INFINITY AS A MULTI-FACETED CONCEPT IN HISTORY AND IN THE MATHEMATICS CLASSROOM

Ferdinando Arzarello(*), Maria G. Bartolini Bussi (**), Ornella Robutti (*)

(*) Dipartimento di Matematica, Università di Torino

(**) Dipartimento di Matematica, Università di Modena e Reggio Emilia

This paper presents the conceptualisation of infinity as a multi-faceted concept, discussing two examples. The first is from history and illustrates the work of Euler, when using infinity in an algebraic context. The second sketches an activity in a school context, namely students who approach the definite integral with symbolic-graphic calculators. Analysing the similarities between the examples, the authors widen the embodied cognition approach to infinity, based on the so called Basic Metaphor of Infinity of Lakoff and Nùñez. In fact, they consider also the manipulation of symbols, the use of virtual and real artefacts (in one case, the algebraic machine, in the other, the calculator) and their interpretation as instruments.

INFINITY

Infinity in the class is a very intriguing guest, which fascinates and challenges pupils and teachers. The existing research (see Boero *et al*., 2003 for some references) underlines the complexity of its conceptualisation, pointing out its multi-faceted sides. For example, it reveals sensible to textual and contextual aspects (Monagham, 2001), to classroom social interaction situations, to the cultural environment lived by pupils (Boero *et al*., ibid.). Also from the epistemological side the infinite reveals intriguing features: many mathematical concepts have been generated speculating on infinite processes and with big jumps between the current ideas in the culture of the time and the new ones (see Jahnke, 2001). Examples of this kind are: the discovery of irrationals in the Greek culture, the creation of the points at infinity in geometry (from XVII c.), the birth and development of infinitesimal calculus (from XVII c.) and of set theory (XIX c.). Roughly speaking, both the cognitive and the epistemological analysis show a persisting conflict between two main approaches to infinity, namely the *potential* and the *actual* one. According to the former, infinite is as an ongoing process that never terminates (e.g. a sequence of decimal approximation of π or $\sqrt{2}$). According to the latter, infinity is conceptualised as a given object (e.g., the number π or $\sqrt{2}$ *per se*).

The history shows moments where the relationships between the two approaches live together in the ideas of the time and moments where the conflict is more apparent within mathematics or between (some parts of) mathematics and other disciplines (e.g. philosophy). We find an example of the first type in the origin of infinitesimal calculus by Leibniz and Newton (since 1670's), while examples of the second type are given by the critique of Bishop Berkeley to infinitesimal methods in 1734 (mainly

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 4 pp 89–96

from the philosophical view) or by the rigour program by Weierstrass and his school in the 1870's (within mathematics).

THEORETICAL FRAMEWORK

Some research has pointed out more or less specific cognitive mechanisms, like analogies and metaphors, which seem to support the relationships between the two approaches and help the transition from one to the other. For important examples, see the pioneer book by Polya (1954, especially Ch. 2), the work by Fischbein (1987, especially Ch. 12) and the recent book by Lakoff and Núñez (2000, Ch. 8).

Lakoff and Núñez have introduced the so-called Basic Metaphor of Infinity (BMI), which arises when one conceptualises actual infinity as the result of an iterative process (Lakoff & Nùñez, 2000, p. 159). The two domains (source and target) of the metaphor are characterised by an ordinary iterative process with an indefinite number of iterations, each of which has an initial state and a resultant state. The crucial effect of the metaphor is to add to the target domain the completion of the process and its resultant state as a unique final state. This metaphor allows to conceptualise infinity in terms of the unique and final result of a process (Lakoff and Nùñez, 2000, p.160):

> Via the BMI, infinity is converted from an open-ended process to a specific, unique entity.

