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EQUIVARIANT LOCALIZATION AND STATIONARY PHASE

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Abstract. Equivariant cohomology in general and the equivariant localization theorems in particular have taken on a role of increasing significance in theoretical physics of late (see e. g. [3], [4] and [10]). These lectures are an attempt to provide a self-contained and elementary introduction to the Cartan model of equivariant cohomology, a complete proof of the simplest of the localization theorems, and, as an application, a proof of the famous Duistermaat–Heckman theorem on exact stationary phase approximations.

1. Stationary Phase Approximation

We consider a compact, oriented, smooth manifold M of dimension $n = 2k$ and denote by ν a volume form on M. Suppose $H\colon M \to \mathbb{R}$ is a Morse function on M, i. e., a smooth function whose critical points p ($\mathrm{d}H(p) = 0$) are all nondegenerate (this means that the Hessian $\mathcal{H}_p\colon T_p(M) \times T_p(M) \to \mathbb{R}$, defined by $\mathcal{H}_p(V_p, W_p) = V_p(W(H))$, where $V_p, W_p \in T_p(M)$ and W is a vector field on M with $W(p) = W_p$, is a nondegenerate bilinear form). Finally, let T denote some real parameter. We consider the integral

$$\int_M \mathrm{e}^{\mathrm{i}TH}\nu \tag{1.1}$$

and are especially interested in its asymptotic behavior as $T \to \infty$. The Stationary Phase Theorem (Chapter I of [6]) asserts roughly that, for large T, the dominant contributions to such an integral come from the critical points of H.

More precisely, one has

$$\int_M e^{iTH}\nu = \sum_{\substack{p\in M \\ dH(p)=0}} \left(\frac{2\pi}{T}\right)^k e^{\pi i(\operatorname{Sgn}\mathcal{H}_p)/4} |\det \mathcal{H}_p(e_i, e_j)|^{-\frac{1}{2}} e^{iTH(p)} + O(T^{-(k+1)}) \tag{1.2}$$

where $\operatorname{Sgn}\mathcal{H}_p$ is the signature (number of positive eigenvalues minus the number of negative eigenvalues) of any matrix representing $\mathcal{H}_p$, $\{e_1, \ldots, e_{2k}\}$ is a basis for $T_p(M)$ with $\nu_p(e_1, \ldots, e_{2k}) = 1$, and $O(T^{-(k+1)})$ stands for terms which, in modulus, are bounded by M/T^{k+1} for some constant M and all T outside some compact set in $\mathbb{R}$. The terms preceding $O(T^{-(k+1)})$ on the right-hand side of (1.2) constitute the **stationary phase approximation** of the integral. These terms arise in the proof of (1.2) from writing H near p as a quadratic function in some coordinates (that this is possible is the content of the **Morse Lemma**) and computing directly the resulting Gaussian integral. It follows from the Morse Lemma that the critical points of a Morse function are isolated. Since M is compact, H can have only finitely many critical points so the sum in (1.2) is necessarily finite.

Let us write out a simple example used by Witten [12] to illustrate the phenomenon we wish to study. We take M to be the 2-sphere S^2 in $\mathbb{R}^3$ and let ν be the usual metric volume form on S^2 (this is the restriction to S^2 of the 2-form $x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$ on $\mathbb{R}^3$). Let $H\colon S^2 \to \mathbb{R}$ be the "height function" ($H(x, y, z) = z$ for any $(x, y, z) \in S^2$). We claim that the critical points of H are the north and south poles, i. e., $N = (0, 0, 1)$ and $S = (0, 0, -1)$, and that both are nondegenerate so H is a Morse function on S^2. For example, on $z > 0$ in S^2, $(x, y, z) \to (x, y)$ is a chart with inverse $(x, y) \to (x, y, (1 - x^2 - y^2)^{\frac{1}{2}})$ and, in these coordinates, $H(x, y) = (1 - x^2 - y^2)^{\frac{1}{2}}$ so $dH(x, y) = -(1 - x^2 - y^2)^{-\frac{1}{2}}(x\,dx + y\,dy)$. Thus, the only critical point in $z > 0$ occurs when $(x, y) = (0, 0)$, i. e., at $N(0, 0, 1)$. Furthermore, the Hessian (which, in any coordinate system, is represented by the matrix of second order partial derivatives) is given by

$$-(1 - x^2 - y^2)^{-\frac{3}{2}} \begin{pmatrix} 1 - y^2 & xy \\ xy & 1 - x^2 \end{pmatrix}.$$

Thus, at $(x, y) = (0, 0)$ we obtain $\mathcal{H}_N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and this is, indeed, nonsingular. The region $z < 0$ on S^2 is, of course, handled in the same way and projecting onto other coordinate planes shows that there are no critical points with $z = 0$.

Now we shall write out the stationary phase approximation for the integral

$$\int_{S^2} \mathrm{e}^{\mathrm{i}Tz}\nu\,. \tag{1.3}$$

From (1.2) with $k = 1$ and $p = N, S$, this is

$$\begin{aligned}\left(\frac{2\pi}{T}\right) \mathrm{e}^{\pi\mathrm{i}(\operatorname{Sgn}\mathcal{H}_N)/4} |\det \mathcal{H}_N(e_i, e_j)|^{-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}Tz(N)} \\ + \left(\frac{2\pi}{T}\right) \mathrm{e}^{\pi\mathrm{i}(\operatorname{Sgn}\mathcal{H}_S)/4} |\det \mathcal{H}_S(e_i, e_j)|^{-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}Tz(S)}.\end{aligned} \tag{1.4}$$

Now, $\operatorname{Sgn}\mathcal{H}_N = \operatorname{Sgn}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = -2$. Next note that evaluating $x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z + z\,\mathrm{d}x \wedge \mathrm{d}y$ at $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ gives z so, at N, $\nu_N(\frac{\partial}{\partial x}(N), \frac{\partial}{\partial y}(N)) = 1$. Thus, $\{e_1, e_2\} = \{\frac{\partial}{\partial x}(N), \frac{\partial}{\partial y}(N)\}$ is a basis of the required type for $T_N(S^2)$ so

$$|\det \mathcal{H}_N(e_i, e_j)|^{-\frac{1}{2}} = \left|\det\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right|^{-\frac{1}{2}} = 1\,.$$

