and attempt to identify their sources. We conclude this section by summarizing some common trends in participants' approaches to the presented questions.

Quantification of results for #1 – considering 0.12122122212… (n=46):

Response category	Number of participants [%]
Correct answer with correct justification	27 [58.7%]
Correct answer with incorrect justification (such as, "this number is irrational because there is an infinite number of digits")	7 [15.2]
Correct answer with no justification	1 [2.2%]
Incorrect answer	6 [13%]
No answer	5 [10.9%]

Quantification of results for #2 – considering 53/83 (n=46):

Response category	Number of participants [%]
Correct answer with correct justification	31 [67.4%]
Correct answer with incorrect justification (such as, "this number is rational because the digits terminate")	7 [15.2]
Correct answer with no justification	2 [4.3%]
Incorrect answer	5 [10.9%]
No answer	1 [2.2%]

As shown in these tables, over 40% of the participants did not recognize the non-repeating decimal representation as a representation of an irrational number. Further, over 30% of the participants either failed to recognize a number represented as a common fraction as being rational or provided incorrect justifications for their claim. It is evident that for a significant number of participants the definitions of rational and irrational numbers were not a part of their active repertoire of knowledge. In the next section we consider the responses of one participant, Steve, that shed light on the possible sources of students' errors and misconceptions.

Focusing on Steve

Steve: [claiming 0.121221222… is irrational] Um hm, I would say because it's not a common, there's not a **common element** repeating there that it would make it a rational...

Interviewer: How about this one, 0.0122222…with 2 repeating endlessly, is this rational of irrational?

Steve: Okay, I would have to say that's irrational real number.

Interviewer: Irrational or rational, I couldn't hear you.

Steve: Irrational. Well oh, the 2 repeats, no but it has to be, then it repeats, even though the 2 repeats, it has to be a **common pattern**, so I would say it's irrational.

Interviewer: Okay, so 0.01222… repeating infinitely is irrational.

Steve: I think so, but I forget if the fact that that, if the 1 there changes, I would have thought it would have to be **012, 012,** … but if it starts repeating later, yeah I can't remember if it starts repeating later, I'm pretty sure it's irrational, but I could be mistaken.

Interviewer: How about the second question, when you consider 53 divided by 83. . .

Steve: Um hm. . .

Interviewer: And let's call this quotient M, and if you perform this division on the calculator the display shows this number, 0.63855421687.

Steve: And I assume it keeps going, that's just what fits on your calculator. . .

Interviewer: Yeah, that's what the calculator shows, that's right. So is M rational or irrational?

Steve: So this is the **quotient** M, yeah I would say it's **irrational**.

Interviewer: Because?

Steve: Because **we can't see a repeating decimal**.

Interviewer: But maybe later, down the road it starts repeating.

Steve: Well that's true, it's possible. . .

Interviewer: So we can't really determine?

Steve: Well I guess we don't, we **wouldn't know** for sure just **from looking at that number on the calculator**, **but chances are that if it hasn't repeated that quickly, then it would be irrational.** I haven't seen a lot of examples where they start repeating with 10 digits or more. I'm sure there are some but... .

Interviewer: Okay, and the fact that it comes from dividing 53 by 83, does that not qualify it as rational?

Steve: Oh so that is a fraction, it's 53/83?

Interviewer: Yeah we, that's how we got this number, so we divided 53 by 83 and called this M. . .

Steve: **53/83 as it's written would be rational**, but yeah, I see what you mean, if you took that decimal, yeah, I guess that's a good point. I see what you're, you're saying that fact that it's 53/83 that is A/B, so that is rational, but then when you take, **if you started dividing. . . It would just go on and on and on and on, so that you would think is irrational.** Yeah, I must say I don't know the answer to that.

