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A C-SPECTRAL SEQUENCE ASSOCIATED WITH FREE BOUNDARY VARIATIONAL PROBLEMS

GIOVANNI MORENO

Department of Mathematics and Informatics, Salerno University
84084 Fisciano, Italy

Abstract. The $\mathcal{C}$–spectral sequence is a cohomological theory naturally associated with a space of infinite jets, which allows to write down many concepts of the variational calculus by using the same logic of the standard differential calculus. In this paper we use such a language (called Secondary Calculus by A. Vinogradov) to describe a delicate aspect of the variational calculus: the appearance of some "natural" boundary conditions in the context of variational problems with free boundary (e.g., transversality conditions). We discover that the Euler–Lagrange operator is actually a graded operator, producing simultaneously the standard Euler–Lagrange equations and these new boundary conditions as different homogeneous components of an unique object. Simple applicative examples will be presented.

1. Introduction

When a system of nonlinear PDEs is formalized as a natural geometrical object, one can use the common tools of differential calculus (e.g., locality, differential cohomology, symmetries, etc.) to reveal some aspects of the equations, which could be hardly accessed by just using analytic techniques (the first steps in this direction were moved by Dedecker 1978, Gel'fand and Dikii 1975, Hordenski 1974, Olver and Shakiban 1978, Tulczyjew 1975, 1977, 1980, and Vinogradov 1977, 1978).

The present work was sustained by the belief that the right geometrical portrait of a system of nonlinear PDEs is the so–called **diffiety** (see [3, 6] and references therein) and that the analitic machinery which is commonly used to look for solutions of PDEs just draws attention away from the more conceptual, and hence more interesting, aspects of the problem.

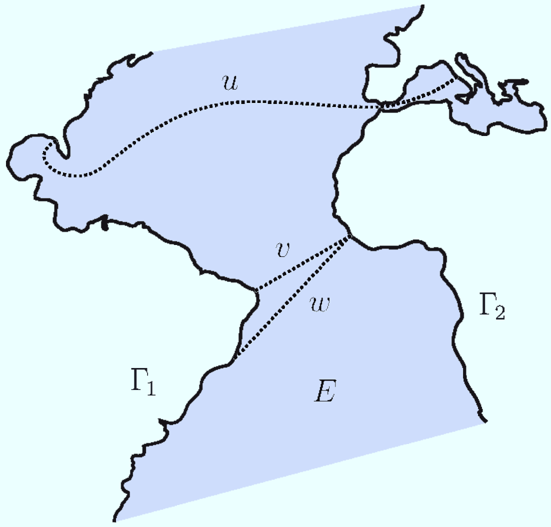

Figure 1. If Columbus had had the possibility of look the Earth from the space, he probably would have chosen a different route than the one he actually took, for it would be the shortest.

For example, when searching for the potential of a closed form ω in $\mathbb{R}^n$, one can choose the coordinate path and build up a family of solutions $F + k$, where the constant k arises from the integration process, or the cohomological way, and observe that the homotopy equivalence of $\mathbb{R}^n$ and $\{0\}$ allows to recast the problem on a zero–dimensional manifold. In this perspective, the above family looks like $k - h(\omega)$, where now the number k is the true solution, and $h(\omega)$ is just an algebraic compensation due to the homotopy formula.

In this paper we present a not–so–trivial circumstance where the use of diffieties and the relative to them cohomological methods is highly advisable, since they avoid the classical analitical proofs and provide far–reaching generalizations.

1.1. A Toy Model

Fig.1 provides a visual proof of the next

Theorem 1 (The problem of Columbus)**.** *Given the curves* Γ_1 *and* Γ_2 *in* $\mathbb{R}^2$, *the problem of finding, among the (non self–intersecting) (smooth) curves which start from a point of* Γ_1 *and ends to a point of* Γ_2 *(without crossing* $\Gamma_1 \cup \Gamma_2$ *in any other point), those whose length is (locally) minimal, admits a unique solution* v.