Similarly, $\operatorname{Sgn}\mathcal{H}_S = 2$ and $|\det \mathcal{H}_S(e_i, e_j)|^{-\frac{1}{2}} = 1$. Substituting all of this into (1.4) gives, for the stationary phase approximation to (1.3),

$$\frac{2\pi\mathrm{i}}{T}\left(\mathrm{e}^{-\mathrm{i}T} - \mathrm{e}^{\mathrm{i}T}\right) = 4\pi\left(\frac{\sin T}{T}\right). \tag{1.5}$$

Next we observe that the integral (1.3) is actually easy to compute exactly. Let $\iota\colon S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion map so that $\nu = \iota^*(x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z + z\,\mathrm{d}x \wedge \mathrm{d}y)$. Define an orientation preserving diffeomorphism φ of $(0,\pi) \times (-\pi,\pi)$ into S^2 by

$$(\iota \circ \varphi)(\phi, \theta) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi).$$

The image of this map covers all of S^2 except a set of measure zero. A simple computation shows that $\varphi^*\nu = \sin\phi\,\mathrm{d}\phi \wedge \mathrm{d}\theta$ and so

$$\varphi^*\left(\mathrm{e}^{\mathrm{i}Tz}\nu\right) = \mathrm{e}^{\mathrm{i}T\cos\phi}\sin\phi\,\mathrm{d}\phi \wedge \mathrm{d}\theta\,.$$

Denoting by $\mathrm{d}m$ Lebesgue measure on the plane we therefore have

$$\begin{aligned}\int_{S^2} \mathrm{e}^{\mathrm{i}Tz}\nu &= \int_{(0,\pi)\times(-\pi,\pi)} \mathrm{e}^{\mathrm{i}T\cos\phi}\sin\phi\,\mathrm{d}\phi \wedge \mathrm{d}\theta \\ &= \int_{[0,\pi]\times[-\pi,\pi]} \mathrm{e}^{\mathrm{i}T\cos\phi}\sin\phi\,\mathrm{d}m\end{aligned}$$

$$= \int_{-\pi}^{\pi} \int_{0}^{\pi} \mathrm{e}^{\mathrm{i}T\cos\phi} \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta$$

$$= -\frac{2\pi}{\mathrm{i}T}\left[\mathrm{e}^{\mathrm{i}T\cos\phi}\right]_0^{\pi}$$

$$= \frac{2\pi\mathrm{i}}{T}\left[\mathrm{e}^{-\mathrm{i}T} - \mathrm{e}^{\mathrm{i}T}\right] = 4\pi\left(\frac{\sin T}{T}\right).$$

We find then that, in this particular case, the stationary phase approximation (1.5) to the integral (1.3) actually gives the exact value of the integral. Our goal now is to uncover the underlying features of this example which account for this exactness of the stationary phase approximation.

2. Hamiltonian Actions on Symplectic Manifolds

We begin with a few observations on the example which concluded the preceding section. Note that the volume form ν on S^2 is also a symplectic form, i. e., a closed, nondegenerate 2-form. Indeed, any volume form ν on any orientable surface is a symplectic form (it is closed because it is a 2-form on a 2-dimensional manifold and nondegenerate because, at each point, an oriented basis $\{e_1, e_2\}$ for the tangent space satisfies $\nu(e_1, e_2) > 0$ so if $v = v^1 e_1 + v^2 e_2 \neq 0$ (say, $v^1 \neq 0$), then $\nu(v, e_2) = v^1 \nu(e_1, e_2) \neq 0$). When thinking of ν as a symplectic form on S^2 we will denote it ω. Now, the height function H, like any smooth, real-valued function on the symplectic manifold (S^2, ω), determines a corresponding **Hamiltonian vector field** V_H on S^2. This is defined to be the unique vector field on S^2 satisfying

$$\iota_{V_H}\omega = \mathrm{d}H \tag{2.1}$$

where ι_{V_H} is interior multiplication by V_H (so that, for any vector field W on S^2, $\mathrm{d}H(W) = \omega(V_H, W)$). We claim that if $\frac{\partial}{\partial\theta}$ is the θ-coordinate velocity field of the spherical coordinate chart on S^2 (taken to be zero at N and S), then

$$V_H = \frac{\partial}{\partial\theta}.$$

First note that (2.1) and the nondegeneracy of ω imply that V_H must vanish at the critical points N and S of H and so it agrees with $\frac{\partial}{\partial\theta}$ there. At any other point, $H(\phi, \theta) = \cos\phi$ so $\mathrm{d}H(\phi, \theta) = -\sin\phi \, \mathrm{d}\phi$. Since $\omega = \sin\phi \, \mathrm{d}\phi \wedge \mathrm{d}\theta$ (here and henceforth we adopt the time-honored custom of omitting references to the diffeomorphism φ whenever it is convenient to do so), we have

$$\begin{aligned}\iota_{\frac{\partial}{\partial\theta}}\,\omega &= \iota_{\frac{\partial}{\partial\theta}}\,(\sin\phi\,\mathrm{d}\phi\otimes\mathrm{d}\theta-\sin\phi\,\mathrm{d}\theta\otimes\mathrm{d}\phi)\\ &= \sin\phi\left(\mathrm{d}\phi\left(\frac{\partial}{\partial\theta}\right)\right)\mathrm{d}\theta-\sin\phi\left(\mathrm{d}\theta\left(\frac{\partial}{\partial\theta}\right)\right)\mathrm{d}\phi\\ &= -\sin\phi\,\mathrm{d}\phi=\mathrm{d}H\end{aligned}$$

as required. The integral curves of $V_H = \frac{\partial}{\partial\theta}$ are then easily found. They are points at N and S and elsewhere they are "horizontal" circles traversed at speed one. The unique one through $p=\varphi(\phi,\theta)$ at time $t=0$ is

$$\alpha_p\,(t)=(\sin\phi\cos{(\theta+t)},\ \sin\phi\sin{(\theta+t)},\ \cos\phi)$$

