Third International Conference on
Geometry, Integrability and Quantization
June 14–23, 2001, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001, pp 431–441

PATH INTEGRALS ON RIEMANNIAN MANIFOLDS WITH SYMMETRY AND STRATIFIED GAUGE STRUCTURE

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Abstract. We study a quantum system in a Riemannian manifold M on which a Lie group G acts isometrically. The path integral on M is decomposed into a family of path integrals on quotient space $Q = M/G$ and the reduced path integrals are completely classified by irreducible unitary representations of G. It is not necessary to assume that the action of G on M is either free or transitive. Hence the quotient space M/G may have orbifold singularities. Stratification geometry, which is a generalization of the concept of principal fiber bundle, is necessarily introduced to describe the path integral on M/G. Using it we show that the reduced path integral is expressed as a product of three factors; the rotational energy amplitude, the vibrational energy amplitude, and the holonomy factor.

1. Basic Observations and the Questions

Let us consider the usual quantum mechanics of a free particle in the one-dimensional space $\mathbb{R}$. A solution for the initial-value problem of the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\phi(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x,t) = \frac{1}{2}\Delta\phi(x,t) \tag{1.1}$$

is given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K(x,y;t)\phi(y,0) \tag{1.2}$$

with the propagator

$$K(x,y;t) = \langle x|\,\mathrm{e}^{-\frac{\mathrm{i}}{2}t\Delta}|y\rangle = \frac{1}{\sqrt{2\pi \mathrm{i} t}} \exp\left[\frac{\mathrm{i}}{2t}(x-y)^2\right]. \tag{1.3}$$

Their physical meanings are clear; the wave function $\phi(x,t)$ represents probability amplitude to find the particle at the location x at the time t. The propagator $K(x,y;t)$ represents transition probability amplitude of the particle to move from y to x in the time interval t.

If the particle is confined in the half line $\mathbb{R}_{\geq 0} = \{x \geq 0\}$, we need to impose a boundary condition on the wave function $\phi(x,t)$ at $x=0$ to make the initial-value problem (1.1) have a unique solution. As one of possibilities we may chose the Neumann boundary condition

$$\frac{\partial \phi}{\partial x}(0,t) = 0\,. \tag{1.4}$$

Then the solution of (1.1) is given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K_N(x,y;t)\phi(y,0) \tag{1.5}$$

with the corresponding propagator

$$K_N(x,y;t) = K(x,y;t) + K(-x,y;t)\,. \tag{1.6}$$

The physical meaning of the propagator $K_N(x,y;t)$ is obvious; the first term $K(x,y;t)$ represents propagation of a wave from y to x while the second term $K(-x,y;t)$ represents propagation of a wave from y to $-x$, which is the mirror image of x. Thus the Neumann propagator $K_N(x,y;t)$ is a superposition of the direct wave with the reflected wave.

As an alternative choice we may impose the Dirichlet boundary condition

$$\phi(0,t) = 0\,. \tag{1.7}$$

Then the solution of (1.1) is given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K_D(x,y;t)\phi(y,0) \tag{1.8}$$

with the corresponding propagator

$$K_D(x,y;t) = K(x,y;t) - K(-x,y;t)\,. \tag{1.9}$$

Thus the Dirichlet propagator $K_D(x,y;t)$ is also a superposition of the direct wave with the reflected wave but reflection changes the sign of the wave.

The half line $\mathbb{R}_{\geq 0}$ can be regarded as an orbifold $\mathbb{R}/\mathbb{Z}_2$. In the above discussion we have assumed the existence of the propagator $K(x,y;t)$ in $\mathbb{R}$ and constructed the propagators in $\mathbb{R}/\mathbb{Z}_2$ from $K(x,y;t)$. There are two inequivalent propagators; the Neumann propagator $K_N(x,y;t)$ obeys the trivial representation of $\mathbb{Z}_2$ whereas the Dirichlet propagator $K_D(x,y;t)$ obeys the defining representation of $\mathbb{Z}_2 = \{+1,-1\}$.