Lakoff and Nùñez point out some important general features in the conceptualisation of the infinity. But their analysis should be refined further, for example adding: "the very idiosyncratic nature of students' individual conceptions" (Sinclair & Schiralli, 2003); the interactions between students and artefacts in a mathematical activity and the cultural environment in which the activity takes place (Rabardel, 1995); the subject's activity, which may be "*learning*, *doing* or *using* mathematics" (Sinclair & Schiralli, 2003). Moreover, not only conceptualisation changes, but also the very nature of the mathematical concept of infinity varies substantially with respect to the context. The infinity of integers, that of the continuum, the infinite limit of a real function, the point at infinity in projective geometry have been generated by very different processes, whose nature can be lost, if one considers only the BMI. Infinity possibly is not a single concept, but a network of concepts: the word itself collects a lot of meanings, each with a different story, and the use of the singular word has a risk, namely to hide its pluralities of meanings.

Our aim is to present infinity as a multi-faceted concept, describing two examples: one from history, namely Euler's work about infinity, the other from a teaching experiment at school, aimed at constructing the concept of definite integral, through approximate area measurements. In the end a comparison between the two examples is sketched, discussing the possible integration of the embodied approach within other theoretical elements, namely the analysis of symbols and artefacts: in fact, both can usefully support the conceptualisation of infinity.

AN EXAMPLE FROM THE HISTORY

This example comes from Euler fascinating book, *Introductio in Analysin Infinitorum* (Euler, 1748), whose title (*Introduction to Analysis of Infinities*) underlines that there are many infinities; in fact, Euler analyses three possible situation in which infinite occurs: infinite series, infinite products and continued fractions. The algebraic context where Euler develops his computations features infinite numbers in a very specific way. Their ultimate meaning is acquired through two levels of performing abstract calculation:

(i) Letters for variables ("which can take any value", *ibid*., #2) are introduced at the level of the mathematical language to represent all possible numbers, infinite ones included; they can be manipulated according to the usual machinery of algebra. Through letters the basic concept of function is introduced, as "an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities" (*ibid*., #5): namely a function is given through its form, which can vary through suitable transformations (Ch. II).

(ii) The very concept of infinity is introduced at the meta-level to manipulate the forms of the functions through the algebraic laws; so one can add or multiply infinite terms. For example (#156), one can add the infinitely small quantity x/j (x finite, j infinite) j/2 times and get the finite term x/2. Or one can express the rational function $a/(\alpha+\beta z)$ as the infinite series that results "by a continuing division procedure" (*ibid*., #60), which gives the value of the function. "Even the nature of transcendental functions seems to be better understood when it is expressed in this form [an infinite series of powers], even though it is an infinite expression" (*ibid*., #59).

It is the interplay between the two levels (i) and (ii) which allows Euler to develop his analysis of infinity, as a study of the different forms of functions. For example, in Chapter VII to express Exponentials and Logarithms through series he writes:

> 114. Since $a^0 = 1$,… it follows that if the exponent is infinitely small and positive, then the power also exceeds 1 by an infinitely small number. Let ω be an infinitely small number, or a fraction so small that, although not equal to zero, still $a^\omega = 1 + \psi$, where ψ is also an infinitely small number. From the preceding chapter we know that that unless ψ were infinitely small, then neither ω would be infinitely small. It follows that $\psi = \omega$, or $\psi > \omega$, or $\psi < \omega$. …so let $\psi = k\omega$. Then we have $a^\omega = 1 + k\omega$, and …we have $\omega = \log(1+k\omega)$.
>
> 115. …we have $a^{j\omega} = (1+k\omega)^j$, whatever value we assign to j. It follows that $a^{j\omega} = 1 + j/1\ k\omega + j(j-1)/1\cdot2\ k^2\omega^2 + j(j-1)(j-2)/1\cdot2\cdot3\ k^3\omega^3 + \ldots$. If now we let $j = z/\omega$, where z denote any finite number, since ω is infinitely small, then j is infinitely large.…When we substitute z/j for then $a^z = 1 + 1/1\ kz + 1(j-1)/1\cdot2j\ k^2z^2 + 1(j-1)(j-2)/1\cdot2j\cdot3j\ k^3z^3 + \ldots$. This equation is true provided an infinitely large number is substituted for j, but then k is a finite number depending on a, as we have just seen.