(we shall also omit references to the inclusion $\iota\colon S^2\hookrightarrow\mathbb{R}^3$). These integral curves are therefore periodic with period 2π. The flow

$$\begin{aligned}\alpha\colon S^2\times\mathbb{R}&\to S^2\\ \alpha(p,t)&=\alpha_p(t)\end{aligned}$$

is therefore also periodic in t. Finally, recall that any symplectic manifold (M^{2k},ω) has a canonical orientation (volume form) ν_ω called the **Liouville form** and defined by

$$\nu_\omega=\frac{1}{k!}\omega\wedge\overset{k}{\cdots}\wedge\omega=\frac{1}{k!}\omega^k.$$

For $k=1$ this is just ω so, in our example on S^2, ν, ω and ν_ω are all the same. Duistermaat and Heckman [5] have shown that the exactness of the stationary phase approximation of $\int_{S^2}\mathrm{e}^{\mathrm{i}Tz}\nu$ is a consequence of the fact that the Hamiltonian vector field of the height function on S^2 has a periodic flow. More generally, we have

Theorem 2.1. (Duistermaat–Heckman) *Let M be a compact manifold of dimension $n=2k$ with symplectic form ω and oriented by the corresponding Liouville form $\nu_\omega=\frac{1}{k!}\omega^k$. Let $H\in C^\infty(M)$ be a Morse function on M and V_H its Hamiltonian vector field ($\iota_{V_H}\omega=\mathrm{d}H$). If the flow of V_H is periodic, then, for any real number $T>0$,*

$$\int_M \mathrm{e}^{\mathrm{i}TH}\nu_\omega=\sum_{\substack{p\in M\\ \mathrm{d}H(p)=0}}\left(\frac{2\pi}{T}\right)^k \mathrm{e}^{\pi\mathrm{i}(\mathrm{Sgn}\,\mathcal{H}_p)/4}\left|\det\mathcal{H}_p(e_i,e_j)\right|^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}TH(p)}$$

where $\mathcal{H}_p\colon T_p(M)\times T_p(M)\to\mathbb{R}$ is the Hessian of H at p and $\{e_1,\dots,e_{2k}\}$ is a basis for $T_p(M)$ with $\nu_\omega(e_1,\dots,e_{2k})=1$.

Note that the set of critical points must be nonempty since M is compact and so H must achieve maximum and minimum values. We intend to provide a complete proof of this result, but will proceed toward it in a rather roundabout fashion. First we return to our example on S^2 and isolate a group action which suggests a more general perspective on the Duistermaat–Heckman Theorem. We formulate this new perspective as a Generalized Duistermaat–Heckman Theorem that concerns itself with Hamiltonian actions on symplectic manifolds and show that this new result implies our Theorem 2.1. Still our perspective is not broad enough, however, and we focus our attention on general group actions on manifolds and their associated equivariant cohomologies. In this context we prove the simplest of the so-called Equivariant Localization Theorems and find that it has as a simple consequence the Generalized Duistermaat–Heckman Theorem and therefore also Theorem 2.1.

Let us then consider again the height function H on the symplectic manifold S^2. Since the Hamiltonian vector field V_H has a periodic flow it gives rise to an obvious action of S^1 on S^2 (rotate points of S^2 around the integral curves containing them). Specifically, if $g = \mathrm{e}^{\mathrm{i}T} \in S^1$ and $p = \varphi(\phi, \theta) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi) \in S^2$, then we define

$$\begin{aligned} g \cdot p &= \mathrm{e}^{\mathrm{i}T} \cdot (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi) \\ &= (\sin\phi\cos(\theta + T),\ \sin\phi\sin(\theta + T),\ \cos\phi) \end{aligned}$$

(if $p = N$ or S we define $g \cdot p = p$ for all $g \in S^1$). This clearly defines a (left) action of S^1 on S^2. As usual, we identify the Lie algebra of S^1 with $\mathrm{i}\mathbb{R}$. Each $\xi = \mathrm{i}a$ in the Lie algebra gives rise to an associated vector field $\xi^\#$ on S^2 defined by

$$\xi^\#(p) = \frac{\mathrm{d}}{\mathrm{d}t}(\exp(-t\xi) \cdot p)|_{t=0} \tag{2.2}$$

(the minus sign is introduced here because we take our actions to be on the left and we want $\xi \to \xi^\#$ to be a Lie algebra homomorphism). It is a simple matter to compute $\xi^\#(p)$ explicitly. At $p = N, S$ it is zero and, otherwise,

$$\begin{aligned} \xi^\#(p) &= \frac{\mathrm{d}}{\mathrm{d}t}(\exp(-t\xi) \cdot p)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\mathrm{i}at} \cdot p)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\sin\phi\cos(\theta - at),\ \sin\phi\sin(\theta - at),\ \cos\phi)|_{t=0} \\ &= -a(-\sin\phi\sin\theta,\ \sin\phi\cos\theta,\ 0) \\ &= -a\frac{\partial}{\partial\theta}(p) \end{aligned}$$

$$= -aV_H(p) = V_{-aH}(p)$$

(the last equality is easy to check by verifying that $\mathrm{d}(-aH) = \iota_{-aV_H}\omega$). Thus,

$$\xi = \mathrm{i}a \Longrightarrow \xi^{\#} = V_{-aH} .$$

In particular, every $\xi^{\#}$ is the Hamiltonian vector field of some smooth function on S^2. We can therefore define a map

$$\mu\colon \mathrm{Lie}(S^1) = \mathrm{i}\mathbb{R} \to C^\infty(S^2)$$

by

$$\mu(\xi) = \mu(\mathrm{i}a) = -aH$$

which has the following properties

1. μ is linear.
2. $\xi^{\#}$ is the Hamiltonian vector field on S^2 determined by $\mu(\xi)$.
3. μ is equivariant with the respect to the natural actions of S^1 on $\mathrm{Lie}(S^1)$ and $C^\infty(S^2)$, i. e.,

$$\mu(g \cdot \xi) = g \cdot \mu(\xi) .$$

Remarks: Regarding (3), the natural action of S^1 on $\mathrm{Lie}(S^1)$ is the adjoint action ($g \cdot \xi = g\xi g^{-1}$) which, in this case, is trivial since S^1 is Abelian. Thus, $\mu(g\cdot\xi) = \mu(\xi)$. The action of S^1 on $C^\infty(S^2)$ is defined by $(g\cdot\psi)(p) = \psi(g^{-1}\cdot p)$ so $(g \cdot \mu(\xi))(p) = \mu(\xi)(g^{-1} \cdot p) = (-aH)(g^{-1} \cdot p) = (-aH)(p) = \mu(\xi)(p)$ because H is constant on the orbits. Thus, $g \cdot \mu(\xi) = \mu(\xi) = \mu(g \cdot \xi)$.

