MATHEMATICAL ABSTRACTION THROUGH SCAFFOLDING

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This paper examines the role of scaffolding in the process of abstraction. An activity-theoretic approach to abstraction in context is taken. This examination is carried out with reference to verbal protocols of two 17 year-old students working together on a task connected to sketching the graph of |f(|x|)|. Examination of the data suggests that abstraction is a difficult activity that can sometimes be beyond students' unassisted efforts, in which case supportive intervention of a scaffolder through several means of assistance is observed to help the students achieve the abstraction.

The issue of abstraction has attracted the attention of many educators (Dienes, 1963; Piaget, 1970; Skemp, 1986). Purely cognitive views see abstraction as ascending from 'concrete' to 'abstract', e.g. "the extraction of what is common to a number of different situations" (Dienes, 1963, p.57). Criticism of this view of abstraction comes from an epistemological point of view which recognises that contextual and social factors are crucial to knowledge acquisition (see van Oers, 2001). Empirical studies of abstraction in context are a relatively recent phenomenon and include Noss and Hoyles' (1996) concept of situated abstraction and Hershkowitz, Schwarz and Dreyfus' (2001) activity-theoretic model of abstraction. This paper works within the Hershkowitz et al.'s model.

Hershkowitz et al.'s (2001) model was inspired by Davydov's (1990) epistemological theory pointing a dialectical connection between abstract and concrete. They provide an operational definition of abstraction as "*an activity of vertically reorganising previously constructed mathematics into a new mathematical structure*." The new structure is the product of three epistemic actions: recognising, building-with, and constructing ('RBC theory of abstraction' hereafter). Recognising a familiar structure occurs when a student realises that a structure is inherent in a given mathematical situation. Building-with consists of combining existing artefacts in order to meet a goal. Constructing consists of assembling knowledge artefacts to produce a new structure. These actions are dynamically nested in such a way that building-with includes recognising, and constructing includes both recognising and building-with.

In empirical studies of abstraction researchers claim to gain insight into students' abstraction processes via the help of a knowledgeable agent e.g. the researcher/ interviewer. Hershkowitz et al. (2001), for example, note that the interviewer in their study aims to induce the student to reflect on what she is doing so that she might progress beyond the point that she would have reached without the interviewer. Others argue that the successful completion of an abstraction process is contingent upon providing the student with 'hinting' (Ohlsson and Regan, 2001) and 'shifting the focus of activities' (van Oers, 2001). All three sets of researchers effectively argue that abstraction is not an easy process which may be beyond the learners' unassisted efforts. This has links with the theoretical concept of 'scaffolding'.

Scaffolding may be defined as any kind of systematic guidance given to a learner to develop and achieve his/her fullest potential, which is beyond his/her actual present ability (Chi, Siler, Jeong, Yamauchi and Hausmann, 2001). The key element of scaffolding is 'the sensitive, supportive intervention of a teacher in the progress of a learner who is actively involved in some specific task, but who is not quite able to manage the task alone" (Mercer, 1995, p.74). Scott (1997) develops the idea of 'sensitive intervention' arguing that throughout scaffolding, in interacting with the learner, the tutor is aware of and responsive to existing modes of and any changes in a learner's thinking, and thus has the opportunity to support the development of the learning goal. Scott breaks the teacher's responsiveness down into three elements: *monitoring* – monitor present performance of the learner, *analysing* – analyse the nature of any differences between present performance and performance required by the learning goal, *assisting* – respond with an appropriate intervention to address differences in performance. When the learner makes progress towards the learning goal, the level of assistance is decreased and responsibility may be handed over to the learner (Bruner, 1983).

Despite the implications of empirical investigations of abstractions for scaffolding, published research to date has not addressed the link between scaffolding and abstraction. This paper examines the role of scaffolding in the process of abstraction within the framework of the RBC theory. The examination is carried out with regard to two 17-year-old girls working together within a scaffolded situation to construct a new mathematical structure. The paper provides a brief description of the girls' joint work and the scaffolder's intervention. It then discusses how scaffolding functions and relates this to the achievement of a mathematical abstraction.

