THE EQUIVALENCE AND ORDERING OF FRACTIONS IN PART-WHOLE AND QUOTIENT SITUATIONS

Ema Mamede – University of Minho

Terezinha Nunes – Oxford Brookes University

Peter Bryant – Oxford Brookes University

This paper describes children's understanding of equivalence, ordering of fractions, and naming of fractions in part-whole and quotient situations. The study involves eighty first-grade children, aged 6 and 7 years from Braga, Portugal. Three questions were addressed: (1) How do children understand the equivalence of fractions in part-whole and quotient situations? (2) How do they master the ordering of fractions in these situations? (3) How do children learn to represent fractions with numbers in these situations? A quantitative analysis showed that the situations in which the concept of fractions is used affected children's; their performance in quotient situations was better than their performance in problems presented in part-whole situations. The strategies used also differed across these two situations.

FRAMEWORK

Research has shown that children bring to school a store of informal knowledge that is relevant to fractions. Pothier and Sawada (1983) have documented that students come to instruction with informal knowledge about partitioning and equivalence; Behr, Wachsmuth, Post and Lesh (1984) have found that children come to instruction with informal knowledge about joining and separating sets and estimating quantities involving fractions; Mack argues that students construct meaning for formal symbols and procedures by building on a conception of fractions that emerges from their informal knowledge (Mack, 1990; 1993); Empson (1999) provided evidence that young children can develop a set of meaningful fraction concepts using their own representations and equal sharing situations as mediating structures. Ball (1993) claims that learning mathematics with understanding entails making connections between informal understandings and more formal mathematical ideas. However, research is needed to explore how students build upon their informal knowledge to improve their understanding of fractions. This includes focusing on which situations can be used in instruction in order to make fractions more meaningful for children.

Several authors have distinguished situations that might offer a fruitful analysis of the concept of fractions. Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Kieren (1988) considers measure, quotient, ratio and operator as mathematical subconstructs of rational number; Mack (2001) proposed a different classification of situations using the term 'partitioning' to cover both part-whole and

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quotient situations. In spite of the differences, part-whole, quotient, measures and operator situations are among the situations identified by all of them. However, there is no unambiguous evidence about whether children behave differently in different situations or not. This paper focuses on the use of fractions in two situations, part-whole and quotient situations, and provides such evidence.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. So, 2/4 in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In a quotient situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). Thus number meanings differ across situations. Do these differences affect children's understanding in the same way that they do in problems with whole numbers?

Fraction knowledge is not a simple extension of whole number knowledge. Regarding the child's conception of natural numbers, Piaget's hypothesis is that number is at the same time both a class and an asymmetrical relation. Extending this analysis of natural numbers to fractions, one has to ask how children come to understand the logic of classes and the system of the asymmetrical relations that define fractions. How do children come to understand that there are classes of equivalent fractions – 1/3, 2/6, 3/9, etc – and that these classes can be ordered – 1/3 > 1/4 > 1/5 etc? (Nunes et al., 2004).

Concerning the ordering of fractions, Nunes et al. (2004) suggest that children have to consider two ideas: (1) that, for the same denominator, the larger the numerator, the larger the fraction, (e.g. 2/4 < 3/4); (2) that, for the same numerator, the larger the denominator, the smaller the fraction, (e.g., 3/2 > 3/4). The first relation is simpler than the second one, because in the second the children have to think of an inverse relation between the denominator and the quantity represented by the fraction. Thus, it is relevant to know under what condition children understand these relations between numerator, denominator and the quantity.

The learning of fractions cannot be dissociated from the words and numbers used to represent them. According to Nunes et al. (2004), in the domain of natural numbers, the fact that two sets are labelled by the same number word – say both sets have six elements – might help children understand the equivalence between two sets. This understanding is probably more difficult with fractions, since equivalent fractions are designated by different words – one half, two quarters – and also different written signs – 1/2, 2/4 – but they still refer to the same quantity.

In this paper we investigate whether the situation in which the concept of fractions is used influences children's performance in problem solving tasks and their procedures. The study was carried out with first-grade children who had not been taught about fractions in school. Three specific questions were investigated. (1) How do children understand the equivalence of fractions in part-whole and quotient situations? (2) How do they master the ordering of fractions in these situations? (3) How do children learn to represent fractions with numbers in these situations? Following the work of Streefland (1993; 1997) and Nunes et al. (2004), it is hypothesised that children's performance in quotient situations will be better than in part-whole situations and that children's procedures will differ across the situations, because quotient situations can be analysed through correspondences more naturally than part-whole situations. Although some research has dealt with part-whole (Sáenz-Ludlow, 1995) and quotient situations (Streefland, 1993) separately, there have been no comparisons between the two situations in research on children's understanding of fractions. Yet in many countries the traditional teaching practice is to use part-whole situations almost exclusively to introduce the fractional language (Behr, Harel, Post & Lesh, 1992). There seems to be an implicit assumption that this is the easiest situation for learning fractional representation; however, there is no supporting evidence for this assumption.