Now a question arises; how is a propagator in a general orbifold M/G constructed? Here M is a Riemannian manifold and G is a compact Lie group that acts on M by isometries. Such an example is easily found; we may take $M = \mathbb{S}^2$ and $G = \mathbb{U}(1)$. Then the quotient space is $M/G = [-1,1]$, which has two boundary points.

Let us turn to another aspect of the propagator, namely, the path-integral expression of the propagator. For the general Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\phi(x,t) = H\phi(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x,t) + V(x)\phi(x,t)\,, \quad x \in \mathbb{R}\,, \tag{1.10}$$

its solution is formally given by

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, K(x,y;t)\phi(y,0)\,. \tag{1.11}$$

The propagator satisfies the composition property

$$K(x'',x;t+t') = \int_{-\infty}^{\infty} \mathrm{d}x'\, K(x'',x';t')K(x',x;t)\,. \tag{1.12}$$

By dividing the time interval $[0,t]$ into short intervals we get

$$K(x_N,x_0;t) = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \mathrm{d}x_{N-1}\cdots \mathrm{d}x_1 K(x_N,x_{N-1};\epsilon)\cdots K(x_1,x_0;\epsilon) \tag{1.13}$$

with $t = N\epsilon$. For a short distance and a short time-interval the propagator asymptotically behaves as

$$K(x+\Delta x,x;\Delta t) \sim \frac{1}{\sqrt{2\pi\mathrm{i}\Delta t}}\exp\left[\frac{\mathrm{i}}{2}\left(\frac{\Delta x}{\Delta t}\right)^2\Delta t - \mathrm{i}V(x)\Delta t\right]. \tag{1.14}$$

Then "the limit $N \to \infty$" gives an infinite-multiplied integration, which is called the path integral,

$$K(x',x;t) = \int_x^{x'} \mathcal{D}x\, \mathrm{e}^{\mathrm{i}\int L\mathrm{d}s} = \int_x^{x'} \mathcal{D}x \exp\left[\mathrm{i}\int_0^t \mathrm{d}s\left(\frac{1}{2}\dot{x}(s)^2 - V(x(s))\right)\right]. \tag{1.15}$$

In a rigorous sense, the limit $N \to \infty$ does not exists but physicists use this expression for convenience. The philosophy of the path integral can be symbolically written as

$$\text{propagation of the wave} = \sum_{\text{trajectories}} \text{motion of the particle}\,. \tag{1.16}$$

We can construct the path integral on the half line $\mathbb{R}_{\geq 0} = \mathbb{R}/\mathbb{Z}_2$ as well:

$$K_N(x', x; t) = \sum_{n=0}^{\infty} \int_x^{x'} \mathcal{D}x\, \mathrm{e}^{\mathrm{i}\int L\,\mathrm{d}s}\,, \tag{1.17}$$

$$K_D(x', x; t) = \sum_{n=0}^{\infty} (-1)^n \int_x^{x'} \mathcal{D}x\, \mathrm{e}^{\mathrm{i}\int L\,\mathrm{d}s}\,, \tag{1.18}$$

where the summations are taken with respect to the number of reflections of the trajectory at the boundary $x = 0$.

Now another question arises; what is the definition of path integrals on a general orbifold M/G? Our main concerns are propagators and path integrals in M/G.

2. Reduction of Quantum System

When a quantum system has a symmetry, it is decomposed into a family of quantum systems that are defined in the subspaces of the original. Here we review the reduction method [5] of quantum system.

A quantum system $(\mathcal{H}, H)$ is defined by a pair of a Hilbert space $\mathcal{H}$ and a Hamiltonian H, which is a self-adjoint operator on $\mathcal{H}$. The symmetry of the quantum system is specified by (G, T), where G is a compact Lie group and T is a unitary representation of G over $\mathcal{H}$. The symmetry implies that $T(g)H = HT(g)$ for all $g \in G$. The compact group G is equipped with the normalized invariant measure $\mathrm{d}g$.

