

# VERIFICATION OR PROOF: JUSTIFICATION OF PYTHAGORAS' THEOREM IN CHINESE MATHEMATICS CLASSROOMS

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*This paper presents key findings of my research on the approaches to justification by investigating how a sample of teachers in Hong Kong and Shanghai taught the topic Pythagoras theorem. In this study, 8 Hong Kong videos taken from TIMSS 1999 Video Study and 11 Shanghai videos videotaped by the researcher comprised the database. It was found that the teachers in Hong Kong and Shanghai emphasized justification of the theorem; one striking difference is the former stressed visual verification, while the latter paid close attention to mathematical proof.*

## INTRODUCTION

The superior performance of students from Confucian Heritage Culture (i.e., CHC)<sup>1</sup> communities in international comparative studies of mathematics achievement (Mullis *et al.*, 2000, 2004) has led researchers to explore what factors possibly account for the superiority of students in CHC (Fan, *et al.*, 2004; Ho, 1986; Leung, 2001). Recently, these studies focused on classroom instruction (Stigler & Hiebert, 1999; Hiebert *et al.*, 2003). Since mathematical reasoning is a key element in school mathematics (National Council of Teachers of Mathematics, 2000), and it was found that Japan and Hong Kong students worked on problems involved more proof than the students in other countries (Hiebert, J. *et al.*, 2003), a contribution to understanding mathematics teaching in CHC, is to describe how proof is dealt with in classrooms.

Although the notions on proof and the ways of introducing proof are diverse, a great body of literatures on the natures of proof and the functions of proof suggests the following functions of proof and proving (Hanna, 1990, 2001; Hersh, 1993): (1) to verify that a statement is true; (2) to explain why a statement is true; (3) to communicate mathematical knowledge; (4) to discover or create new mathematics, or (5) to systematize statements into an axiomatic system.

The above statements seem to suggest that the justification should include convincing, explaining and understanding. Justification refers to providing reasons why the theorem is true. The validity of a proof does not depend on a formal presentation within a more or less axiomatic-deductive setting, not on the written form, but on the logical coherence of conceptual relationships, that serve not only to

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<sup>1</sup> Confucian Heritage Culture communities refer to Chinese Taiwan, Korea, Japan, and Singapore (Biggs & Watkins, 1996).

convince others that the theorem is true, but also to explain why it is true (Cooney *et al.*, 1996; Hanna, 1990, 2001). “Proof” in this study is grouped into two categories. One is called *mathematical proof*, in which the theorem is proved deductively and logically by using geometrical properties and theories or the operations of algebraic expressions. The other is called “*verification*”, in which the theorem is shown to be true by using certain evidences such as solving a puzzle or demonstrating some cases.

In the research, the focus will be on the practice of teaching of proof in Chinese classrooms (Hong Kong and Shanghai) when Pythagoras theorem was taught. It attempts to address the following two questions: (1) How the theorem was justified in the lessons? (2) How were those justifications carried out?

## METHOD

In this study, the Chinese cities of Hong Kong and Shanghai were selected as two cases to be investigated. Within each setting, several lessons were selected, as discussed below. The Hong Kong videos from the TIMSS 1999 Video Study on the teaching of Pythagoras’ theorem were chosen. Thus, the Hong Kong data for the present study consists of eight CDs of the sampled lessons and relevant documents such as questionnaires for understanding teachers’ background, sample of students’ work in the lessons etc. Correspondingly, adopting the procedure of videotaping designed by the TIMSS 1999 Video Study, eleven Shanghai lessons which come from compatible Shanghai schools were videotaped. As in Hong Kong, supplementary documents were collected.

The CDs of the Hong Kong lessons and relevant multimedia database with English transcripts from LessonLab, Inc. of Los Angeles were already available to the researchers. For Shanghai, all videos of the lessons were digitized into CDs. The teachers who delivered the lessons respectively transcribed the CDs verbatim in Chinese and the data analysis mainly depended on the Chinese transcripts and the original CDs. During the process of data analysis, the CDs were referred to from time to time, to ensure that the description represented the reality as closely as possible.