Above result contradicts the fact that the Euler–Lagrange equations associated with the length functional are second order. However, by manipulating the first variational formula, it has been discovered that the extremal v in fact fulfills some hidden boundary conditions, which where called **transversality conditions**.

We will show, in a natural geometric language, that such conditions in fact arise for a large class $\mathcal{P}$ of variational problems, yet keep using the problem of Columbus as a toy model.

1.2. The Main Problem $\mathcal{P}$

The formal definition of our main problem reads as follows

Definition 1. *Given*

- *a manifold* E *with* non–empty boundary ∂E
- *an integer* $n < \dim E$
- *the set* $\mathrm{Adm}(\mathcal{P}) = \{L\}$ *such that*
  - $*$ L *is an* n*–dimensional compact connected submanifolds of* E
  - $*$ L *is* nowhere tangent to ∂E
  - $*$ *and* ∂L *is* non–empty *and coincides with* $L \cap \partial E$
- *and, finally, an* horizontal n–form $\omega \in \overline{\Lambda}^n(J^\infty(E, n))$

the problem of finding the (local) extrema for the ***action***

$$\mathrm{Adm}(\mathcal{P}) \ni L \mapsto \int_L j_\infty(L)^*(\omega) \in \mathbb{R} \tag{1}$$

determined by ω *on the elements of* $\mathrm{Adm}(\mathcal{P})$*, is called a* ***free boundary variational problem*** $\mathcal{P}$.

1.3. The Natural Strategy

In a coordinate–free approach, the first attack to the problem of Columbus would be to transform the domain E into the total space of the trivial bundle $\pi : [0,1] \times (0,1) \to [0,1]$. Since our theory is a natural construction, such a **rectified problem** of Columbus isequivalent to the original one, and it can be stated as follows

- $E = [0,1] \times (0,1)$, and $\partial E = \{0,1\} \times (0,1)$
- $n = 1$
- $\mathrm{Adm}(\mathcal{P}) = \Gamma(\pi)$
- the *horizontal one–form* $\omega \in \overline{\Lambda}^1(J^\infty(\pi))$ is given by $\omega = f(x, y, y')\mathrm{d}x$, where $f = \sqrt{1 + (y')^2}$

and we want to find the (local) extrema for the action

$$\Gamma(\pi) \ni u \mapsto \int_0^1 j_\infty(u)^*(\omega) \in \mathbb{R}. \tag{2}$$

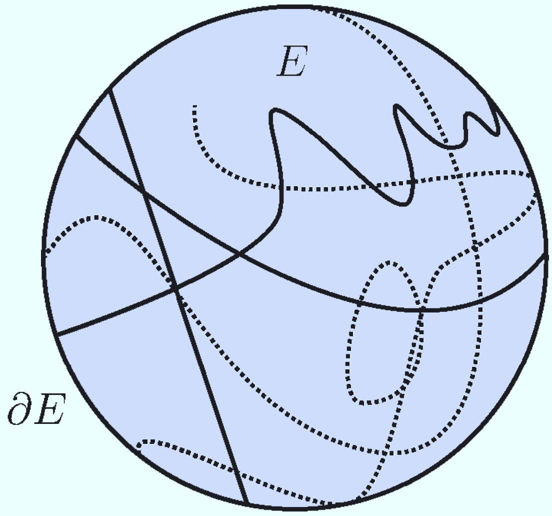

Figure 2. A variational problem in the two–dimensional disk.

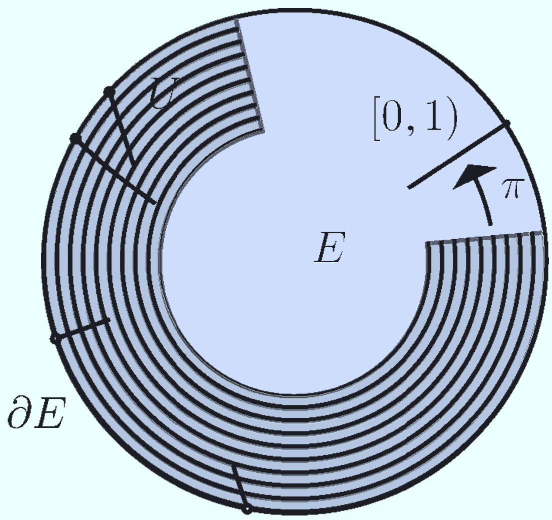

Figure 3. Any problem $\mathcal{P}$ has locally a fibered structure.