To decompose $(\mathcal{H}, H)$ into a family of reduced quantum systems, we introduce $(\mathcal{H}^\chi, \rho^\chi)$, where $\mathcal{H}^\chi$ is a finite dimensional Hilbert space of the dimensions $d^\chi = \dim \mathcal{H}^\chi$. Besides, ρ^χ is an irreducible unitary representation of G over $\mathcal{H}^\chi$. The set $\{\chi\}$ labels all the inequivalent representations. For each $g \in G$, $\rho^\chi(g) \otimes T(g)$ acts on $\mathcal{H}^\chi \otimes \mathcal{H}$ and defines the tensor product representation. The **reduced Hilbert space** is defined as the subspace of the invariant vectors of $\mathcal{H}^\chi \otimes \mathcal{H}$,

$$(\mathcal{H}^\chi \otimes \mathcal{H})^G := \{\psi \in \mathcal{H}^\chi \otimes \mathcal{H}\,;\, \forall h \in G,\, (\rho^\chi(h) \otimes T(h))\psi = \psi\}\,. \tag{2.1}$$

Let the set $\{e_1^\chi, \ldots, e_d^\chi\}$ be an orthonormal basis of $\mathcal{H}^\chi$. Then the *reduction operator* $S_i^\chi \colon \mathcal{H} \to (\mathcal{H}^\chi \otimes \mathcal{H})^G$ is defined by

$$f \in \mathcal{H} \mapsto S_i^\chi f := \sqrt{d_\chi} \int_G \mathrm{d}g \, (\rho^\chi(g) e_i^\chi) \otimes (T(g) f) \,. \tag{2.2}$$

Theorem 2.1. *S_i^χ is a partial isometry. Namely, $(S_i^\chi)^* S_i^\chi$ is an orthogonal projection operator acting on $\mathcal{H}$ while $S_i^\chi (S_i^\chi)^*$ is the identity operator on $(\mathcal{H}^\chi \otimes \mathcal{H})^G$.*

Theorem 2.2. *The family of the projections $\{(S_i^\chi)^* S_i^\chi\}$ forms a resolution of the identity as*

$$\sum_{\chi,i} (S_i^\chi)^* S_i^\chi = I_{\mathcal{H}} \,. \tag{2.3}$$

Hence, the Hilbert space is decomposed as

$$\mathcal{H} = \bigoplus_{\chi,i} \mathrm{Im}(S_i^\chi)^* S_i^\chi \cong \bigoplus_{\chi,i} (\mathcal{H}^\chi \otimes \mathcal{H})^G \tag{2.4}$$

and this decomposition is compatible with the Hamiltonian action. Namely, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{S_i^\chi} & (\mathcal{H}^\chi \otimes \mathcal{H})^G \\ {\scriptstyle H}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id} \otimes H} \\ \mathcal{H} & \xrightarrow{S_i^\chi} & (\mathcal{H}^\chi \otimes \mathcal{H})^G \end{array} \tag{2.5}$$

*Then $((\mathcal{H}^\chi \otimes \mathcal{H})^G, \mathrm{Id} \otimes H)$ defines a **reduced quantum system**.*

The projection $P^\chi \colon \mathcal{H}^\chi \otimes \mathcal{H} \to (\mathcal{H}^\chi \otimes \mathcal{H})^G$ onto the reduced space is defined by

$$P^\chi := \int_G \mathrm{d}g \, \rho^\chi(g) \otimes T(g) \,. \tag{2.6}$$

The **reduced time-evolution operator** of the reduced system is

$$U^\chi := P^\chi (\mathrm{Id} \otimes \mathrm{e}^{-\mathrm{i}Ht}) \,. \tag{2.7}$$

Theorems 2.1 and 2.2 are easily proved by an application of the Peter–Weyl theorem, which states that the set of the matrix elements of irreducible unitary representations $\{\sqrt{d_\chi}\, \rho_{ij}^\chi(g)\}_{\chi,i,j}$ forms a complete orthonormal set of $L_2(G)$. Our main purpose is to give a path-integral expression to the time-evolution operator U^χ. To describe it we need to introduce some related notions.