In the beginning of the interview Steve claims correctly that an infinite non-repeating decimal fraction represents an irrational number. However, his use of the words "common element" prompts an inquiry into his perception of "common". This perception is clarified in Steve's incorrect claim that 0.0122222… is also irrational. Steve is looking for a common *pattern*, and the repeating digit of 2 does not seem to fit his perception of a pattern. For the next question Steve is presented with a fraction 53/83 and distracted by its display on a calculator. Focusing on the decimal representation rather than the common fraction representation, his first response – this quotient is irrational – presents an oxymoron. It is based on the inability to "see" the repeating pattern. The underlying assumption here is that a repeating pattern, if it exists, has a short and easily detectable repeating cycle. This perception is confronted by the interviewer in directing Steve's attention to the number representation as a fraction, 53/83. From his reply it appears that Steve believes that whether the number is rational or irrational depends on how it is written; that is, a common fraction represents a rational number, but its equivalent decimal representation could be irrational.

In what follows we demonstrate several frequent erroneous beliefs expressed by the participants, some of which have been exemplified in the excerpt from the interview with Steve. There are two, interrelated sources of conflict responsible for these erroneous beliefs: applying incorrect or incomplete definition and not understanding the relationship between fractions and their decimal representations.

Applying incorrect or incomplete definition

- If there is a pattern, then the number is rational. Therefore 0.12122122212… is rational, (similarly, 0.100200300… is rational, but 0.745555… is not, because there is no pattern).
- 53/83 is irrational because there is no pattern in the decimal 0.63855421687.
- 53/83 is rational because it terminates (calculator shows 0.63855421687)
- 53/83 could be rational or irrational – I cannot tell whether digits will repeat because too few digits are shown. They might repeat or they might not.

The first illustration above echoes Steve's reliance on a personal interpretation of "pattern", but is ignoring the required repetition of digits. The other three responses demonstrate participants' dependence on a calculator and preference towards decimal representation, which is misinterpreted as either terminating or having no repeating pattern, or treated as ambiguous.

Interrelations of fractions and repeating decimals

- There is no way of telling if 53/83 is rational - unless you actually do the division which could take you forever. Digits might terminate at a millionth place or they might start repeating after a millionth place.

- It is possible that a number is rational and irrational at the same time. For example, there are fractions that have non-repeating non-terminating decimals, yet they can be represented as a/b
- It is easy to turn a fraction into a decimal. But there is no easy, general way of turning a decimal into a fraction. Looking at a decimal, unless it is a terminating decimal, you cannot tell if it is rational or not.
- 0.012222… is not rational. I cannot think of any two numbers to divide to get that decimal.

These approaches are mostly procedural in their focus on carrying out the operation of division or performing conversion, rather than attending to the structure of the given representation. It is apparent that the connection between fractions and repeating decimals is not recognized.

SUMMARY AND CONCLUSION

We investigated the understanding of irrational numbers of the group of preservice secondary mathematics teachers. In this report we focused on the role that representations play in concluding rationality or irrationality of a number.

Though the majority of participants provided correct and appropriately justified responses attending to the provided representation, incorrect responses of the minority are troublesome, especially taking into account participants' formal mathematical background. For this significant minority,

- the definitions of irrational, as well as rational, numbers were not in the "active" repertoire of their knowledge;
- there was a tendency to rely on a calculator and participants expressed preference towards decimal representation over the common fraction representation;
- there was a confusion between irrationality and infinite decimal representation, regardless of the structure of this representation
- the idea of "repeating pattern" in decimal representation of numbers was at times overgeneralized to mean any pattern.

From our theoretical perspective, we say that the transparent features of the given representations were either not recognized or not attended to. A possible obstacle to students' understanding is that the equivalence of the two definitions of irrational numbers given in school mathematics – nonexistence of representation as *a/b*, where *a* is an integer and *b* is a nonzero integer and infinite non-repeating decimal representation – is not recognized. We consider this as a *missing link* that is rooted in understanding of rational numbers, that is, the understanding of how and when the division of whole numbers gives rise to repeating decimals, and conversely, that every repeating decimal can be represented as a ratio of two integers.

A general suggestion for teaching practice calls for a tighter emphasis on representations and conclusions that can be derived from considering them. In particular, attending to the connections between decimal (binary, etc.) and other representations (geometric, symbolic, common fraction, and even continued fractions) of a number can be an asset. Simply put, we suggest that by directing explicit attention of students to representations and to mathematical connections that render the two representations equivalent, teachers can help students acquire a more profound understanding of number.

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