Problems $\mathcal{P}$ in which only sections of a fiber bundle π are involved belongs to the so–called **fibered case**. A problem not belonging to such a case is illustrated in Fig.2. There E is the two–dimensional closed disk D^2, ∂E is the circle S^1, and $n = 1$. Then only the full lines belong to $\mathrm{Adm}(\mathcal{P})$, while does not any of the dotted ones, for either it is tangent to ∂E in some point, or its boundary *is not the set of its common points with* ∂E, or it has got **empty boundary**.

The natural way to attack such a problem relies on the localizability of our theory. As suggested by Fig.3, consider the subset U of E.

And observe that, nearby a point of ∂E, an admissible submanifold for the problem $\mathcal{P}$ coincides with the graph of a (local) section of π in a neighborhood of the

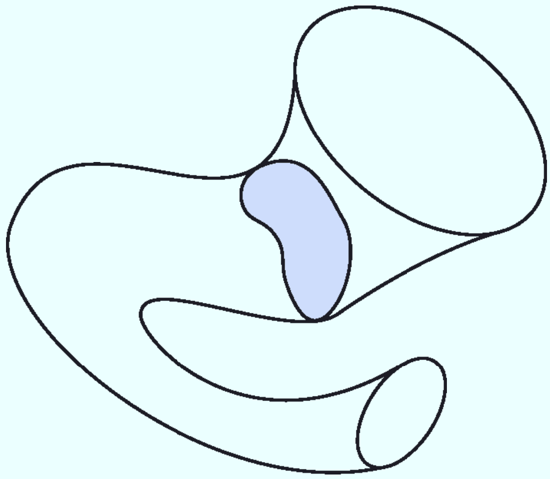

Figure 4. The naturality of our theory allows to reduce he problem of finding the least area membrane in a pipe of arbitrary shape, to a very trivial one.

origin. The reader should now be persuaded that our toy model is not so restrictive, since by localization and naturality, any problem $\mathcal{P}$ can be reduced to the fibered case (see Fig.4). The remainder of the paper is devoted to show that the theory which describes the problems $\mathcal{P}$ is just an aspect of the *Secondary Calculus* (introduced by Vinogradov, see [6]) over a certain diffiety, wich we will call $(B, \mathcal{C})$, naturally associated to the problem $\mathcal{P}$, much as the problem $\mathrm{d}F = \omega$ is an aspect of the differential cohomology of $\mathbb{R}^n$. We stress that *natural* here means that if the data of the problem $\mathcal{P}$ undergo a transformation, then the objects and morphisms which constitute the theory which describe $\mathcal{P}$ also undergo a transformation, and the theory which describe the transformed problem is obtained.

2. Main Tools

Details about standard terminology and results of Secondary Calculus can be found in [6]. To give a problem $\mathcal{P}$ a natural location in the framework of Secondary Calculus, we have to introduce new tools.

Definition 2. *$(B, \mathcal{C})$, where $B \stackrel{\text{def}}{=} J^\infty(\pi)$, $\pi : E \to M$, and $\mathcal{C}$ is the Cartan distribution, is called the* ***main diffiety*** *associated with $\mathcal{P}$.*

Definition 3. *$(\partial B, \mathcal{C}_{\partial B})$, where $\partial B \stackrel{\text{def}}{=} \pi_\infty^{-1}(\partial M)$, and $\mathcal{C}_{\partial B}$ is the restriction of $\mathcal{C}$ to ∂B, is called the* ***sub–diffiety*** *associated with $\mathcal{P}$.*

Notice that $\partial B \stackrel{\text{def}}{=} \pi_\infty^{-1}(\partial M)$ is a sort of "infinite prolongation" (see, e.g. [4]) of ∂M. From now on, the problem $\mathcal{P}$ may be identified with the pair of diffieties $(B, \partial B)$, or simply with the main one B.

