MATHEMATICALLY GIFTED STUDENTS' GEOMETRICAL REASONING AND INFORMAL PROOF

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This paper provides an analysis of a teaching experiment designed to foster students' geometrical reasoning and verification in small group. The purposes of the teaching experiment in this paper were to characterize gifted students' proof constructions and to contribute to the theoretical body of knowledge about gifted students' mathematical thinking. The experiment was conducted as part of education for gifted sixth-graders (12 years of age). The analysis of the students' responses in this paper documents the evolution of the students' proving ability as they participated in activities from an instructional sequence designed to support geometrical reasoning. Three types of reasoning (pragmatic, semantic, intellectual) and creative informal proofs were identified in the analysis. In order for mathematically gifted students to develop their proving ability, teachers need to draw explicit attention to the value of informal proofs. Likewise, in order for students to develop their sense of geometrical reasoning, they need a lot of experience in conjecturing, testing, and then verifying in a mathematical way.

INTRODUCTION

There is a widespread agreement that students have difficulties with constructing proofs (Senk, S. L., 1982; Chazan, D. & Lehrer, R. (Eds.), 1998; English, L. D. (Ed.), 1997;Weber, K. 2001). A great deal of educational research investigating students' proving abilities were conducted. Much of the research on proof has examined both the valid and invalid proofs. More recently, some researchers have paid less attention to the proofs that students produce and have focused instead on the processes that students use to create those proofs (Graves, B., & Zack, V., 1997;Artzt, A., & Yoloz-Femia, S., 1999;Weber, K., 2003). The aim of this study is also to describe and investigate the proving processes that students produce.

The traditional view of proof has been and still is, largely determined by a kind of philosophical rationalism, namely, that the formalist view that mathematics in general (and proof in particular) is absolutely precise, rigorous and certain. Although this rationalistic view has been strongly challenged in recent years by the fallibilist views of, for example, Lakatos (1976), Davis and Hersh (1986), and Ernest (1991), it is probably still held by the vast majority of mathematics teachers and mathematicians. In an extreme version of this view, the function (or purpose) of proof is seen as only that of the verification (conviction or justification) of the correctness of the mathematical statements (Chazan, D. & Lehrer, R. (eds), 1998).

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This paper is based on the fallibilist perspective and Brousseau's (1997) theory of didactic situation. According to Lakatos (1976), mathematics is not a purely deductive science proceeding from accepted axioms to established truths. And he believes that the uncertainty in proofs is the basis for mathematical activity, not an impediment to it, because uncertainty makes possible the process of proof analysis. On the other hand, Brousseau (1997) documented that an attitude of proof exists and is developed by particular didactic situations. According to Brousseau, proof must be formulated and presented while being considered, and therefore most often written, and must be able to be compared with other written proofs also dealing with the same situation. The didactic situation in this study was designed to motivate students to discuss and favor the formulation of their implicit validations and informal proofs, even if the students' reasoning is incorrect or imperfect.

METHODOLOGY

Participants in the study were two groups of mathematically gifted sixth-graders of 16 students per group, studied geometry for three hours each day emphasizing mathematical argumentation and validation. The students' individual written proof constructions were collected. Classroom instruction and task-based, semi-structured individual interviews with 3 students were videotaped. Students were asked to explain their reasoning and challenge other students' explanations and validations in the instruction. The instructor asked to find the answers or representations, even though it consists of one word or visual images only. The necessity of proof was proposed by students who met uncertainty about the truth of mathematical propositions. In this process, students' pragmatic, semantic, intellectual reasoning, which Brousseau (1997) has distinguished as such, and informal proofs were identified and discussed.

Analysis was conducted by the theory of didactic situations as described by Brousseau (1997) and the interpretive framework developed by Strauss, A., & Corbin, J. (1990). Transcriptions of the classes and interviews, written proof constructions were summarized per student per task. Analyses of these summaries were discussed with the instructor in order to minimize inappropriate interpretations. The analyses led to the identification of many creative mathematical ideas, reasoning and informal proofs.

The task below provides an example of the question sets presented to the students. Observing, conjecturing, testing, generalizing and validating occurred while confronting the problematic issues in these questions.