## RESULT

### Approaches to justification

The approaches to justification can be summarized in Table 1. The most prominent difference between these two cities regarding justification is that the Shanghai teachers paid considerable attention to the introduction of **mathematical proofs**. On the contrary, the Hong Kong teachers seemed to hold different attitudes toward justification. Six out of the eight teachers tended to **verify the theorem** either through exploring activities for discovering the theorem or certain other activities for verifying the theorem once it was found.

Types of Justification	Hong Kong* (8)	Shanghai (11)
Visual verification	6	0
Single mathematical proof	2	3
Multiple mathematical proofs	1	8

\*The sum of the ways of justification is not equal to the total of the teachers since there are some teachers who gave different kinds of justifications.

Table 1: Distribution of the ways of justification

### Verification

It was found that six out of the eight Hong Kong teachers verified the theorem visually. The typical way of verification is which serves as both discovering and verifying.

Given a bag containing five pieces of puzzles and a diagram as shown in Figure 1(1), students are asked to (1) fit the 5 pieces into square C in the diagram in the shortest time; (2) rearrange the above 5 pieces into two smaller squares A and B in the diagram

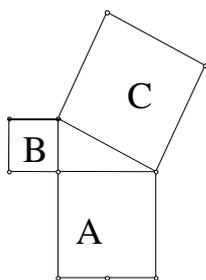


Figure 1(1)

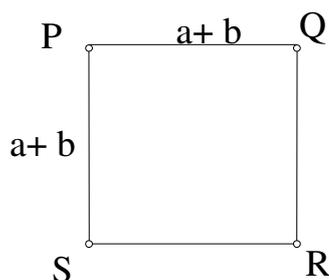


Figure 1(2)

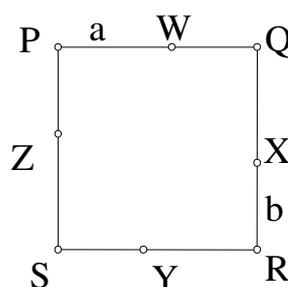


Figure 1(3)

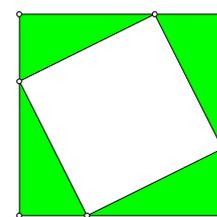


Figure 1(4)

It is expected that the students by solving the puzzle in different ways would discern the invariant area relationship – the area of the square on the hypotenuse is equal to the sum of the areas of the two smaller squares on the two adjacent sides. Moreover, by associating with the area formula of the square, the relationship among the areas of the three squares was transformed into the relationship among the lengths of the three sides of the right-angled triangle, which is Pythagoras' Theorem.

### Mathematical proof

All the Shanghai teachers introduced mathematical proof. In particular, three quarter of teachers introduced more than two mathematical proofs. However, all these different proofs are based on a strategy, i.e., calculating the area of the same figure by using different methods. Although only a half of the Hong Kong teachers introduced mathematical proof, there are different strategies to prove. The following is one example.

Consider a square  $PQRS$  (Figure 1(2)) with side  $a + b$  and prove Pythagoras' Theorem by finding the area of  $PQRS$  with two different methods.

Method 1: Find the area of  $PQRS$  directly.

Method 2: Hint: Join  $WX$ ,  $XY$ ,  $YZ$ , and  $ZW$  (Figure 1(3)).  $PQRS$  is divided into four \_\_\_\_\_ and one \_\_\_\_\_  $WXYZ$ .

Let the hypotenuse of the right-angled triangle  $WPZ$  be  $c$ , then Area of  $PQRS$  = \_\_\_\_\_.

In the Shanghai lessons, the students were asked to fit the squares together by using four congruent right-angled triangles. Then, they were asked to calculate the areas of the squares in different ways. Based on these kinds of activity, several proof were found. The following is one typical method (Figure 1(4)):

$$S_{\text{large square}} = S_{\text{small squares}} + 4S_{\text{triangles}}$$

$$(a + b)^2 = c^2 + 4 \times \frac{1}{2} ab$$

Simplifying:  $a^2 + b^2 = c^2$

From a mathematical point of view, the above two proofs are essentially the same. However, it may make differences in students' learning when the proof is introduced differently. The following excerpts demonstrate the kind of classroom interactions when one method of calculating the area of the square was introduced.