And later to sum infinite series he writes in Chapter X:

> 165. If $1 + Az + Bz^2 + Cz^3 + Dz^4 + \ldots = (1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)\ldots$, then these factors, whether they be finite or infinite in number, must produce the expression, when they are actually multiplied. It follows that the coefficient A is equal to the sum $\alpha + \beta + \gamma + \delta + \varepsilon + \ldots$ The coefficient B is equal to the sum of the products taken two at a time. Hence $B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \ldots$ All of this is clear from ordinary algebra.

As it is well known, Euler uses this type of arguments to prove that: $\pi^2/6=1+1/4+1/9+1/16+\ldots$(*ibid.,* #167). The excerpts clearly show that the possibility of managing the new entities within a suitable symbolic register allows Euler to acquire new mathematical results in the field of transcendental functions and new operative insights in the concept of infinity. The symbolic register is the machinery of algebra, used to make infinite sums and products (the infinite is at the meta-level previously described). Euler is particularly attentive to make only finite local computations: it is at the level of a global insight that he uses infinite to make general considerations, which allow him to introduce the new result within the old frame.

This analysis goes a step beyond the usual comments (Polya, 1954; Steiner, 1975; Fischbein, 1987), which underline the analogy that Euler puts forward between the finite and the infinite, extending an algebraic rule "from equations of finite degrees to an equation of an infinite degree" (Fischbein, p. 132). However, the extension of the law is built up controlling its meaningfulness with respect to the algebraic manipulations of the formulas, and not because of abstract 'transfer' principles. Lakoff & Nùñez (2000) underline that "Infinite Sums Are Limits of Infinite Sequences of Partial Sums" (p. 197). This aspect of approximation is present in Euler more times, but he is seeking for understanding the infinite by the algebraic exactness (Euler, Preface):

> Although Analysis does not require an exhaustive knowledge of algebra, even of all the algebraic techniques so far discovered, still there are topics whose consideration prepares a student for deeper understanding" and can allow people avoiding the "strange ideas [that they entertain] about the concept of infinity.

AN EXAMPLE FROM THE CLASSROOM

We describe now an example from the classroom, aimed at showing the conceptualisation of definite integral; it has been carried out through various activities based on approximate measures of areas under curves in the Cartesian plane, using before paper and pencil, then a technological artefact, namely the calculator TI89 (Robutti, 2003). The students, at the 12^{th} grade of a scientific-oriented Italian school, are used to work in small groups, then to share the results in a class discussion led by the teacher.

In the example described here, the students are working to evaluate the area under the graph of a given function. The task consists in the determination of the work made by a perfect gas during an isothermal transformation, represented by a hyperbola on the Cartesian plane (p,V) [1]. From the discussion about different procedures (obtained by the students in the groups) to determine the work (the area under the hyperbola), the need of an algorithmic formula arose. A formula has an advantage respect to other non-algorithmic methods: it can be implemented in a program on the calculator. The teacher guides the various students' interventions to converge on the method of rectangles under and over the function to approximate the area. After that, the students use the program (Figure 1) based on this calculation, in a group activity, to evaluate the area under the graph with different numbers of rectangles [2].

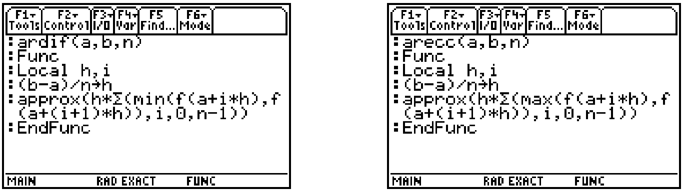

Figure 1

The discussion following this activity was aimed to reflect on the degree of approximation with respect to the number of rectangles.

Teacher: "Which was the best we said?"

Andrea: "The last!"

Teacher: "Why?"

Andrea: "Because it has more intervals and then ... "

Stella: "Because it gets nearer to the area"

Teacher: "But why is it so precise, if there are more intervals?"