1. Let's observe a football. You need to imagine a football as a polyhedron made of regular pentagons and regular hexagons. Then how many regular pentagons and regular hexagons are used?
2. How many vertices are there? Explain how you found it.
3. How many edges are there? Explain how you found it.

4. Looking at one vertex, what kinds of polygons are there? Is it all the same for every vertex? What is the sum of the interior angles collected at one vertex?
5. How many spherical solids can be made if we use regular triangles?
6. How many spherical solids can be made if we use squares?
7. How many spherical solids can be made if we use regular pentagons?
8. How many spherical solids can be made if we use regular hexagons?
9. How many spherical solids can be made if we use two kinds of regular polygons such as a football?

There are many possibilities for making spherical solids, so it is necessary to establish criteria for constructional method and their justification or verification. They may discuss what kind of regular polygons can and cannot be used simultaneously for making a spherical polyhedron. Of course they can investigate and announce the reasons, and predict another possible form for a football, discuss their strengths and weaknesses. The intent of the instructional sequence is to support students' development of sophisticated ways to reason geometrically about polyhedron in space and represent the consequences mathematically. The instructor encouraged students to observe regularity, pattern, or law and yield worthwhile results by insight. Any particular case or consequence was actively examined and verified, so that students acquired the credit of the conjecture they produced.

RESULTS AND DISCUSSION

The responses were marked and coded in terms of the number of reasoning types (intellectual, semantic, pragmatic) and in terms of the process of reasoning, whether it was attempted, incomplete and invalid. The transcriptions were categorized in terms of the central issue being considered.

Questions	Pragmatic	Semantic	Intellectual
How many regular pentagons and regular hexagons are used?	20/32 (62.5%) counted directly	3/32 (9.4%) examined other students' answers	9/32 (28.1%) logic and calculation
How many vertices are there?	19/32 (59.4%) counted	4/32 (12.5%) systematic counting	9/32(28.1%) 12 times 5
How many edges are there?	9/32 (28.1%) Counted	12/32 (37.5%) systematic counting	11/32 (34.4%) sum of 60, 30
Is it all the same for every vertex?	21/32 (65.6%) constructed	0/32 (0%)	11/32 (34.4%) what if not

How many spherical solids if we use regular triangles?	4/32 (12.5%) constructed	18/32 (56.3%) systematic	10/32 (31.3%) sum of angles
How many spherical solids if we use squares?	3/32 (9.4%) constructed	6/32 (2.2%) systematic	23/32 (71.9%) sum of angles
How many spherical solids if we use regular pentagons?	1/32 (3.1%) constructed	4/32 (12.5%) systematic	27/32 (84.4%) sum of angles
How many spherical solids if we use regular hexagons?	0/32 (0%)	2/32 (6.3%) systematic	30/32 (93.8%) sum of angles
How many spherical solids if we use two kinds of regular polygons such as a football?	28/32 (87.5%) constructed	2/32 (6.3%) systematic	2/32 (6.3%) sum of angles analogy induction

Table 1: Percentages of reasoning type students used (N=32)

The above table shows percentages of reasoning types followed by students per question. None of the students used pragmatic reasoning after similar questions were presented repeatedly (see shaded parts in Table 1). This indicates that students could judge the value or the level of reasoning and reflected their reasoning processes when they proceeded (see the last row of Table 1). However, if the problem situation would be changed completely, most students went back to pragmatic reasoning. Pragmatic reasoning is not perfect and often clumsy but foster students to revise their idea or think alternatively. The manipulative material named "Polydron" was used in this study, which helped students construct solids, guess, reflect on their construction, and test their guesses in various ways. It is argued that appropriate manipulative material can foster pragmatic reasoning that facilitates semantic and intellectual reasoning.

It was extremely difficult to distinguish between semantic and intellectual reasoning because students often reasoned in a mixed way. Both semantic and intellectual reasoning given by one or a group of students usually were discussed for a long time and were connected to creative informal proofs. The episodes described here provide explanations on that.

Episode: pragmatic reasoning and semantic reasoning

The episode below, which lasts about 3 minutes, comes from discussions in a group of five students. These students have studied independently on the first question, and now are comparing their answers.