2.1. The Term E_0 of the Relative $\mathcal{C}$–spectral Sequence

Observe that the new objects B and ∂B came with an evident interrelationship, which influences the cohomological theories associated with them. Indeed, much as to any submanifold it is possible to associate a short exact sequence of differential forms, to the sub–diffiety ∂B it is possible to associate a short exact sequence of $\mathcal{C}$–spectral sequences.

Definition 4.

$$0 \to E_0^p(B, \partial B) \xrightarrow{i} E_0^p \xrightarrow{\alpha} E_0^p(\partial B) \to 0 \tag{3}$$

is called the short exact sequence of E_0 terms of $\mathcal{C}$–spectral sequences associated with $\mathcal{P}$

The term $E_0^p(B, \partial B)$, which is defined as

$$E_0^p(B, \partial B) \stackrel{\text{def}}{=} \frac{\mathcal{C}^p \cap \Lambda(B, \partial B) + \mathcal{C}^{p+1}}{\mathcal{C}^{p+1}} \tag{4}$$

can be understood as the sub–complex of E_0^p whose elements vanish when they are restricted to ∂B, in fully accordance with the definition of a relative (with respect to the boundary) form on a standard manifold.

Inspired by this analogy, we give the next

Definition 5. *Complex* $E_0^p(B, \partial B)$ *defined by* (4) *is called the* E_0 *term of the* ***relative*** (*with respect to* ∂B) ***C–spectral sequence*** *associated with* B.

2.2. The Long Exact Sequence of E_1 Terms

The analogy with standard manifolds, when we find the cohomology long exact sequence, goes even further, and the next result is immediately proven.

Lemma 1. *The triangle*

$$\begin{array}{ccc} E_1^p(B, \partial B) & \xrightarrow{\quad H(i) \quad} & E_1^p \\ & {}_{\partial}\nwarrow \quad \swarrow_{H(\alpha)} & \\ & E_1^p(\partial B). & \end{array} \tag{5}$$

is exact.

Definition 6. *The triangle* (5) *is called a* ***long exact sequence*** *of* E_1 *terms associated with* B.

2.3. The Main Object Associated with $\mathcal{P}$

Complex $E_1^p(B, \partial B)$ appearing at the left vertex of (5), is just a column of the E_1 term of a whole spectral sequence $E(B, \partial B)$, whose other terms are beyond the scope of this paper.

Definition 7. *$E(B, \partial B)$ is called the* ***relative C–spectral sequence****, associated with the problem $\mathcal{P}$.*

All the objects needed to build up the theory are now ready. It is worth stressing that, together with objects, we have also introduced (in a more or less explicit way) new differentials and morphisms.

In the next section we will see how such objects and differentials and morphisms encode all the known information about the problem $\mathcal{P}$, and also reveal some new aspects.

3. Main Results

Once inscribed in the framework of Secondary Calculus, the whole theory of the free boundary variational problems depends on a good definition of the Lagrangian. All the rest, is an (almost) algorithmic consequence of this definition.

3.1. The Lagrangian Associated with $\mathcal{P}$

In the statement of $\mathcal{P}$, only the Lagrangian density ω was involved (see Definition 1). Observe that, for any $u \in \Gamma(\pi)$, the map $j_\infty(u)$ sends ∂M into ∂B. So, if ω is the differential of some form vanishing on ∂B, then $j_\infty(u)^*(\omega)$ will be the differential of some form vanishing on ∂M. By Stokes formula, the action determined by ω, evaluated on M, is zero.