Now we abstract these properties of our example and formulate general definitions. Let (M, ω) be a compact symplectic manifold of dimension $n = 2k$ and G a compact Lie group (with Lie algebra $\mathfrak{g}$) that acts smoothly on M on the left (we will write such an action as $\sigma\colon G \times M \to M$ with $\sigma(g,p) = g \cdot p = \sigma_g(p) = \sigma_p(g)$). The action is said to be **Hamiltonian** if there is a map

$$\mu\colon \mathfrak{g} \to C^\infty(M)$$

such that

1. μ is linear.
2. For each $\xi \in \mathfrak{g}$ the vector field $\xi^{\#}$ on M (defined by (2.2)) is the Hamiltonian vector field associated with $\mu(\xi)$, i. e.,

$$\mathrm{d}\mu(\xi) = \iota_{\xi^{\#}}\omega . \tag{2.3}$$

3. μ is equivariant, i. e.,

$$\mu(g \cdot \xi) = g \cdot \mu(\xi)\,.$$

The function $\mu(\xi)$ is called the **symplectic moment** of ξ and one defines the associated **moment map**

$$\Phi\colon M \to \mathfrak{g}^*$$

(where $\mathfrak{g}^*$ is the dual of the vector space $\mathfrak{g}$) by

$$(\Phi(p))(\xi) = (\mu(\xi))(p)\,.$$

Although they will play no role in our story here, these moment maps have many striking and beautiful properties (see, e. g., [1] and [7]).

Notice that it follows at once from (2) and the nondegeneracy of ω that *the critical points of* $\mu(\xi)$ *coincide with the zeros of* $\xi^{\#}$. Moreover, every fixed point of the G-action is, by (2.2), a zero of every $\xi^{\#}$ (and so, a critical point of every $\mu(\xi)$). If $\xi \in \mathfrak{g}$ has the property that $\xi^{\#}$ vanishes *only* at the fixed points of the G-action, then ξ is said to be nondegenerate and, in this case, one can show that $\mu(\xi)$ is necessarily a Morse function (see [7]).

Remark: Before proceeding we must recall that for any action of a compact Lie group G on a manifold M it is always possible to construct a Riemannian metric $\langle\,,\rangle_G$ on M that is G-invariant, i. e., for which the diffeomorphisms $\sigma_g\colon M \to M$ are all isometries. Roughly, this is done by selecting some Riemannian metric $\langle\,,\rangle$ on M and, at each point $p \in M$, averaging over G relative to some invariant measure $\mathrm{d}G$ on G, i. e., defining, for all $V_p, W_p \in T_p(M)$,

$$\langle V_p, W_p\rangle_G = \int\limits_G \langle(\sigma_g)_{*p}(V_p), (\sigma_g)_{*p}(W_p)\rangle\,\mathrm{d}G\,.$$

We assume that some such invariant Riemannian metric $\langle\,,\rangle_G$ on M has been selected and note that any vector field $\xi^{\#}$ defined by (2.2) for some $\xi \in \mathfrak{g}$ is then necessarily a Killing vector field for $\langle\,,\rangle_G$, i. e.,

$$\mathcal{L}_{\xi^{\#}}\langle\,,\rangle_G = 0$$

where $\mathcal{L}_{\xi^{\#}}$ denotes the Lie derivative with respect to $\xi^{\#}$. This last condition can be written equivalently as

$$\xi^{\#}\left(\langle V, W\rangle_G\right) = \langle[\xi^{\#}, V], W\rangle_G + \left\langle V, [\xi^{\#}, W]\right\rangle_G \tag{2.4}$$

for all vector fields V and W on M.

Now, fix a $\xi \in \mathfrak{g}$. We denote by $Z(\xi^{\#})$ the set of zeros of the vector field $\xi^{\#}$. For each $p \in Z(\xi^{\#})$ we define a linear transformation

$$L_p(\xi)\colon T_p(M) \to T_p(M)$$

by

$$L_p(\xi)(V_p) = (\mathcal{L}_{\xi^{\#}} V)_p = [\xi^{\#}, V]_p \tag{2.5}$$

where V is any vector field on M with $V(p) = V_p$. By writing out the definition of the Lie derivative explicitly one obtains the following alternative expression for $L_p(\xi)(V_p)$.

$$L_p(\xi)(V_p) = -\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{\exp(-t\xi)})_{*p}(V_p)\,. \tag{2.6}$$

Note that, since $\xi^{\#}(p) = 0$, $\sigma_{\exp(-t\xi)}(p) = p$ for every t so $(\sigma_{\exp(-t\xi)})_{*p}\colon T_p(M) \to T_p(M)$ and the derivative in (2.6) is computed in the single tangent space $T_p(M)$. We claim that $L_p(\xi)$ is *skew-symmetric* with respect to the inner product on $T_p(M)$ supplied by $\langle\,,\rangle_G$, i. e., that

$$\langle L_p(\xi)(V_p), W_p\rangle_G = -\langle V_p, L_p(\xi)(W_p)\rangle_G \tag{2.7}$$

for all V_p, $W_p \in T_p(M)$. To see this one simply evaluates (2.4) at p and uses the fact that $\xi^{\#}(p) = 0$ and the definition (2.5) of $L_p(\xi)$. Next we will require a lemma which follows from a simple manipulation of well-known identities from differential geometry, but, since we use the result several times, we supply a proof.

Lemma 2.1. *Let H be an arbitrary smooth function on the symplectic manifold (M,ω) and V_H its Hamiltonian vector field ($\iota_{V_H}\omega = \mathrm{d}H$). Suppose $p \in M$ and $V_H(p) = 0$. Then, for any V_p, $W_p \in T_p(M)$,*

$$\mathcal{H}_p(V_p, W_p) = -\omega((\mathcal{L}_{V_H} V)_p, W_p) \tag{2.8}$$

where $\mathcal{H}_p$ is the Hessian of H at p, $\mathcal{L}_{V_H}$ is the Lie derivative with respect to V_H and V is any vector field on M with $V(p) = V_p$.