[Codes]

1 S1: So what is your answer? Mine is 12.
2 S2: Regular pentagon? 12. I counted the regular hexagon first. It's 19.
3 S3: So did I, but in my case, it's different, it's 20.
4 S1: How did you count?
5 S2: Well, I started at this face, let's count again, one, two, …, twenty.
6 Oh, it's strange, what's happening here! **[Pragmatic]**
7 S4: I think 20 is correct because there were no mistakes before. Maybe you've
8. missed one.
9 S2: I need to count once again. By the way, all of you got 20?
10 S1, S5: Yeah.
11 S1: Why don't you count by following different directions? It might be
12. helpful. **[Semantic]**
13 S5: [speaking to S1] Directions? Why do we consider directions?
14 S2: If we collect lots of evidence, then we can believe a lot. Is it correct?
15 S1: In addition to that, there would not be mistakes if we insist on a direction
16 while counting.
17 S5: Oh! That's a good idea. Then we had better investigate how many directions are there.
18 S1: [speaking to teacher] We have discovered interesting aspects.

S2 explicitly states that he reasoned pragmatically not by mathematical calculation but by direct counting only ("I started at this face." Line 5) and he tried to prove by counting in front of the peers again. It reveals how he reasoned and how he felt certainty in his thinking at the same time. When S1 says that 20 is correct because there were no mistakes in S2's counting (Line 7), it indicates that S1 observed S2's way of counting and sought mathematical or systematic approach. His remark on directions (Line 11-12) is sufficient to show that he proceeded to semantic reasoning and promote other students' understanding of the problem.

In the above episode, S1 not only proposed a good solution to the problem but also presented a systematic way of counting which can diminish uncertainty in their processes. His description of the counting pattern does not need to be tested and is sufficient to suggest that they found out interesting aspects of semi-regular solids. In the next episode, on the spherical solids made of squares, another example of pragmatic and semantic reasoning is provided

[Codes]

1 S6: Only cube, cube can be made if we use squares.
2 S7: How do you know?
3 S6: Just because, …, at any rate, I remember what I did. **[Pragmatic]**
4 S8: We need to explain mathematically. I am thinking on that.

S9: It is a sort of order. I mean we have to consider each one from the case with three squares at each vertex by increasing squares. **[Semantic]**

S10: Four. Then, four.

S9: Okay. Isn't it obvious? There are no solids with four squares at each vertex.

[S1 is trying to make a plane with four squares]

S6: Okay, okay. It always becomes a plane if we use four squares. So we are done, it was proved finally.

Although the above episode occurred after students spent considerable time to guess, test, and verify, S6 still seems to be in the pragmatic thinking level (Line 3). However, S8 and S9 helped him quickly grasp key ideas of justification (Line 4-6), so that he describes or formulates the problem situation meaningfully (Line 11). The pragmatic reasoning itself is insufficient but can be seen as an important part of mathematical reasoning from the above episode. When S9 states an order of consideration ("a sort of order" Line 5), S6 not only understood what she meant but was also convinced that it is sufficient for verification.

Episode: intellectual reasoning

According to Brousseau (1997), students adopt false theories, accept insufficient or false proofs and the didactic situation must lead them to evolve, revise their opinion and replace their false theory with a true one. He emphasized that a mathematical theory is progressively constructed. This point of view was identified in the previous section on pragmatic reasoning. Students who participated in this study often presented false conjectures and tested them insufficiently. It was impossible to find students who can reason intellectually from the beginning. The next episode occurred after students spent much time reasoning pragmatically or semantically.

[Codes]

S1: It is easy to find the number of regular hexagons by using the number of regular pentagons in a football.

S2: How?

S1: 12 times 5 divided by 3, and then we have 20. Because, uh, uh, every regular pentagon is surrounded by regular hexagons and every regular hexagon has three regular pentagons. **[Intellectual]**

S3: Fantastic! Fantastic! It makes sense.

S4: Say that again please. Why do you divide 60 by 3?

S1: Because we counted three times.

S1 arrives at a mathematical expression which is an essential idea for developing geometrical reasoning in the given problem situation ("12*5÷3=20" in his worksheet & Line 3). Led by him, students check their solution and expressions in terms of efficiency or sufficiency. This intellectual work of S1 must similar to mathematicians' activity, which is a faithful reproduction of a scientific activity, as Brousseau (1997) emphasized. S9 too, reasons intellectually in the next episode when she solves the last question on semi-regular polyhedron.