In other words, the action of ω is given only by its cohomology class modulo ∂B. Then we can say that the

Definition 8. *The Lagrangian associated with $\mathcal{P}$ is the relative to ∂B cohomology class $[\omega] \in E_1^{0,n}(B, \partial B)$.*

From the point of view of Secondary Calculus, all the information about $\mathcal{P}$ can be retrieved from the element $[\omega]$ of $E_1^{0,n}(B, \partial B)$, which should be called the **secondary function** associated with $\mathcal{P}$.

3.2. The Graded Euler–Lagrange Equations

Following the same logic governing the standard calculus, one must apply the differential $\mathrm{d}_{1,\mathrm{rel}}^{0,n}$ to $[\omega]$ in order to get the equations for the extrema of $\mathcal{P}$.

Definition 9.

$$\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega]) \in E_1^{1,n}(B,\partial B). \tag{6}$$

is called the (left–hand side of the) ***relative*** *(or* ***"graded"****)* ***Euler–Lagrange equations*** *associated with $\mathcal{P}$.*

The theory will be complete when we show that "inside" $\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega])$ there are both the (standard) Euler–Lagrange equations and the (generalized) transversality conditions.

3.3. The Structure of $\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega])$

Even though Secondary Calculus assures that the relative (or "graded") Euler–Lagrange equation

$$\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega]) = 0 \tag{7}$$

is satisfied by the extrema of $\mathcal{P}$, the left–hand side of (7) cannot even called an "equation". The problem is that, in general, $E_1^{1,n}(B,\partial B)$ is not a module, so $\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega])$ cannot be interpreted as a differential operator.

The structure of ∂B, is clarified by the next result, whose proof is omitted (see [5] for more datails).

Theorem 2 (Structure Theorem). *$\pi^{-1}(\partial B)$ is isomorphic to the infinite jet space $J^\infty(\xi)$, where ξ is a special ∞–dimensional bundle over ∂M, called a* ***normal jets bundle****.*

As a consequence, the diffiety ∂B fulfills the Vinogradov one–line Theorem (see [6]).

The one–line Theorem for B and of Theorem 2, combined, produce the next

Corollary 1. *The one–line Theorem holds for the relative $\mathcal{C}$–spectral sequence as well.*

Proof: Choreographic: first represent vertices of triangle (5) in the (p,q)–plane, then take into account the degrees of the maps into play, and finally observe that there is no other possibility for the relative $\mathcal{C}$–spectral sequence but to be one–line (see Fig.5). □

3.4. The Short Exact Sequence of E_1 Terms

Paraphrasing Corollary 1, we can say that the long exact sequence (5) is, in fact, a short one. The component $p = 1$ plays a prominent role in our theory.

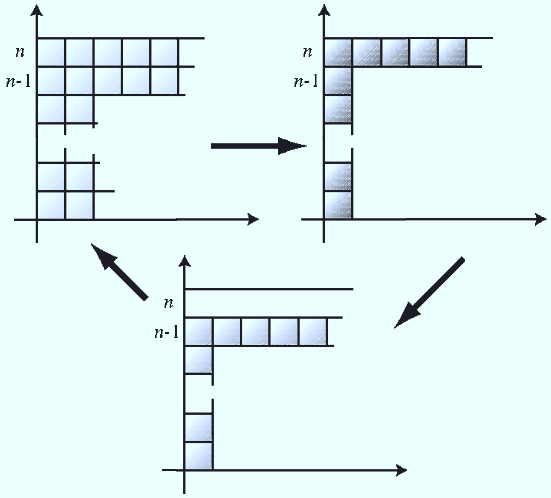

Figure 5. Geographical proof of the relative one–line Theorem.

Definition 10. *Sequence of vectors spaces*

$$0 \longrightarrow \widehat{\varkappa}(\partial B) \xrightarrow{\partial} E_1^{1,n}(B, \partial B) \xrightarrow{H(i)} \widehat{\varkappa} \longrightarrow 0 \tag{8}$$

is called the ***short exact sequence*** *of E_1 terms associated with $\mathcal{P}$.*

Lemma 2. *The sequence* (8) *splits.*

Proof: An easy consequence of the Green $\mathcal{C}$–formula (see [6]). □

Such a splitting is crucial for interpreting $\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega])$ as a pair of equations.