BACKGROUND

The study: The study presented in this paper is part of a larger study focusing on the role that scaffolding and students' interaction play in the formation of mathematical abstraction within the framework of the RBC theory. For the study, data were collected from the students working on four tasks connected with sketching the graphs of absolute values of linear functions. The students were selected on two criteria: (1) they had the prerequisite knowledge needed to embark on the tasks; (2) they were not to be acquainted with the intended abstractions. In order to select a sample that met the criteria, a diagnostic test was prepared and applied in Turkey to 134 students aged 16-18. 20 students were selected and organised so that 14 worked in pairs and six worked alone. Four pairs of students and three individuals worked within a scaffolded environment, the rest without. Verbal protocols were audio-recorded.

The task: Four tasks were prepared and applied on four successive days. The overall aim of the first, second and fourth task was to construct a method to draw the graphs of, respectively, $|f(x)|$, $f(|x|)$ and $|f(|x|)|$ by using the graph of $f(x)$. The organisational structure of these three tasks was identical apart from the mathematical objects i.e. $|f(x)|$, $f(|x|)$ and $|f(|x|)|$. The third task was prepared to consolidate the first and second tasks. This paper reports on protocols generated in the fourth task, which had five questions. In the first question the students were asked to draw the graph of $|f(|x|)|=|(|x|-4)|$ and to comment on any patterns in the graph. In the second question, they were asked if they saw any relationship between the graph of $|f(|x|)|=|(|x|-4)|$ and the graph of $f(x)=x–4$. In the third question, the graph of $f(x)=x+3$ was given and the students were asked if they could draw the graph of $|f(|x|)|=|(|x|+3)|$ by using the given graph as an aid. In the fourth question, four linear graphs, without equations, were given and the students were asked to obtain the graph of $|f(|x|)|$ for each one. In the fifth question, students were asked to explain how to draw the graph of $|f(|x|)|$ by drawing on the graph of $f(x)$. By the end of the task, the students were expected to construct a method to sketch the graph of $|f(|x|)|$ by using the graph of $f(x)$, which will be referred to as 'the structure of $|f(|x|)|$' in the paper.

THE DATA

Part of the two girls' verbal protocol on the fourth task, parsed into episodes, is presented in this section. Please note that some comments have been inserted in square brackets to assist the reader to follow the interaction amongst the participants.

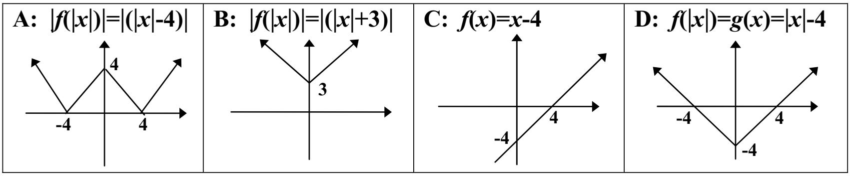

Figure 1: The graphs obtained by the students.

Episode 1:

At the beginning of the task the interviewer did not intervene, in order to observe how far they could go without his assistance. The students obtained the graphs of $|f(|x|)|$ for the first and third questions (see figure 1A and 1B) correctly by substituting into the given equations for different values of x. After considering these graphs together, as can be seen in the excerpt below, they stated that they were unable to find a general method.

141H: *I don't think we can ever understand how to use f(x) to draw the graph of |f(|x|)|…*

142S: *The first graph [figure 1A] was something like W-shaped… but this graph [figure 1B] is V-shaped…*

143H: *They are totally different! How can we speak in a general way… even this question made things worse… rather than helping us.*

144S: *We'd better stick to substituting… we can answer the next question by substituting.*

This excerpt clearly shows that the structure of $|f(|x|)|$ is beyond the students unassisted collaborative efforts. At this point the interviewer intervened and suggested that the students return to the first question. Between utterances 149 and 164 (not shown), the interviewer helped the students recognise what they know about the graphs of $|f(x)|$ and $f(|x|)$.