METHODS

Participants

Portuguese first-grade children (N=80), aged 6 and 7 years, from the city of Braga, in Portugal, were assigned randomly to work in part-whole situations or quotient situations with the restriction that the same number of children in each level was assigned to each condition in each of the schools. All the participants gave informed consent and permission for the study was obtained from their teachers, as required by the Ethics Committee of Oxford Brookes University. The children had not been taught about fractions in school, although the words 'metade' (half) and 'um-quarto' (a quarter) may have been familiar in other social settings. The two participating schools are attended by children from a range of socio-economic backgrounds.

The tasks

The tasks presented to children working both in quotient situations and in part-whole situations were related to: (1) equivalence of fractions; (2) ordering of fractions; and (3) naming fractions. An example of each type of task is presented below (Table 1). The instructions were presented orally; the children worked on booklets which contained drawings that illustrated the situations described. The children were seen individually by an experimenter, a native Portuguese speaker.

Table 1: Types of problem presented to the children

Problem	Situation	Example
Equivalence	Part-whole	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar in 2 equal parts and eats 1 of them; Ann breaks hers into 4 equal parts and eats 2 of them. Does Bill eat more, the same, or less than Ann?
	Quotient	Group A, formed by 2 children have to share 1 bar of chocolate fairly; group B, comprising of 4 children have to share 2 chocolates fairly. Do the children in group A eat the same, more, or less than the children in group B?
Ordering	Part-whole	Bill and Ann each have a bar of chocolate the same size; Bill breaks his bar into 2 equal parts and eats 1 of them; Ann breaks hers into 3 equal parts and eats 1 of them. Who eats more, Bill or Ann?
	Quotient	Group A, formed by 2 children has to share 1 bar of chocolate fairly; group B which consists of 3 children has to share 1 chocolate fairly. Who eats more, the children of group A, or the children of group B?
Naming	Part-whole	Children write a half and indicate what the numbers mean. The researcher summarizes the children's description, ensuring that they realize that ½ means "you cut something into two equal parts and take one". The children are then asked to represent a child cutting a chocolate into three parts and taking one: what fraction will s/he get?
	Quotient	Children write a half and indicate what the numbers mean. The researcher summarizes the children's description, ensuring that they realize that ½ means "you share a chocolate between two children". The children are then asked to represent a child sharing a chocolate between three children: what fraction will each get?

Design

Children were randomly assigned to either the part-whole or the quotient situation.

The six equivalence items and the six ordering items were presented in a block in random ordered at the beginning of the session. The naming items were presented in a fixed order at the end of the session. In the naming tasks, the children were taught how to write fractions at the start of the tasks; this teaching covered four examples (1/2, 1/3, 1/4, 1/5) and two further examples of non-unitary fractions, such as 2/3 and

3/7 were given. These were followed by four test items where the children were asked to name the fractions. No feedback was given for any of the test items. The numerical values were controlled for across situations.

RESULTS

Descriptive statistics for the performances on the tasks for each working situation are presented in Table 2.

Table 2 – Proportions of correct answers and (standard deviation) by task

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	0.35(0.25)	0.49 (0.26)	0.09 (0.11)	0.1 (0.08)
Ordering	0.55 (0.35)	0.71 (0.22)	0.24 (0.23)	0.2 (0.14)
Naming	0.88 (0.28)	0.88 (0.24)	0.58 (0.23)	0.6 (0.27)

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and problem solving situation (quotient *vs* part-whole) as between-subjects factor, and tasks (Equivalence, Ordering, Naming) as within-subjects factor. Because the tasks presented to children involved 6 problems of equivalence of fractions, 6 problems of ordering of fractions and 4 problems of name fractions, proportional scores were analysed and an arcsine transformation was applied.

There was a significant main effect of the problem situation, $F(1,76) = 125.69$, $p<0.001$, indicating that children's performance when problems are presented to them in quotient situations is significantly better than when problems are presented in part-whole situations. There was a significant tasks effect, $F(2,152) = 82.37$, $p<0.001$. Simple contrasts showed that the children were significantly better at naming fractions than at ordering fractions, $F(1,76) = 69.93$, $p<0.001$, and significantly better at ordering than at judging the equivalence of fractions $F(1,76)=20.06$. There were no other significant effects.

