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DIRAC AND SEIBERG–WITTEN MONOPOLES

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Abstract. Dirac magnetic monopoles, which may or may not exist in nature, seem to exist everywhere in mathematics. They are in one-to-one correspondence with the natural connections on principal $U(1)$-bundles over S^2 and, moreover, appear as solutions to the field equations of $SU(2)$ Yang–Mills–Higgs theory on $\mathbb{R}^3$ as well as Seiberg–Witten theory and its non-Abelian generalization on Minkowski spacetime. This talk will present an informal survey of the situation.

1. Classical Dirac Monopoles

We begin with the source-free Maxwell equations written in complex form as

$$\nabla \cdot (\vec{E} + \mathrm{i}\vec{B}) = 0, \qquad \frac{\partial}{\partial t}(\vec{E} + \mathrm{i}\vec{B}) + \mathrm{i}\nabla \times (\vec{E} + \mathrm{i}\vec{B}) = \vec{0}\,. \tag{1.1}$$

These equations have a great many well-known symmetries. They are, for example, Lorentz invariant, gauge invariant and conformally invariant, but they also possess what might be called a "duality symmetry". Specifically, if $\vec{E}+\mathrm{i}\vec{B}$ is a solution to (1.1), then so is $\mathrm{e}^{\mathrm{i}\varphi}(\vec{E} + \mathrm{i}\vec{B})$ for any complex number $\mathrm{e}^{\mathrm{i}\varphi}$ of modulus one. When $\varphi = \pi/2$ this reduces to the familiar fact that the substitutions $\vec{B} \to \vec{E}$ and $\vec{E} \to -\vec{B}$ carry one solution into another.

This last symmetry is lost, of course, if one includes charge densities and currents in Maxwell's equations, but Dirac [3] realized that it could be reinstated by including also (hypothetical) magnetic charges and currents. For this he introduced the magnetic analogue of a Coulomb field defined, on $\mathbb{R}^3\backslash\{0\}$, by

$$\vec{E} = \vec{0}\,, \qquad \vec{B} = \frac{n/2}{\rho^2}\,\hat{e}_\rho\,, \tag{1.2}$$