Episode 2:

165I: *Ok, if you pay a closer attention to the equation… I mean look at the expression itself, |f(|x|)|, it is a combination of these two [of |f(x)| and f(|x|)]. Do you see that?*

166H: *Yes, that's right. We already mentioned about this at the beginning…*

167S: *Yeah, this [|f(|x|)|] is a combination of f(|x|) and |f(x)|… for example [the graph of] |f(x)| never goes under the x-axis…*

168I: *Ok, let's think about it and consider what you know. How can we use our knowledge to obtain this graph [of |f(|x|)|]?*

169S: *Look, it makes sense… I mean [the graph of] |f(x)| doesn't pass under the x-axis as [values of] y is always positive and also the graph of f(|x|) is symmetric in the y-axis… so the graph of |f(|x|)| doesn't take negative value and symmetric in the y-axis.*

170H: *Yeah, it makes sense now… look, if |f(|x|)| is a combination of f(|x|) and |f(x)|, we can think about it like a computation with parenthesis…*

171I: *Computation with parenthesis?*

172H: *I mean for example when we are doing computations with some parentheses like… let's say for example… err ... (7-(4+2)), then we follow a certain order…*

173S: *Right, I understood what you mean… we need to first deal with the parenthesis inside of the expression, is that what you mean?*

174H: *Yeah, I think it is somehow similar in here, I can sense it but I can't clarify…*

175S: *I know what you mean but how could we determine the parentheses in here?*

176I: *You both made an excellent point. OK, let's think about it together. In the expression |f(|x|)|, can we think about the absolute value sign at the outside of the whole expression as if a larger parenthesis, which includes another one just inside?*

In this episode, the interviewer stresses that $|f(|x|)|$ could be seen as a combination of $f(|x|)$ and $|f(x)|$. This prompts S to recognise some properties of the graphs of $f(|x|)$ and $|f(x)|$ in relation to the graphs of $|f(|x|)|$ (167 and 169). In 168, the interviewer asks the students to think about how to obtain the graph of $|f(|x|)|$ by making use of what they already know and thus sets a subgoal to develop a strategy about how to obtain the graph of $|f(|x|)|$. In 170, H recognises the order of the operational priority of computations including parentheses and proposes that $|f(|x|)|$ may be treated the same way. In 171, the interviewer probes H to understand her intention. Based on H's interaction with S, the interviewer has an opportunity to monitor and analyse their performance. The students, between 172 and 175, are developing an appropriate strategy but they are not sure if their approach is reasonable or how they might determine the 'parentheses' in the expression of $|f(|x|)|$. In 176 the interviewer intervenes to keep the students in pursuit of the subgoal and gives positive feedback indicating that their approach is reasonable. He further accentuates how the absolute value signs in the expression $|f(|x|)|$ might be used in the similar way to parentheses.

Episode 3:

177H: *Aha, I got it… I know what we will do.*

178I: *Could you please tell us?*

179H: *We can consider f(|x|) as if it was the smaller parenthesis!*

180I: *Smaller parenthesis?*

181H: *I mean it should be the first thing that we need to deal with.*

182S: *Yeah, I agree… I think we should begin with the graph of f(|x|) and first draw it.*
183H: *But what next?*
184S: *Then we can use the absolute value at the outside… in the similar way of doing computations.*
185H: *But we will be drawing graphs? Can we really do this?*
186S: *I am not too sure if we can… but it sounds plausible…*
187I: *What you are doing here is not computation of course… but you are making an analogy, I mean you are making some certain logical assumptions based on your earlier experiences… and I see no problem with that… let's draw the graph by considering what we've just talked about and then decide if it will work or not?*

In this episode, the interviewer's help in 176 prompts H to start to build a plan as to how to execute the strategy by using what they recognised in the second episode. The interviewer here asks some probing questions (178 and 180) to gain insights into the students' explanations in order to monitor how the given assistance in 176 is taken up. The interviewer later observes the students and analyses their performance on the basis of their interaction. The students put forward that they could first draw the graph of $f(|x|)$ and then consider the absolute value sign at the outside of the expression of $|f(|x|)|$. Yet, they are not sure if they can do so or if this approach works. In 187, the interviewer gives the students positive feedback and explains why their approach is reasonable. He also assures them that he does not see any problem with their approach. After that he sets another subgoal to the students and asks them to draw the graph of $|f(|x|)|$ by considering what they have just discussed.