3.5. The Graded Euler–Lagrange Equations

Now we know that the graded Euler–Lagrange equations $\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega])$ look like an element

$$(\ell'_\omega, \ell^*_\omega(1))$$

of the graded object $\widehat{\varkappa}(\partial B) \oplus \widehat{\varkappa}$.

The second component is the (left–hand side of the) well–known Euler–Lagrange equations associated to ω.

Definition 11. *The new entry ℓ'_ω appearing as the first component of* $\mathrm{d}_{1,\mathrm{rel}}^{0,n}([\omega])$, *is called the* (*left–hand side of the*) ***transversality conditions*** *associated with $\mathcal{P}$.*

Such a choice of the name allowed us to be consistent with the already established terminology.

Thanks to the Structure Theorem, ℓ'_ω is an element of a vector bundle over ∂M. As such, it will admit a coordinate expression, which we show below, though quite complicated

$$\sum_{j=1}^{m}\sum_{k=0}^{\infty}\sum_{\tau\in\mathbb{N}_0^{n-1}}\sum_{l=k}^{\infty}N(\tau,l,i)D_\tau\left(\left.\left(D_n^{l-k}\left(\frac{\partial f}{\partial u^j_{\tau+(l+1)1_n}}\right)\right)\right|_{\partial B}\right)D_\emptyset^{(k,j)} \quad (9)$$

where m is the dimension of π.

Here $f\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$ is the local representation of ω, and $x^n=0$ is the equation for ∂M. The D's are (compositions of) the total derivatives.

3.6. Example of Application to the Toy Model

Consider the setting of the problem of Columbus, but allow the Lagrangian density $f\mathrm{d}x$ on E, where $f=f(x,y,y')$, to be arbitrary. Denote the corresponding Lagrangian on π by $\omega=g\mathrm{d}x$, for a suitable function $g=g(x,y,y')$.

Formula (9) tells that ℓ'_ω is simply $\left.\frac{\partial g}{\partial y'}\right|_{\partial B}$, so that the transversality conditions (accordingly to our theory) look like

$$\left.\frac{\partial g}{\partial y'}\right|_{\partial B}=0. \quad (10)$$

If we pull–back the last expression on E, we get the following expression

$$\left(f-\frac{\partial f}{\partial y'}\right)x^\Gamma+\frac{\partial f}{\partial y'}y^\Gamma=0 \quad (11)$$

where (x^Γ,y^Γ) is a tangent to ∂E vector, which is the classical formulation of the transversality conditions (see, e.g. [1] and [2]).

When $f=\sqrt{1+(y')^2}$ is the (restriction to E of the) **length functional** (on $\mathbb{R}^2$), equation (11) tell us exactly that the curve u must form a right angle with Γ_1 and Γ_2, while $\ell^*_{f\mathrm{d}x}(1)=0$ is equivalent to $y''=0$.

It is surprising that two conditions of such an heterogeneous natures (they are differential equations imposed on sections of bundles over different bases and with non–isomorphic fibers are in fact the homogeneus components of the same graded object $\mathrm{d}^{0,1}_{1,\mathrm{rel}}([f\mathrm{d}x])$.

3.7. Conclusions

Before this moment, formula (11), which we obtained just by using the naturality of a purely cohomological theory, could not be derived without introducing ad–hoc technicalities. Analogous formulae for two–dimensional problems (see Fig. 4) need even more computations, and it is not hard to imagine that their difficulty grows exponentially as the dimension gets larger.

On the contrary, thanks to the robustness of Secondary Calculus, we have managed to provide a simple description for any problem $\mathcal{P}$ which belongs to much more wide and general class of problems, where we have Lagrangians of any order, and no restrictions on the topology of E.

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