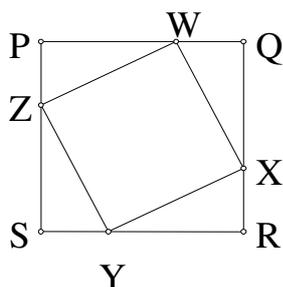


Figure 2(1)

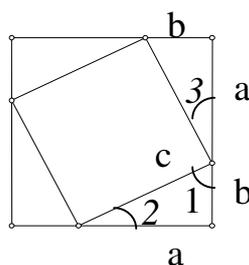


Figure 2(2)

In Hong Kong, the first method (namely, calculating the area of  $PQRS$  directly) was discussed. After the students have found that the square  $PQRS$  was divided into four congruent right-angled triangles (see figure 2(1)), the discussion then moved on to the following episode:

- 1 T: Congruent. Right. So you may say,  $PQRS$ , now this time, after you joined the four sides, you will have to divide it into four congruent right-angled triangles.
- 2 T: So  $PQRS$  is divided into four congruent right-angled triangles and also, another figure,  $WXYZ$ , that's a?
- 3 Ss: Square.
- 4 T: Square. So now, this is the second method. The second method in finding about the area of this square with the side  $a$  plus  $b$ , so now, this time, how to find about its area?

- 5 T: There are four congruent triangles. [They are] Right-angled triangles with the side, base is  $a$ , height is?
- 6 Ss:  $b$ .
- 7 T:  $b$ . You know the area  $ab$  over two for each triangle. And then four of them. Plus?
- 8 Ss:  $c$  square.
- 9 T:  $c$  square, right. The smaller square that is inside. So you may find, they're two  $ab$  plus  $c$  square. Two  $ab$  plus  $c$  square. How about the last step? By considering the two different methods in finding the area of the same square, what do you find?
- 10 T: Method one, you find  $a$  square plus two  $ab$  plus  $b$  square. How about the second one? Method two you find two  $ab$  plus  $c$  square. So what do you find?
- 11 T: Oh, both sides they're two  $ab$ , so you cancel this, and finally you find  $a$  squared plus  $b$  squared is  $c$  squared. Is it?

The Hong Kong episode shows that the teacher led students to achieve the proof expected by her through asking a sequence of simple and close questionings (2, 5, and 7). Even though, sometimes, the teacher asked an open-ended question (4), finally, the question was rephrased into two closed questions (5, 7). Meanwhile, the teacher often answered her own questions (9~11). Thus, the teachers tended to *tell* her students the proof.

In Shanghai lesson, after students created a figure by using four given congruent right-angled triangles, the teacher presented the diagram on a small blackboard (see figure 2(2)). The students were then asked to calculate the area of the diagrams using different methods, as shown below:

- 1 T: This is what he put. He cut a big square outside and a small square inside. Then put the big one on the small one. Ensure that the vertexes of the small square are on the four sides of the big one. Would you please prove it? (Nominating a student)
- 2 S: Sorry, I can't
- 3 T: You have put it (together), but you can't prove it? Who can? *You, please.*
- 4 S: (Coming to the front and proving on the blackboard)
- $$S_{\text{large square}} = (a + b)^2 .$$
- 5 T: How do you know that this side is  $a$ ?
- 6 S: Because the two triangles are congruent.
- 7 T: How to prove they are congruent? Don't be nervous.
- 8 S: I've forgotten.
- 9 T: He's forgotten. Please give him a hand. Go back to your seat, please. Tao Li.
- 10 S: I've forgotten.

- 11 T: You've all forgotten. How to prove? Liu Wenju, would you please prove it? Be brave.
- 12 S: There's one that is equal.  $\angle 1 + \angle 2 = 90^\circ$ ,  $\angle 1 + \angle 3 = 90^\circ$ ,  $\angle 2 = \angle 3$ .
- 13 T: What's the reason?
- 14 S: (Together) the complementary angles are equal.
- 15 S: We can get congruence according to ASA.
- 16 T: With the same reason, we can get the four triangles equal. Now continue, please.
- 17 S:  $S_{\text{large square}} = (a+b)^2$ .