Proof: By definition, $\mathcal{H}_p(V_p, W_p) = V_p(W(H))$ and

$$W(H) = \mathcal{L}_W H = \mathrm{d}H(W) = (\iota_{V_H}\omega)(W) = (\iota_W \circ \iota_{V_H})(\omega)\,.$$

Now compute

$$\begin{aligned} V(W(H)) &= \mathcal{L}_V(W(H)) = \mathcal{L}_V \circ \iota_W \circ \iota_{V_H}(\omega) \\ &= (\iota_{[V,W]} + \iota_W \circ \mathcal{L}_V) \circ \iota_{V_H}(\omega) \\ &= \iota_{[V,W]} \circ \iota_{V_H}(\omega) + \iota_W \circ \mathcal{L}_V \circ \iota_{V_H}(\omega) \\ &= \iota_{[V,W]} \circ \iota_{V_H}(\omega) + \iota_W \circ \left(\iota_{[V,V_H]} + \iota_{V_H} \circ \mathcal{L}_V\right)(\omega) \\ &= \omega(V_H,[V,W]) - \omega([V_H,V],W) + (\mathcal{L}_V\omega)(V_H,W). \end{aligned}$$

Now, evaluate at p and use $V(p) = 0$ to obtain

$$V_p(W(H)) = 0 - \omega([V_H,V]_p,W_p) + 0$$

i. e.,

$$\mathcal{H}_p(V_p,W_p) = -\omega((\mathcal{L}_{V_H}V)_p,W_p)$$

as required. □

Proposition 2.1. *Let (M,ω) be a symplectic manifold with a Hamiltonian action of the compact Lie group G. Let $\xi \in \mathfrak{g}$ be nondegenerate and $p \in Z(\xi^\#)$. Then $L_p(\xi)\colon T_p(M) \to T_p(M)$ is nonsingular.*

Proof: We apply Lemma 2.2 to $H = \mu(\xi)$. Then $V_H = V_{\mu(\xi)} = \xi^\#$. For any $V_p \in T_p(M)$ we select a vector field V on M with $V(p) = V_p$. Then $(\mathcal{L}_{V_H}V)_p = (\mathcal{L}_{\xi^\#}V)_p = L_p(\xi)(V_p)$. Since ξ is nondegenerate, $\mu(\xi)$ is a Morse function so its Hessian $\mathcal{H}_p$ is nondegenerate. Thus, the equality

$$\mathcal{H}_p(V_p,W_p) = -\omega(L_p(\xi)(V_p),W_p)$$

implies that $L_p(\xi)(V_p)$ cannot be zero unless $V_p = 0$. □

Remark: We will improve this result shortly by showing that if G is any compact Lie group acting on any (not necessarily symplectic) manifold M and if $\xi \in \mathfrak{g}$ has the property that $\xi^\#$ (defined by (2.2)) has only isolated zeros, then each $L_p(\xi)$ (defined by (2.5)) is nonsingular.

Now, let us assume that $\xi \in \mathfrak{g}$ is nondegenerate and $p \in Z(\xi^\#)$. Then $L_p(\xi)$ is nonsingular and skew-symmetric with respect to $\langle\,,\rangle_G$. Thus, we can find a basis $\{e_1,\ldots,e_{2k}\}$ for $T_p(M)$ that is orthonormal with respect to $\langle\,,\rangle_G$ and oriented (with respect to the Liouville form) and relative to which the matrix of $L_p(\xi)$ is of the form

$$\begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_k \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_k & 0 \end{pmatrix} \tag{2.9}$$

with $\lambda_j \in \mathbb{R}\backslash\{0\}$ for $j = 1, \ldots, k$. We define a square root of the determinant of $L_p(\xi)$ by taking

$$[\det(L_p(\xi))]^{\frac{1}{2}} = \lambda_1 \lambda_2 \cdots \lambda_k \,. \tag{2.10}$$

Remark: This is, in fact, the Pfaffian of the matrix (2.9). Although this observation will play no role in what we do here it is crucial in formulating more general localization theorems than the one we will prove since these involve the so-called equivariant Euler class of a certain (equivariant) vector bundle and this is constructed, *a la* Chern–Weil, from the Pfaffian.

With this we are prepared to formulate what we will call the **Generalized Duistermaat–Heckman Theorem**.

Theorem 2.2. *Let (M, ω) be a compact, symplectic manifold of dimension $n = 2k$ with a Hamiltonian action of a compact Lie group G and corresponding symplectic moments given by $\mu \colon \mathfrak{g} \to C^\infty(M)$. Orient M with the Liouville form $\nu_\omega = \frac{1}{k!}\omega^k$ (and assume that a G-invariant Riemannian metric $\langle\,,\rangle_G$ on M has been chosen). If $\xi \in \mathfrak{g}$ is nondegenerate, then*

$$\int_M e^{i\mu(\xi)} \nu_\omega = \sum_{\substack{p \in M \\ \xi^\#(p) = 0}} (2\pi i)^k \, [\det(L_p(\xi))]^{-\frac{1}{2}} \, e^{i\mu(\xi)(p)}. \tag{2.11}$$

Remarks: Since ξ is nondegenerate, $\mu(\xi)$ is Morse and so has (at least two and at most) finitely many critical points. These critical points coincide with the zeros of $\xi^\#$ so the sum in (2.11) is (nonvacuous and) finite. Furthermore, $L_p(\xi)$ is nonsingular so $\det(L_p(\xi)) \neq 0$ and the right-hand side of (2.11) is meaningful.