Assume that the base space M is equipped with the measure $\mathrm{d}x$. Then the space of the square-integrable functions $L_2(M)$ becomes a Hilbert space $\mathcal{H}$. Moreover, assume that the compact Lie group G acts on M preserving the measure $\mathrm{d}x$. Then $g \in G$ is represented by the unitary operator $T(g)$ on $f \in L_2(M)$ by

$$(T(g)f)(x) := f(g^{-1}x)\,. \tag{2.8}$$

Let $p\colon M \to Q = M/G$ be the canonical projection map. Then a measure $\mathrm{d}q$ of $Q = M/G$ is induced by the following way. Let $\phi(q)$ be a function on Q such that $\phi(p(x))$ is a measurable function on M. The induced measure $\mathrm{d}q$ of Q is then defined by

$$\int_Q \mathrm{d}q\,\phi(q) := \int_M \mathrm{d}x\,\phi(p(x))\,. \tag{2.9}$$

On the other hand, suppose that the time-evolution operator $\mathbb{U}(t) := \mathrm{e}^{-\mathrm{i}Ht}$ is expressed in terms of an integral kernel $K\colon M \times M \times \mathbb{R}_{>0} \to \mathbb{C}$ as

$$(\mathbb{U}(t)f)(x) = \int_M \mathrm{d}y\,K(x,y;t)f(y) \tag{2.10}$$

for any $f(x) \in L_2(M)$.

Let us turn to the reduced Hilbert space (2.1) and characterize it for the case $\mathcal{H} = L_2(M)$. A vector $\psi \in \mathcal{H}^\chi \otimes L_2(M)$ can be identified with a measurable map $\psi\colon M \to \mathcal{H}^\chi$. The tensor product $\rho^\chi(g) \otimes T(g)$ acts on ψ as

$$((\rho^\chi(g) \otimes T(g))\psi)(x) = \rho^\chi(g)\psi(g^{-1}x)\,, \qquad g \in G \tag{2.11}$$

via the definition (2.8). The definition (2.1) of the invariant vector $\psi \in (\mathcal{H}^\chi \otimes L_2(M))^G$ implies

$$((\rho^\chi(g) \otimes T(g))\psi)(x) = \rho^\chi(g)\psi(g^{-1}x) = \psi(x)\,, \tag{2.12}$$

which is equivalent to

$$\psi(gx) = \rho^\chi(g)\psi(x)\,. \tag{2.13}$$

A function $\psi\colon M \to \mathcal{H}^\chi$ satisfying the above property is called an **equivariant function**. Hence the reduced Hilbert space is identified with the space of the equivariant functions $L_2(M,\mathcal{H}^\chi)^G$.

The projection operator $P^\chi\colon L_2(M;\mathcal{H}^\chi) \to L_2(M,\mathcal{H}^\chi)^G$, is now given by

$$(P^\chi\psi)(x) = \int_G \mathrm{d}g\,\rho^\chi(g)\psi(g^{-1}x)\,. \tag{2.14}$$

From (2.7–2.10) and (2.14) the reduced time-evolution operator is given by

$$(U^\chi(t)\psi)(x) = \int_G \mathrm{d}g \int_M \mathrm{d}y\, \rho^\chi(g) K(g^{-1}x, y; t)\psi(y) \tag{2.15}$$

and thus the corresponding **reduced propagator** is $K^\chi\colon M \times M \times \mathbb{R}_{>0} \to \operatorname{End}\mathcal{H}^\chi$ is defined by

$$K^\chi(x, y; t) := \int_G \mathrm{d}g\, \rho^\chi(g) K(g^{-1}x, y; t)\,. \tag{2.16}$$

Our aim is to express the reduced propagator in terms of path integrals.

3. Stratification Geometry

To write down a concrete form of the path integral we need to equip the base space M with a Riemannian structure. Namely, now we assume that M is a differential manifold equipped with a Riemannian metric g_M and that the Lie group G acts on M preserving the metric g_M. Then the volume form induced from the metric defines an invariant measure $\mathrm{d}x$ of M. We do *not* assume that the action of G on M is free. Therefore $p\colon M \to M/G$ is not necessarily a principal bundle.