<u>Episode 4:</u>

188H: *What are we doing now?*
189S: *We will draw first the graph of f(|x|).*
190H: *Ok let's draw the graph now… [They draw the graph of f(|x|) (see figure 1D) by using the graph of f(x) (see figure 1C)].*
191I: *Alright, you drew the graph of f(|x|). But this is not what we expected to find, is it?*
192S: *No… we will now draw |f(|x|)|.*
193H: *Do you know how? Well, the next step is not too clear to me!*
194I: *Ok, just to make your job a bit easier, let's rename f(|x|) as g(x). So what you need to find turns into [he stops]…*
195S: *|g(x)|*
196H: *Aha! I can see it now…*
197I: *What is it*?
198H: *That means we will draw the absolute value of this graph… I mean we need to take the absolute value of this graph… oh it is so clear now, do you understand?*
199S: *Of course, but renaming the expression helped me see it clearly now…*
200I: *Ok, let's think about it now, how can we apply absolute value to this graph?*
201S: *|g(x)| never takes negative values… I mean it never passes under the x-axis.*
202H: *We will be taking the symmetry of the rays [she refers to the line segments (see figure 1D)] under the x-axis.*
203S: *Yes.*
204H: *Ok then, let's draw it now. We are now drawing the graph of |f(|x|)|.*

205S: *We were taking the symmetry of this part [the line segment on the fourth quadrant (see figure 1D)] in the x-axis… and we should also take the symmetry of that part as well [the line segment on the third quadrant (see figure 1D)]… according to x-axis.*

In this episode, the students satisfy the subgoal set by the interviewer, to draw the graph of $|f(|x|)|$, through two steps: (1) by drawing the graph of $f(|x|)$ and then (2) by drawing the absolute value graph of $f(|x|)$. Regarding the first step, the students recognise the structure of $f(|x|)$ that they constructed in the second task and use it to draw the graph of $f(|x|)$ (190). They have, however, some difficulties in seeing the second step (193). The interviewer realises this and assists them by renaming the expression of $f(|x|)$ as $g(x)$. He then invites the students to discuss how to apply the structure of $|f(x)|$, which they constructed in the first task, to the graph of $f(|x|)$ (200). In doing so the interviewer sets a sub-subgoal to the students, which is to draw the absolute value graph of $g(x)$. The interviewer seems to break down the subgoal, which was set at the end of the third episode, into further sub-subgoals. Satisfaction of these goals requires the students to reorganise their earlier constructions of $|f(x)|$ and $f(|x|)$ to draw the graph of $|f(|x|)|$. To do so, they recognise the structures of $|f(x)|$ and $f(|x|)$, appeal to the features of these two structures, and apply it to build the intended graph (e.g., 201, 202, 203, 204, and 205). In doing so, the students construct a method to obtain the graph of $|f(|x|)|$ from the graph of $f(x)$.

Episode 5:

Between the utterance of 206-227 (not shown) the students draw the graph of $|f(|x|)|$ for the third question by first drawing the graph of $f(|x|)=g(x)=|x|+3$ and then drawing the graph of $|g(x)|=|f(|x|)|$. They obtain the same graph of $|f(|x|)|$ as they obtained previously by substituting.

228I: *Right let's go on to the fourth question…what will you do in this question?*
229H: *We will draw the graphs with the same method again.*
230S: *Yes*
231H: *[They talk about a graph of f(x) given in the 4th question] Ok… now… first of all…*
232S: *The graph of f(x) at the positive values of x will remain the same*
233H: *First we obtain the graph of f(|x|)…*
234S: *Yes [they are drawing the graph of f(|x|)]*
235H: *Now we will draw its absolute value graph.*
236S: *That means we will take the symmetry in the x-axis.*
237H: *All of the parts over the x-axis remain as they are and…*
238S: *The parts under the x-axis will be cancelled and their symmetries will be taken in the x-axis [they drew the graph of |f(|x|)| successfully].*

In this episode, the students are expected to draw the graph of $|f(|x|)|$ for each of the given graphs of $f(x)$ in the fourth question. As can be seen from the students' interaction, they are able to regulate themselves and proceed without any help from the interviewer. The interviewer is still monitoring and analysing the students' performance but he does not feel the need to intervene. Thus in this episode the interviewer hands responsibility over to the students. After the students manage to reach to the new structure of $|f(|x|)|$, they become relatively self-regulated. Having drawn the graphs asked in the fourth question correctly, the students go on to the fifth question where they are asked to give a method to obtain the graph of $|f(|x|)|$. As can be seen from the excerpt below, the students have constructed the structure of $|f(|x|)|$.