Andrea: "Because … with more intervals … it is possible to give a better approximation of the curve with a line going to a more … microscopic, and then … nearer"

The last phrase is interesting, because it reveals a passage from the global to the local properties of functions, as if Andrea could notice the local properties of a graph, after having observed the global ones, thanks to the sub-division of the interval on the x-axis. The student has the intuition that the more the intervals, the better is the approximation of a curve with segments, which are closer to the curve. This intuition marks a first step in the conceptualisation of definite integral. The word "microscopic" reminds to the local approximation of curves with lines, that is the theoretical base of Calculus. The discussion continues with the next excerpts:

Teacher: "The last is more precise: what does it mean saying more precise?"

Andrea: "That it gets nearer to the average value"

Students: "That it gets nearer to the real value". "That it gets closer to the real value"

The students come to a second step in the conceptualisation process: the idea that the last result of the program, which approximates the area, is more precise than all the previous results, because “it gets nearer to the real value”. This step is characterised by the consciousness that there exists a “real value” for the area, even if they do not have it, at the moment, because they have seen a succession of values approximating the area, but not ‘the end of the story’.

In the collective discussion after the group activity, the students are guided by the teacher toward connecting the approximate evaluation of the area with a theoretical content, which was developed in the previous year (the concept of real number as a pair of contiguous classes).

Teacher: “What do we remember thinking back to this situation?”

Stella: “The square root of 2”

Teacher: “The square root of 2. That is, when did we construct what?”

Francesco: “The contiguous classes”

In the process of evaluating the area of the rectangles, the students recognise the construction of a real number, namely $\sqrt{2}$, and this is the third step in the conceptualisation. But it is not sufficient, because, if they understand the analogy between the approximate measure of the area and a real number, they are unable to bridge the gap between the approximation process of evaluation and the exact value of the area, namely between finite and infinite.

The students need to extend the possibilities of the real calculator in order to reach infinity, because at a certain moment Francesco says, substituting *n* with the symbol ∞ in the program of rectangles on the real calculator:

Francesco: “I put infinite instead of a number *n*, and the calculator answers *undef*” [3]).

And when the response of the calculator is *undef*, because it is unable to produce this number, Francesco shares his surprise with his mates.

In order to help students to pass to infinity, the teacher introduces an ideal calculator:

Teacher: “Now I am in an ideal calculator, which doesn’t exist of course, and I imagine doing the calculation”

Francesco: “At the end we will have a root”

Teacher: “A root?”

Francesco: “No, a number … What is the name of those numbers?”

Teacher: “Real”

Through the ideal calculator, conceived as an instrument (Rabardel, 1995) that does the same calculation as those done by the real calculator, but without limitations, neither in quantities, nor in the number of operations, it is possible to bridge the gap between finite and infinite. This is the fourth step in the conceptualisation: to recognise the analogy between the exact measure of the area and the concept of real number.