In the Shanghai lessons, the teacher encouraged her students to focus on critical aspects, express their understanding, and finally find their solutions through asking a sequence of open-ended questions (1, 11, and 13). Moreover, when students faced difficulties in answering teacher's questions, the teacher asked other students until eliciting desirable answers (3, 7, and 11). Thus, in the Shanghai lesson, the teachers tried to encourage and foster students to *construct their proof* instead of telling them the proof as the Hong Kong teacher did.

There are two critical aspects in this proof. One aspect is to dissect the square into one smaller square and four congruent right-angled triangles. Another aspect is to calculate the area of the square by two different methods and simplify the expression of an area relationship into a side relationship. Regarding the first aspect, in the Hong Kong lesson, the students got used to following the teacher's instructions and the worksheet seemed to limit students' thinking. Furthermore, the teacher did not provide justification as to why the central figure was a square. However, in the Shanghai lesson, the students were not only asked to present the diagram, but also to justify the statement on the diagram. In this sense, there is an obvious difference in experiencing the diagram: **taking for granted and justifying the diagram logically**. Regarding the second aspect, basically the Hong Kong teacher stated the relevant formulae and simplified the expression on his own. Thus, the students were seldom given a chance to express their own understanding and thinking. On the contrary, the Shanghai teacher always let the students express the formula and the relevant transformation verbally. In this regard, the Shanghai lesson seemed to have created more **space for students to verbalize the process of deductive reasoning**.

## DISCUSSION AND CONCLUSION

The findings of this study show both similarities and differences in terms of the approach to the justification of Pythagoras theorem between Hong Kong and Shanghai. Although the teachers in both places emphasize on justification of the theorem by various activities, the following differences are noticeable: the Hong Kong teachers are visual verification-orientated, while the Shanghai teachers are mathematical proof-orientated. Moreover, compared with the Hong Kong teachers,

the Shanghai teachers made more efforts to encourage students to speak and construct the proof.

What are the possible explanations of the finding that the Hong Kong teachers emphasized visual verification, while the Shanghai teachers emphasized mathematical proof? Firstly, it was repeatedly found that Chinese and Japanese students tended to use more abstract and symbolic representation while the U.S. students tended to use more concrete and visual representation (Cai, 2001). However, within Chinese culture, the Hong Kong teachers valued visual presentation more than the Shanghai teachers did which might be partly due to the British cultural impact on Hong Kong for more than one century. Secondly, even in the current mathematics curriculum in Shanghai, the emphasis is put on abstract presentation and logical deductive thinking. It is an expected phenomenon that when teaching geometry, particularly when teaching Pythagoras' Theorem, geometric proof is stressed since it is believed that geometric proof is a good way "to master the mathematical method and to develop the ability to think" (Mannana & Villani, 1998, p.256). In the official textbook, two different proofs of Pythagoras' Theorem are introduced. The textbooks used Hong Kong, some visual exploring activities are presented and one proof is introduced as well. However, the teachers in Hong Kong seemed to pay more attention to verifying the theorem through exploring activities, rather than proving the theorem.

It seems that there is a consensus that deductive reasoning (or in the classroom jargon 'proving') still has a central role in geometry learning. However, the classical approach is now enriched by new facets and roles such as verification, convincing and explanation (Mannana & Villani, 1998, p.31). The teacher's classroom challenge is to exploit the excitement and enjoyment of exploration to motivate students to supply a proof, or at least to make an effort to follow a proof supplied (Hanna, 2000, p.14).

As demonstrated in this study, there are multiple approaches to justification: visual verification, one mathematical proof, multiple mathematical proofs etc.. If regarding the ways of justification as continuity spectrum in terms of the rigor: one pole is the visual reasoning and the opposite is the deductive reasoning, then this study shows that the approaches in Hong Kong seem to be quite near to the visual reasoning, while those in Shanghai seems to be near to the deductive reasoning side. However, it seems to suggest that the teachers in Hong Kong and Shanghai are making effort to strike a balance between visual reasoning and deductive reasoning which seems to be a direction to pursue. From the classroom practice in those Chinese mathematics lessons at eight-grade, the students are able to explore the theorem and prove it in certain ways if the teachers organize the teaching appropriately. Although we have no intention to apply these findings to any other cultures, we try to argue that effective teaching of justifications in Chinese classrooms is possible which might contribute to high mathematical performance of Chinese students.

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