For each point $x \in M$, $G_x := \{g \in G\,;\, gx = x\}$ is called the **isotropy group** of x and $\mathcal{O}_x := \{gx \,|\, g \in G\}$ is the **orbit** through x. It is easy to see that $\mathcal{O}_x \cong G/G_x$. Note that the dimensions of the orbit $\mathcal{O}_x$ can change suddenly when the point $x \in M$ is moved. The subspace of the tangent space T_xM, $V_x := T_x\mathcal{O}_x$, is called the **vertical subspace** and its orthogonal complement $H_x := (V_x)^\perp$ is called the **horizontal subspace**. $P_V\colon T_xM \to V_x$ is the **vertical projection** while $P_H\colon T_xM \to H_x$ is the **horizontal projection**. A curve in M whose tangent vector always lies in the horizontal subspace is called a **horizontal curve**. Although these terms have been introduced in the theory of principal fiber bundle, we use them for a more general manifold that admits group action.

Let $\mathfrak{g}$ denote the Lie algebra of the group G. For each $x \in M$, $\mathfrak{g}_x$ is the Lie subalgebra of the isotropy group G_x. The group action $G \times M \to M$ induces infinitesimal transformations $\mathfrak{g} \times M \to TM$ by differentiation. The induced linear map $\theta_x\colon \mathfrak{g} \to T_xM$ has $\ker\theta_x = \mathfrak{g}_x$ and $\Im\theta_x = V_x$. Then it defines an isomorphism $\tilde{\theta}_x\colon \mathfrak{g}/\mathfrak{g}_x \to V_x$. Now we define the **stratified connection form** ω by

$$\omega_x := (\tilde{\theta}_x)^{-1} \circ P_V : T_xM \to \mathfrak{g}/\mathfrak{g}_x\,. \tag{3.1}$$

Actually ω is not smooth over the whole M but it is smooth on each stratum of M.

4. Reduction of Path Integral

The Riemannian structure (M, g_M) defines the Laplacian Δ_M. Suppose that $V\colon M \to \mathbb{R}$ is a potential function such that $V(gx) = V(x)$ for all $x \in M$, $g \in G$. Then the Hamiltonian $H = \frac{1}{2}\Delta_M + V(x)$, which acts on $L_2(M)$, commutes with the action of G, which is defined in (2.8). Let us assume that the path integral in M is formally given by

$$K(x', x; t) = \int_x^{x'} \mathcal{D}x \exp\left[\mathrm{i}\int_0^t \mathrm{d}s \left(\frac{1}{2}\|\dot{x}(s)\|^2 - V(x(s))\right)\right]. \tag{4.1}$$

Now we repeat our question; what is the path-integral expression for the reduced propagator (2.16) on $Q = M/G$? The answer is our main result which is given below.

Theorem 4.1. *The reduced path integral on $Q = M/G$ is*

$$\begin{aligned} K^\chi(x', x; t) = &\int_q^{q'} \mathcal{D}q\, \rho^\chi(\gamma)\rho^\chi_*\left(\mathcal{P}\exp\left[-\frac{\mathrm{i}}{2}\int_0^t \mathrm{d}s\, \Lambda(\tilde{q}(s))\right]\right) \\ &\times \exp\left[\mathrm{i}\int_0^t \mathrm{d}s \left(\frac{1}{2}\|\dot{q}(s)\|^2 - V(q(s))\right)\right]. \end{aligned} \tag{4.2}$$

To read the above equation we need explanation of the symbols. The canonical projection map $p\colon M \to Q = M/G$ induces the metric g_Q of Q by asserting that the map p is a stratified Riemannian submersion. For $x, x' \in M$ we put $q = p(x)$ and $q' = p(x')$. The map $q\colon [0,t] \to Q$ is a curve connecting $q = q(0)$ and $q' = q(t)$. The map $\tilde{q}\colon [0,t] \to M$ is a horizontal curve such that $\tilde{q}(0) = x$ and $p(\tilde{q}(s)) = q(s)$ for $s \in [0,t]$. The element $\gamma \in G$ is a holonomy defined by $x' = \gamma \cdot \tilde{q}(t)$.