As we mentioned earlier we shall eventually derive Theorem 2.4 as a consequence of our equivariant localization theorem. Our task for the present is simply to show that Theorem 2.4 implies Theorem 2.1. Thus, we begin with a compact, symplectic manifold (M, ω) of dimension $n = 2k$ and oriented by the the Liouville form ν_ω. We let $H \in C^\infty(M)$ be a Morse function and V_H the corresponding Hamiltonian vector field on M. The assumption of Theorem 2.1 is that the flow of V_H is periodic. By rescaling we may assume that the period is 2π. Now, just as for our example on S^2, this gives rise to a circle action on M with the property that

$$\xi = \mathrm{i}a \in \mathrm{Lie}(S^1) \Longrightarrow \xi^\# = V_{-aH} \,. \tag{2.12}$$

In somewhat more detail, the action moves $p \in M$ along the integral curve of V_H that begins at p so

$$(\mathrm{i}(-1))^\#(p) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t(-\mathrm{i})} \cdot p)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{t\mathrm{i}} \cdot p)|_{t=0} = V_H(p) \,.$$

Moreover, $V_{-aH} = -aV_H$ because $\mathrm{d}(-aH) = -a\,\mathrm{d}H = -a\iota_{V_H}\omega = \iota_{-aV_H}\omega$ so

$$\xi^\#(p) = (\mathrm{i}a)^\#(p) = -a(\mathrm{i}(-1))^\#(p) = -aV_H(p) = V_{-aH}(p) \,.$$

Consequently, the S^1-action is Hamiltonian with symplectic moments given by

$$\mu(\xi) = \mu(\mathrm{i}a) = -aH$$

(equivariance is proved in the same way as for the S^2 example). Next we record a simple, but crucial fact about S^1-actions in general.

Lemma 2.2. *Let M be a smooth manifold and suppose S^1 acts smoothly on M on the left. Then, for any nonzero ξ in the Lie algebra of S^1, the zero set $Z(\xi^\#)$ of the vector field $\xi^\#$ (defined by (2.2)) coincides with the fixed point set of the S^1-action.*

Proof: Since $\xi \neq 0$ and S^1 is 1-dimensional, ξ spans the Lie algebra of S^1, i. e., $\mathrm{Lie}(S^1) = \{-t\xi\,;\, t \in \mathbb{R}\}$. The exponential map of $\mathrm{Lie}(S^1)$ to S^1 is onto so the orbit of any $p \in M$ coincides with $\{\exp(-t\xi)\cdot p\,;\, t \in \mathbb{R}\}$, i. e., with the integral curve of $\xi^\#$ through p. If $\xi^\#(p) = 0$, then this integral curve is a point and therefore the orbit of p is a point, i. e., p is a fixed point for the S^1-action. Since a fixed point is obviously a zero of any $\xi^\#$, the result follows.

Returning to the derivation of Theorem 2.1 from Theorem 2.4 we now have that any nonzero ξ in the Lie algebra of S^1 is nondegenerate. In particular, for any $T > 0$, we may apply Theorem 2.4 to $\xi = \mathrm{i}(-T)$ to obtain

$$\int_M e^{iTH}\nu_\omega = \sum_{\substack{p\in M\\ \xi^{\#}(p)=0}} (2\pi i)^k [\det(L_p(-iT))]^{-\frac{1}{2}}\, e^{iTH(p)}$$
$$= \sum_{\substack{p\in M\\ dH(p)=0}} \left(\frac{2\pi}{T}\right)^k (iT)^k [\det(L_p(-iT))]^{-\frac{1}{2}}\, e^{iTH(p)}.$$

Comparing this with the conclusion of Theorem 2.1 we find that we need only show

$$(iT)^k [\det(L_p(-iT))]^{-\frac{1}{2}} = e^{\pi i(\mathrm{Sgn}\,\mathcal{H}_p)/4} |\det \mathcal{H}_p(e_i, e_j)|^{-\frac{1}{2}} \tag{2.13}$$

to complete the proof (here $\{e_1, \ldots, e_{2k}\}$ is a basis for $T_p(M)$ that satisfies $\frac{1}{k!}(\omega\wedge \overset{k}{\cdots} \wedge\omega)(e_1, \ldots, e_{2k}) = 1$). While largely computational, the proof of (2.13) relies on one nontrivial result so we shall go through it in some detail. First note that

$$L_p(-iT)(V_p) = (\mathcal{L}_{(-iT)^{\#}} V)_p = T(\mathcal{L}_{(i(-1))^{\#}} V)_p = T(\mathcal{L}_{V_H} V)_p.$$

Thus, (2.8) can be written

$$T\mathcal{H}_p(V_p, W_p) = -\omega(L_p(-iT)(V_p), W_p)\,. \tag{2.14}$$

In particular, if $\{e_1, \ldots, e_{2k}\}$ is any basis for $T_p(M)$,

$$T\mathcal{H}_p(e_i, e_j) = -\omega(L_p(-iT)(e_i), e_j)$$

and if we write $L_p(-iT)(e_i) = L_i^l e_l$, then

$$T\mathcal{H}_p(e_i, e_j) = -L_i^l \omega(e_l, e_j)$$

for all $i, j = 1, \ldots, 2k$. As a matrix product this is

$$\begin{pmatrix} T\mathcal{H}_p(e_1, e_1) & \cdots & T\mathcal{H}_p(e_1, e_{2k}) \\ \vdots & & \vdots \\ T\mathcal{H}_p(e_{2k}, e_1) & \cdots & T\mathcal{H}_p(e_{2k}, e_{2k}) \end{pmatrix}$$
$$= \begin{pmatrix} -L_1^1 & \cdots & -L_1^{2k} \\ \vdots & & \vdots \\ -L_{2k}^1 & \cdots & -L_{2k}^{2k} \end{pmatrix} \begin{pmatrix} \omega(e_1, e_1) & \cdots & \omega(e_1, e_{2k}) \\ \vdots & & \vdots \\ \omega(e_{2k}, e_1) & \cdots & \omega(e_{2k}, e_{2k}) \end{pmatrix}. \tag{2.15}$$

□

Now we will make a particular choice of basis. The classical Darboux Theorem guarantees the existence of an oriented, orthonormal basis for $T_p(M)$ relative to which

$$(\omega(e_i,e_j)) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix} \tag{2.16}$$

and $\frac{1}{k!}(\omega\wedge \overset{k}{\cdots} \wedge\omega)(e_1,\ldots,e_{2k}) = 1$. We know also that we can find an oriented, orthonormal basis in which the matrix of $L_p(-\mathrm{i}T)$ has the form (2.9). It so happens that for circle actions (and, more generally, for torus actions) it is possible to do all of this simultaneously, i. e., to find one oriented, orthonormal basis $\{e_1,\ldots,e_{2k}\}$ for $T_p(M)$ in which (2.9), (2.16) and $\nu_\omega(e_1,\ldots,e_{2k}) = 1$ are all satisfied (see Section 32 of [7]). Making such a choice of basis, substituting (2.9) and (2.16) into (2.15) and taking determinants gives