243S: *First of all, by making use of the graph of f(x) we obtain the graph of f(|x|)=g(x) and then obtain |g(x)|…*

244H: *To do this, first when drawing f(|x|), part of f(x) at the positive [values of] x remains unchanged… umm then this part is taken symmetry in the y-axis and err and also part of f(x) at the negative [values of]* x *is cancelled. After that, we apply absolute value to this graph, and for this… umm… negative values of y are taken symmetry in the x-axis and thus we obtain the graph of |f(|x|)|.*

DISCUSSION

Construction is central to the RBC theory of abstraction in that without it abstraction cannot be claimed to take place. It requires students to combine and reorganise already constructed mathematical structures so as to create a new one. The protocols suggest that when the construction of a new structure is beyond the students' unassisted efforts, supportive and sensitive intervention of a scaffolder to direct the students' actions and attentions, and thus regulate their work and effort, is likely to induce them to make progress towards abstraction. The presented data reveal three major facets of scaffolding in the process of abstraction which are discussed below.

1. Based on the monitoring and analysing, the scaffolder assists the students through several means: The scaffolder's acts here appear to consist mainly in a continuous cycle around three elements: monitoring, analysing, and assisting. He constantly monitored the learners' performance when they took action in response to the given assistance and also when they were interacting with each other. This monitoring helped him analyse the learners' situation to determine the difference between their existing level and the intended level of performance. Based on the analysis he decided on the type of (and adjusted the amount of) assistance. It should be noted that there are many types of assistance in scaffolding such as explaining, questioning, feedback, and hinting (see Chi et al., 2001; Tharp and Gallimore, 1988). However, the scaffolder's selection of the type of assistance was completely subjective and dependent on his perception and interpretation of the situation. We thus posit that the type of assistance is not the central element to successful scaffolding as long as it is provided at the right time and results in progress on the part of the students towards the intended abstraction.

2. The scaffolder regulates the students by organising the main goal of the activity into subgoals and sub-subgoals: The scaffolder had the vision of the target goal of the activity and expected performance. This helped him to regulate the students towards the achievement of the main goal of the activity by setting new subgoals in such a way that attainment of each subgoal moved the students closer to the construction of a new structure. In order to get the students to attend to a subgoal, he even broke it down into sub-subgoals. Pre-determination of these subgoals and sub-subgoals is not possible. Quite the contrary, they emerge as the interaction amongst the participants evolves. Therefore the structure of a subgoal was negotiated in the interaction itself and required the scaffolder to make dynamical adjustment of it depending on his monitoring and analysing. In order to achieve these subgoals, the students at times needed to recognise a structure(s) constructed earlier (recognising); at other times to use these recognised structures to satisfy the subgoal(s) (building-with); and still other times to assemble and reorganise previously constructed knowledge artefacts to produce a new one (constructing). It seems that goals that students have, or are given, determine the nature of and initiate a series of epistemic actions that are required to attain the goal. For example, in the presented data, when the goal was to draw the graph of $f(|x|)$ the students needed to recognise and use the structure of $f(|x|)$ that they had constructed earlier. However, when the goal was to sketch the graph of $|f(|x|)|$, they needed to reorganise the structure of $|f(x)|$ and integrate it with the structure of $f(|x|)$ (see Episode 4).

3. The scaffolder steadily reduces the amount of assistance and gradually hands the responsibility over to the students as they make progress towards the main goal of the activity: As a result of scaffolder's monitoring and analysing, when he felt that the learners could proceed through the task without needing assistance from him, he gradually reduced the amount of help to the level of none and handed the responsibility over completely to the learners. This was the case after the students have constructed the structure of $|f(|x|)|$ (see Episode 5). The complete handover of the responsibility is thus likely to indicate that the students have become acquainted with a new structure and that the main goal of the activity, abstraction, is attained.

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