To describe the symbol Λ, which is called the **rotational energy operator**, we need more explanation. The metric $g_M : TM \otimes TM \to \mathbb{R}$ defines an isomorphism $\hat{g}_M : TM \to T^*M$. Then its inverse map $\hat{g}_M^{-1} : T^*M \to TM$ defines a symmetric tensor field $g_M^{-1} : M \to TM \otimes TM$. Thus combining it with the stratified connection $\omega_x : T_xM \to \mathfrak{g}/\mathfrak{g}_x$ we define the rotational energy operator by

$$\Lambda(x) := -(\omega_x \otimes \omega_x) \circ g_M^{-1}(x) \in (\mathfrak{g}/\mathfrak{g}_x) \otimes (\mathfrak{g}/\mathfrak{g}_x). \tag{4.3}$$

The unitary representation ρ^χ of the group G in $\mathcal{H}^\chi$ induces the representation ρ^χ_* of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then we have $\rho^\chi_*(\Lambda(x)) \in \operatorname{End}\mathcal{H}^\chi$. Moreover,

$$\lambda(\tau) = \rho_*^\chi\left(\mathcal{P}\exp\left[-\frac{\mathrm{i}}{2}\int_0^\tau \mathrm{d}s\,\Lambda(\tilde{q}(s))\right]\right) \in \operatorname{End}\mathcal{H}^\chi \tag{4.4}$$

is defined as a solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\lambda(\tau) = -\frac{\mathrm{i}}{2}\rho_*^\chi\left(\Lambda(\tilde{q}(\tau))\right)\lambda(\tau)\,, \qquad \lambda(0) = I \in \operatorname{End}\mathcal{H}^\chi\,. \tag{4.5}$$

Now we can read off the physical meaning of the reduced path integral (4.2). The path integral is expressed as a product of three factors:

i) the rotational energy amplitude $\exp[-\frac{\mathrm{i}}{2}\int_0^t \mathrm{d}s\,\Lambda(\tilde{q}(s))]$, which represents motion of the particle along the vertical directions of $p\colon M \to M/G$;
ii) the vibrational energy amplitude $\exp[\mathrm{i}\int_0^t \mathrm{d}s\,(\frac{1}{2}\|\dot{q}(s)\|^2 - V(q(s)))]$, which represents motion of the particle along the horizontal directions;
iii) the holonomy factor γ, which is caused by non-integrability of the horizontal distributions.

Here we give the outline of the proof of the main Theorem 4.1. For the detail see the reference [6]. Essentially, it is only a matter of calculation; from the path integral on M (4.1)

$$K(x', x; t) = \int_x^{x'} \mathcal{D}x\, \mathrm{e}^{\mathrm{i}I[x]}\,, \qquad I[x] = \int_0^t \mathrm{d}s\,\left(\frac{1}{2}\|\dot{x}(s)\|^2 - V(x(s))\right) \tag{4.6}$$

with the reduction procedure (2.16) we get

$$\begin{aligned} K^\chi(x', x; t) &:= \int_G \mathrm{d}h\,\rho^\chi(h) K(h^{-1}x', x; t) = \int_G \mathrm{d}h\,\rho^\chi(h) \int_x^{h^{-1}x'} \mathcal{D}x\,\mathrm{e}^{\mathrm{i}I[x]} \\ &= \int_G \mathrm{d}h\,\rho^\chi(h) \int_q^{q'} \mathcal{D}q \int_e^{h^{-1}\gamma} \mathcal{D}g\,\mathrm{e}^{\mathrm{i}I[g\tilde{q}]} = \int_q^{q'} \mathcal{D}q \int_G \mathrm{d}h\,\rho^\chi(h) \int_e^{h^{-1}\gamma} \mathcal{D}g\,\mathrm{e}^{\mathrm{i}I[g\tilde{q}]} \\ &= \int_q^{q'} \mathcal{D}q \int_G \mathrm{d}h\,\rho^\chi(\gamma h) \int_e^{h^{-1}} \mathcal{D}g\,\mathrm{e}^{\mathrm{i}I[g\tilde{q}]} \\ &= \int_q^{q'} \mathcal{D}q\rho^\chi(\gamma) \int_G \mathrm{d}h\,\rho^\chi(h) \int_e^{h^{-1}} \mathcal{D}g\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}s\,\frac{1}{2}\|\dot{g}\|^2}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}s\,\{\frac{1}{2}\|\dot{q}\|^2 - V(q)\}} \\ &= \int_q^{q'} \mathcal{D}q\rho^\chi(\gamma)\rho_*^\chi\left(\mathcal{P}\exp\left[-\frac{\mathrm{i}}{2}\int_0^t \mathrm{d}s\,\Lambda(\tilde{q}(s))\right]\right)\mathrm{e}^{\mathrm{i}\int \mathrm{d}s\,\{\frac{1}{2}\|\dot{q}\|^2 - V(q)\}}\,. \end{aligned} \tag{4.7}$$