$$T^{2k}\det(\mathcal{H}_p(e_i,e_j)) = \lambda_1^2\cdots\lambda_k^2\,.$$

Thus,

$$\begin{aligned} T^k|\det(\mathcal{H}_p(e_i,e_j))|^{\frac{1}{2}} &= \mathrm{Sign}(\lambda_1\cdots\lambda_k)\lambda_1\cdots\lambda_k \\ &= \mathrm{Sign}(\lambda_1\cdots\lambda_k)[\det(L_p(-\mathrm{i}T))]^{\frac{1}{2}} \end{aligned}$$

where $\mathrm{Sign}(\lambda_1\cdots\lambda_k) = 1$ if $\lambda_1\cdots\lambda_k > 0$ and $\mathrm{Sign}(\lambda_1\cdots\lambda_k) = -1$ if $\lambda_1\cdots\lambda_k < 0$ and so

$$T^k\,[\det(L_p(-\mathrm{i}T))]^{-\frac{1}{2}} = \mathrm{Sign}(\lambda_1\cdots\lambda_k)|\det(\mathcal{H}_p(e_i,e_j))|^{-\frac{1}{2}}. \tag{2.17}$$

Comparing (2.17) and (2.13) we see that all that remains is to prove

$$\mathrm{Sign}(\lambda_1\cdots\lambda_k) = (-\mathrm{i})^k\,\mathrm{e}^{\pi\mathrm{i}(\mathrm{Sgn}\,\mathcal{H}_p(e_i,e_j))/4}. \tag{2.18}$$

This will follow easily by induction if we can show that it is true for $k=1$. In this case, (2.15) gives

$$\begin{pmatrix} T\mathcal{H}_p(e_1,e_1) & T\mathcal{H}_p(e_1,e_2) \\ T\mathcal{H}_p(e_2,e_1) & T\mathcal{H}_p(e_2,e_2) \end{pmatrix} = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so

$$\begin{pmatrix} \mathcal{H}_p(e_1,e_1) & \mathcal{H}_p(e_1,e_2) \\ \mathcal{H}_p(e_2,e_1) & \mathcal{H}_p(e_2,e_2) \end{pmatrix} = \begin{pmatrix} \lambda_1/T & 0 \\ 0 & \lambda_1/T \end{pmatrix}.$$

Now, if $\lambda_1 > 0$ (Sign $\lambda_1 = 1$), then $\operatorname{Sgn}\mathcal{H}_p(e_i, e_j) = 2$ so

$$(-\mathrm{i})^1\, \mathrm{e}^{\pi \mathrm{i}(\operatorname{Sgn}\mathcal{H}_p(e_i,e_j))/4} = -\mathrm{i}\, \mathrm{e}^{\pi \mathrm{i}/2} = 1 = \operatorname{Sign}\lambda_1$$

and, similarly, if $\lambda_1 < 0$ (Sign $\lambda_1 = -1$), then $\operatorname{Sgn}\mathcal{H}_p(e_i, e_j) = -2$ so

$$(-\mathrm{i})^1\, \mathrm{e}^{\pi \mathrm{i}(\operatorname{Sgn}\mathcal{H}_p(e_i,e_j))/4} = -\mathrm{i}\, \mathrm{e}^{-\pi \mathrm{i}/2} = -1 = \operatorname{Sign}\lambda_1 \,.$$

Leaving the induction to the reader this completes the proof of (2.18) and therefore the derivation of the Duistermaat–Heckman Theorem 2.1 from the Generalized Duistermaat–Heckman Theorem 2.4. We shall find that Theorem 2.4 itself is a simple consequence of a beautiful localization theorem in equivariant cohomology, but for this we must briefly digress.

3. The Cartan Model of Equivariant Cohomology

Equivariant cohomology arose from attempts to understand the topology of the orbit space M/G of a topological space M on which some topological group G acts. We will be concerned only with the case in which M is a smooth manifold and G is a compact Lie group (for an introduction to the more general subject, see Chapter 1 of [8]). In this case there are a number of (equivalent) algebraic constructions which give rise to the appropriate cohomology groups, but we will describe only one (see [9] for a concise description of the others).

Thus, we begin with a smooth manifold M and a compact, connected Lie group G (which, for simplicity, we assume is a matrix group). The Lie algebra of G will be denoted $\mathfrak{g}$. Assume also that there is a smooth left action of G on M, for which we employ the usual notation:

$$\sigma\colon G \times M \to M$$
$$\sigma(g,p) = g \cdot p = \sigma_g(p) = \sigma_p(g)\,.$$

The action of G on M induces various other actions of interest to us. First we denote by $\Omega^*(M)$ the graded algebra of complex-valued differential forms on M and define, for any $\varphi \in \Omega^*(M)$ and any $g \in G$,

$$g \cdot \varphi = \sigma^*_{g^{-1}}\varphi \tag{3.1}$$

(the reason for the inverse is that we wish to define a left action). An element φ of $\Omega^*(M)$ is **G-invariant** if $g \cdot \varphi = \varphi$ for every $g \in G$. Notice that, since the exterior derivative d commutes with pullback, d of a G-invariant form is G-invariant. The subalgebra of $\Omega^*(M)$ consisting of the G-invariant elements will be denoted

$$\Omega^*(M)^G.$$

One can show that, because G is compact, G-invariance is equivalent to $\mathfrak{g}$-invariance, i. e., $\varphi \in \Omega^*(M)$ is G-invariant if and only if, for each $\xi \in \mathfrak{g}$,

$$\mathcal{L}_{\xi^\#}\varphi = 0$$

where $\mathcal{L}_{\xi^\#}$ is the Lie derivative with respect to the vector field $\xi^\#$ defined by (2.2). Finally, we observe that any $\alpha \in \Omega^*(M)$ can be "G-invariantized" in the sense that there is a chain map $I\colon \Omega^*(M) \to \Omega^*(M)^G$ which reduces to the identity on $\Omega^*(M)^G \subseteq \Omega^*(M)$ ("chain map" means $\mathrm{d} \circ I = I \circ \mathrm{d}$). The map is constructed in much the same way as the G-invariant Riemannian metric in Section 2, i. e., by averaging over the group. In somewhat more detail, we choose an invariant measure $\mathrm{d}G$ on G and, for $\alpha \in \Omega^i(M)$, $p \in M$ and $v_1, \dots, v_i \in T_p(M)$, define

$$\begin{aligned}(I(\alpha))_p\,(v_1,\dots,v_i) &= \int_G (\sigma_g^*\alpha)_p(v_1,\dots,v_i)\,\mathrm{d}G \\ &= \int_G \alpha_p\left((\sigma_g)_{*p}(v_1),\dots,(\sigma_g)_{*p}(v_i)\right)\mathrm{d}G\,.\end{aligned}$$