DISCUSSION

The conceptualisation process described above reveals the Basic Metaphor of Infinity, in the particular case: Infinite Sums Are Limits of Infinite Sequences of Partial Sums (Lakoff & Nùñez, 2000; p. 197). But the BMI is not sufficient. We have shown that in both Euler's and students other ingredients are essential to understand the process. The core of Euler's process is the use of infinite sums and products, in which the coefficients must be equal, in that they represent different forms of the same function. More specifically, Euler did not use an identity principle in infinite formulas, but the necessity that the calculations produce the same results, showing in that a semiotic need, more than a structural one. And in doing this, he found his famous outcome about the powers of π as infinite sums of the inverses of powers of naturals (Euler, 1748, #168). The students, in the process of approximation, can use different values in the number of rectangles. In this way, they use the letter n in the program as a symbol for a variable (see in Figure 1, where *arecc* and *ardif* are programs depending on the endpoints of the interval, a and b, and on the number n of subdivision of the interval). They use different numbers, bigger and bigger, to reach a better approximation for the area, till Francesco tries to substitute ∞. And his attempt to insert ∞ in the calculation process corresponds to a need of algebraic exactness, as in Euler's process. The second element is the *ideal calculator*, which leads the students towards the measure of the area, thought of as a real number. The students already know $\sqrt{2}$ as a real number, and they are constructing a calculation process with the areas of rectangles, to approximate the area under a graph. The link between the two objects (the exact area and the real number), which represent the same concept, is 'embodied' by the ideal calculator, introduced by the teacher. A kind of ideal calculator exists also in Euler's process, when he uses infinite algebraic computations as if they were finite. Euler's ideal calculator is constituted by the algebraic computations extended to infinity, together with the use of infinity at the meta-level, as pointed out by him several times. Both the protagonists of the two examples (Euler and the classroom) had at disposal artefacts: in the former case, it was the set of algorithms of algebraic computation; in the latter it was the TI89. During the process the artefacts went through the process of instrumentation (Rabardel 1995) that transformed them into instruments. The role of the artefacts is essential: they work at the meta-level, and help the subjects to "manipulate" or "conceive" infinity as well. The cases we have presented in this paper have been chosen in a list of examples that concern the parallel analysis of historical processes and of didactical processes concerning infinity. The artefacts that lead the manipulation and the conception of infinity may be different when different meanings of infinity are into play. Hence a local analysis is needed that adds the needed elements to the Basic Metaphor of Infinity (for examples see the case of perspectograph in Bartolini *et al.*, in press; the case of abacus in Bartolini & Boni, 2003).

Endnotes

[+] Research funded by the MIUR and the Università di Torino and the Università di Modena e Reggio Emilia (COFIN03 n. 2003011072).

[1] p and V mean pressure and volume of a gas respectively.

[2] *ardif* indicates the area of the rectangles under the graph (defect approximation), while *arecc* refers to the rectangles over the graph (excess approximation).

[3] The response *undef* means *undefined*, in the sense that the calculator has no possibility to produce an answer.

References

Bartolini Bussi M. G. & Boni M. (2003). Instruments for semiotic mediation in primary school classrooms, *For the Learning of Mathematics*, 23 (2), 12-19.

Bartolini Bussi M. G., Mariotti M. A. & Ferri F. (to appear). Semiotic mediation in primary school: Durer's glass in Hoffmann H., Lenhard J. & Seeger F. (eds.) *Activity and Sign – Grounding Mathematics Education,* Kluwer Academic Publishers.

Boero, P., Douek, N. & Garuti, R. (2003). Children's Conceptions Of Infinity Of Numbers In A Fifth Grade Classroom Discussion Context, *Proceedings of the 27th PME Conference*, Honolulu, Hawaii, vol. 2, 121-128.

Euler, L. (1748). *Introductio in Analysin Infinitorum*, Lausanne. (English translation by J.D. Blanton: *Introduction to Analysis of the Infinite*, Springer, Berlin, 1988).

Fischbein, E. (1987). *Intuition in Science and Mathematics*, Reidel, Dordrecht.

Jahnke, H. N. (2001). 'Cantor's cardinal and ordinal infinity', *Educational Studies in Mathematics*, *48* (2-3) 175-197.

Lakoff, G. & Núñez, R. (2000). *Where Mathematics comes from*, Basic Books, New York.

Monaghan, J. (2001). Young people's ideas of infinity, *Educational Studies in Mathematics*, *48* (2-3) 239-257.

Polya, G. (1954). *Induction and analogy in mathematics*, Princeton University Press, Princeton, NJ.

Rabardel, P. (1995). *Les hommes & les technologies. Approche cognitive des instruments contemporains*. Arman Colin Éditeur, Paris.

Robutti, O. (2003). Real and virtual calculator: from measurement to definite integral. *CERME 3*, http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft/.

Sinclair, N. & Schiralli, M. (2003). A constructive response to 'Where mathematics comes from', *Educational Studies in Mathematics*, *52*(1), 79-91.

Steiner, M. (1975). *Mathematical Knowledge*, Cornell University Press, London.