[Codes]

1	S9:	I am thinking of a good expression for them. Always the same kind of polygons	
2		are at each vertex, …, in this case, three squares, in case of …	
3	S6:	Right. In case of a football, there are two regular hexagons and one	
4		regular pentagon.	
5	S10:	That kind of information varies in all situations. I don't know.	
6	S6:	That kind of information varies… in all situations?	
7	S9:	It's not important … I mean… If I say … five, six, six, then all of you	
8		are able to know what solid I am thinking. Okay, let's express it like this	
9		from now on. [writes as "(5, 6, 6)" and shows it to other members] Then	
10		we are ready to find another solid like a football.	**[Intellectual]**

S9 drew a table and check several cases which can be semi-regular solids using her own expressions (Line 9), that made her discover meaningful ideas. Intellectual reasoning is usually accompanied by useful expressions. S9 proposed excellent expressions or informal proofs while solving other problems, too. (see Figure 1) She demonstrated that not all squares have circumscribed circles by Figure 1. She said in the interview after the class, "Triangles are very special polygons because we can find circumscribed circles in any of them." Her intellectual reasoning was informal and insufficient as in the above episode or Figure 1, but it contributed to successive verification and justification, which is essential for developing geometrical reasoning.

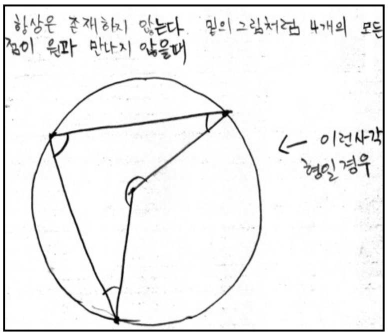

Figure 1

CONCLUSION

The mathematically gifted students demonstrated pragmatic, semantic, and intellectual reasoning in this study. Nine (including S1, S9 in the previous sections) of 32 students consistently reasoned intellectually in most of all the settings via pragmatic and semantic reasoning. Furthermore, they presented crucial ideas for formal proving or validation. These ideas gave chances to discuss what proof means and why proof is needed in mathematical situations. The students who had reasoned pragmatically at first have learned from discussions with other students on verifications, and consequently changed their ways of reasoning. Thus, it is argued that the didactic situation in this study permitted the evolution and the organization of informal but close to formal proofs by means of various kinds of reasoning. The example episodes also illustrate that peer interaction in small group is critical to leading students in which they reflect and simplify their idea while explaining to peers. The theoretical purposes of the teaching experiment were to characterize gifted students' proof constructions and to contribute to the theoretical body of knowledge about gifted students' mathematical thinking. The connection of informal and formal proofs by students is an area in need of additional research.

References

Artzt, A., & Yoloz-Femia, S. (1999). Mathematical reasoning during small-group problem solving. In L. V. Stiff (Ed.), *Developing mathematical reasoning in grades K-12: 1999 Yearbook*, (pp. 115-126). Reston, VA: National Council of Teachers of Mathematics.

Brousseau, G. (1997). *Theory of Didactical Situation in Mathematics Education*. Dordrecht: Kluwer Academic Publishers.

Chazan, D. & Lehrer, R. (Eds.) (1998). *Designing Learning Environments for Developing Understanding of Geometry and Space*. Mahwah, NJ: Lawrence Erlbaum Associates.

Davis, P. J. & Hersh, R. (1986). *Descartes' dream*. New York: Harcourt, Brace, & Jovanovich.

English, L. D. (1997). *Mathematical reasoning: Analogies, metaphors and images*. Mahwah, NJ: Lawrence Erlbaum Associates.

Ernest, P. (1991). *The philosophy of mathematics education* . Bristol: Falmer.

Graves, B., & Zack, V. (1997). Collaborative mathematical reasoning in an inquiry classroom. In Pehkonnen, E. (Ed.) *Proc. 21th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 17-24). Lahti, Finland: PME.

Hanna, G. (1983). *Rigorous proof in mathematics education*. Toronto: OISE Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. New York: Cambridge University Press.

Senk, S. How well do students write geometry proofs? *Mathematics Teacher*, 78, 448-456.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48, 101-119.

Weber, K. (2003). A procedural route toward understanding the concept of proof. In Pateman, N., Daugherty, B., and Zilliox, J. (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 395-401). Honolulu, HI: PME.