5. Example

Finally, we show an example of application of our formulation. Let us begin with the plane $M = \mathbb{R}^2$, which has the standard metric $g_M = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. It admits the symmetry action of $G = \mathbb{SO}(2)$. The quotient space is a half line $Q = \mathbb{R}^2/\mathbb{SO}(2) = \mathbb{R}_{\geq 0}$. The invariant potential is a function $V(r)$ only of r.

The group action

$$\mathbb{SO}(2) \times \mathbb{R}^2 \to \mathbb{R}^2; \quad \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.1}$$

induces the action of the Lie algebra

$$\mathfrak{so}(2) \times \mathbb{R}^2 \to T\mathbb{R}^2; \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.2}$$

which defines the vertical distribution

$$\theta\colon \mathfrak{so}(2) \times \mathbb{R}^2 \to T\mathbb{R}^2; \quad \left(\begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \phi \frac{\partial}{\partial\theta}\,. \tag{5.3}$$

Then the stratified connection becomes

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\theta\,. \tag{5.4}$$

In the cotangent space the metric is given as

$$(g_M)^{-1} = \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial\theta} \otimes \frac{\partial}{\partial\theta}\,. \tag{5.5}$$

The rotational energy operator is

$$\Lambda = -(\omega \otimes \omega) \circ (g_M)^{-1} = -\frac{1}{r^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.6}$$

The irreducible unitary representations of $\mathbb{SO}(2)$ are labeled by the integers $n \in \mathbb{Z}$ and defined by

$$\rho_n\colon \mathbb{SO}(2) \to \mathbb{U}(1); \quad \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mapsto \mathrm{e}^{\mathrm{i}n\phi}\,. \tag{5.7}$$

The differential representation of the Lie algebra of $\mathbb{SO}(2)$ is

$$(\rho_n)_*\colon \mathfrak{so}(2) \to \mathfrak{u}(1); \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mapsto \mathrm{i}n\phi\,. \tag{5.8}$$

The rotational energy operator is then represented as

$$(\rho_n)_*(\Lambda) = -\frac{(\mathrm{i}n)^2}{r^2} = \frac{n^2}{r^2} \, . \tag{5.9}$$

Finally the reduced path integral is given by

$$\begin{aligned} K_n(r', \theta', r, \theta; t) = & \int_r^{r'} \mathcal{D}r \, \mathrm{e}^{\mathrm{i}n(\theta' - \theta)} \\ & \times \exp\left[\mathrm{i} \int_0^t \mathrm{d}s \left\{ -\frac{n^2}{2r^2} + \frac{1}{2}\dot{r}^2 - V(r) \right\}\right] . \end{aligned} \tag{5.10}$$

So the effective potential for the radius coordinate r is given by

$$V_{\text{eff}}(r) = V(r) + \frac{n^2}{2r^2} \, , \tag{5.11}$$

where the second term represents the centrifugal force.

Acknowledgements

Thanks are due organizers for the stimulating conference at the beautiful site, Sts. Constantine and Elena in Bulgaria. I wish to thank I. Mladenov especially for his effort to keep touch with me for a long time from a great distance. I have benefited also from discussions with P. Exner and C. Herald. This work is supported by a Grant-in-Aid for Scientific Research (#12047216) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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