An analysis of children's procedures allowed the identification of correspondence and partitioning as the most used procedures by children to solve the problems. Correspondence is defined as an association established between the shared parts and each recipient; partitioning is defined as the division of an item into parts. Examples of children's use of correspondence and partitioning are in Figure 1.

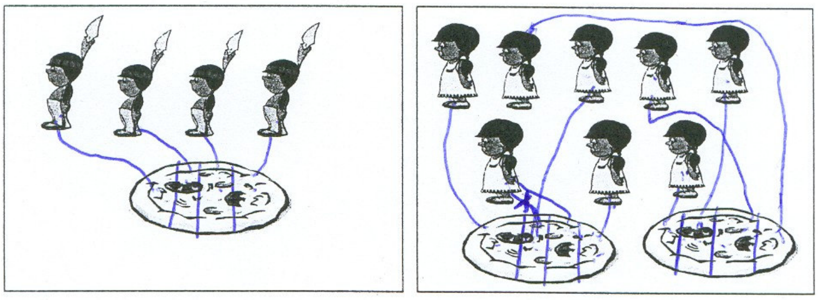

Figure 1 – The use of correspondence and partitioning by a child solving a problem in quotient situation.

Table 3 – Proportion and (standard deviation) of the use of correspondence and partitioning by task and problem situation.

	Quotient				Part-whole	
	Correspondence		Partitioning		Partitioning	
Task	6 yrs	7 yrs	6 yrs	7 yrs	6 yrs	7 yrs
Equivalence	0.24(0.25)	0.33(0.30)	0.53(0.39)	0.6(0.33)	0.35(0.46)	0(0)
Ordering	0.33(0.34)	0.43(0.36)	0.43(0.40)	0.66(0.4)	0.34(0.45)	0(0)
Naming	0.02(0.07)	0.02(0.07)	0.03(0.1)	0.05(0.22)	0.05(0.16)	0(0)

Table 3 shows that children from both age groups used correspondence when problems were presented to them in quotient situations. No children used correspondence in part-whole situations. Partitioning was used in quotient situations more often than in part-whole situations. In quotient situations many children combined correspondence and partitioning to solve the problems (see Figure 1). In part-whole situations partitioning was never combined with correspondence.

DISCUSSION

Children's ability to solve problems of equivalence of fractions and ordering of fractions is better in quotient situations than in part-whole situations. The analysis of the procedures used by children suggests that correspondence play a role in children's understanding of fractions. The use of correspondence helped children to solve problems correctly when working in quotient situations. In opposite to this, when problems were presented to children in part-whole situations the correspondence seemed not to be an acceptable procedure by children. This might be due to the different reasoning expressed by correspondence in part-whole and quotient situations. Nunes et al. (2004) argues that in part-whole situations the values in the fractions refer to quantities of the same nature, expressing a relation between the part and the whole. Here, one-to-many correspondence reasoning expresses the ratio between the number of parts taken and the number of parts left, which is different from the fraction representation that is based on the number of parts into which the whole was cut and the number of parts taken. In quotient situations the values in the fraction refer to two variables of a different nature; the numerator indicates one quantity - the item being shared, the denominator indicates other quantity – the number of recipients. This explains why quotient situations can be analyzed through correspondences more naturally than part-whole situations. Moreover, the use of correspondence allowed children to use sharing activities, i.e. to use distribution of the items among recipients, (Nunes & Bryant, 1996; Correa, Nunes & Bryant, 1998) which demands the recognition of three number meanings: the number of items to share, the number of equal shared parts and the number of recipients. To solve problems of equivalence and ordering of fractions children have to recognize the direct relation between the number of items being shared and the size of the shared parts, and the inverse relation between the size of the shared parts and the number of recipients. These relations are extremely relevant to the understanding of fractions.

Also when children were asked to represent fractions symbolically, their level of success differed between part-whole and quotient situations. There is also evidence that children's ability to learn name fractions in quotient situations is easier than in part-whole situations. The results suggest that children can construct appropriate meanings for mathematical symbols when problems presented to them are closely matched to problems drawn on their informal knowledge.

CONCLUSION

There is evidence that the situation in which the concept of fraction is used influences children's performance in problem solving tasks. The levels of success in children's performance on equivalence and order of fractions using quotient situations supports the idea that children have some informal knowledge of the logic of fractions, i.e., children have some knowledge of the logic of fractions that was developed in their everyday life, without instruction in school. Children's performance in problems presented in quotient situations is better than their performance in problems presented in part-whole situations. Nevertheless, traditional teaching practices use part-whole and not quotient situations to introduce the concept of fractions. Thus, maybe we should rethink which is the best situation to introduce children to fractions in the classroom.

This study involved tasks in which children were working only in part-whole and quotient situations. We also need to know how children would perform and what strategies they would use to solve fractional problems in other situations.

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