The conclusion we need to draw from the existence of the map I is contained in

Lemma 3.1. *If $\eta \in \Omega^{i+1}(M)^G$ is exact, then there exists a $\varphi \in \Omega^i(M)^G$ with $\mathrm{d}\varphi = \eta$.*

Proof: Since η is exact there is an $\alpha \in \Omega^i(M)$ such that $\mathrm{d}\alpha = \eta$. Let $\varphi = I(\alpha) \in \Omega^i(M)^G$. Then

$$\eta = I(\eta) = I(\mathrm{d}\alpha) = \mathrm{d}(I(\alpha)) = \mathrm{d}\varphi\,.$$

□

Next we consider the graded algebra $\mathbb{C}[\mathfrak{g}]$ of complex-valued polynomial functions on the Lie algebra $\mathfrak{g}$. This can be thought of as the complexification $S[\mathfrak{g}^*] \otimes \mathbb{C}$ of the symmetric algebra $S[\mathfrak{g}^*]$ of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Thus, if $\{\xi_1, \dots, \xi_t\}$ is a basis for $\mathfrak{g}$ and $\{x^1, \dots, x^t\}$ the corresponding dual basis for $\mathfrak{g}^*$, we think of the x^a as linear functions on $\mathfrak{g}$ and $S[\mathfrak{g}^*] = S[x^1, \dots, x^t]$ is the polynomial algebra generated by $\{x^1, \dots, x^t\}$ with real coefficients. The corresponding algebra with complex coefficients is $\mathbb{C}[\mathfrak{g}]$. We define a left action of G on $\mathbb{C}[\mathfrak{g}]$ by conjugating the variable (in $\mathfrak{g}$). In more detail, if $\mathcal{P} \in \mathbb{C}[\mathfrak{g}]$ and $g \in G$ we define $g \cdot \mathcal{P} \in \mathbb{C}[\mathfrak{g}]$ by

$$(g \cdot \mathcal{P})(\xi) = \mathcal{P}(g^{-1}\xi g) \tag{3.2}$$

for each $\xi \in \mathfrak{g}$. Those $\mathcal{P} \in \mathbb{C}[\mathfrak{g}]$ for which $g \cdot \mathcal{P} = \mathcal{P}$ for every $g \in G$ are said to be **G-invariant** and the subalgebra of $\mathbb{C}[\mathfrak{g}]$ consisting of all such is denoted

$$\mathbb{C}[\mathfrak{g}]^G$$

(this is the domain of the classical Chern–Weil map in the theory of characteristic classes). Notice that if G is Abelian (i. e., the circle S^1 or some higher dimensional torus), then $g^{-1}\xi g = \xi$ for all $g \in G$ and all $\xi \in \mathfrak{g}$ so $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{g}]$.

The object of real interest to us is the tensor product $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ of the two preceding examples. The elements of $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ are sums of terms of the form $\mathcal{P} \otimes \varphi$ (which we will call **basic**), where $\mathcal{P} \in \mathbb{C}[\mathfrak{g}]$ and φ is a (homogeneous) form on M. Such an $\alpha = \mathcal{P} \otimes \varphi$ is most conveniently thought of as a form-valued polynomial on $\mathfrak{g}$ whose value at $\xi \in \mathfrak{g}$ is

$$\alpha(\xi) = (\mathcal{P} \otimes \varphi)(\xi) = \mathcal{P}(\xi)\varphi\,.$$

Now, $\mathbb{C}[\mathfrak{g}] = \bigoplus_j \mathbb{C}^j[\mathfrak{g}]$ and $\Omega^*(M) = \bigoplus_i \Omega^i(M)$ are graded by algebraic and cohomological degree in the usual way, but, rather than the usual tensor product grading on $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ we wish to "double the degrees" in $\mathbb{C}[\mathfrak{g}]$

$$\deg(\alpha) = \deg(\mathcal{P} \otimes \varphi) = 2\deg \mathcal{P} + \deg \varphi$$

so that

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M) = \bigoplus_{2j+i=k} \mathbb{C}^j[\mathfrak{g}] \otimes \Omega^i(M)$$

(the reason will become clear shortly).

The actions of G on $\mathbb{C}[\mathfrak{g}]$ and $\Omega^*(M)$ combine to give a left action of G on $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$. Specifically, if $\alpha = \mathcal{P} \otimes \varphi$ is basic, and $g \in G$, we define $g \cdot \alpha$ to be the $\Omega^*(M)$-valued polynomial on $\mathfrak{g}$ whose value at $\xi \in \mathfrak{g}$ is

$$(g \cdot \alpha)(\xi) = g \cdot (\mathcal{P} \otimes \varphi)(\xi) = \mathcal{P}\left(g^{-1}\xi g\right) \sigma^*_{g^{-1}}\varphi\,. \tag{3.3}$$

An $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ is **G-invariant** if $g \cdot \alpha = \alpha$ for each $g \in G$ and the subalgebra of all such is denoted

$$[\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)]^G\,.$$

It is easy to verify that $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ is G-invariant if and only if

$$\alpha(g \cdot \xi) = g \cdot \alpha(\xi), \quad \text{i. e.} \quad \alpha\left(g\xi g^{-1}\right) = \sigma^*_{g^{-1}}\alpha(\xi) \tag{3.4}$$

for every $\xi \in \mathfrak{g}$ and every $g \in G$. Being an element of $\Omega^*(M)$ each $\alpha(\xi)$ can be written as

$$\alpha(\xi) = \alpha(\xi)_{[0]} + \alpha(\xi)_{[1]} + \cdots + \alpha(\xi)_{[n]}$$

where $\alpha(\xi)_{[i]}$ is a form of degree i and $n = \dim M$.

Let us write out a few concrete examples. We consider the standard action of $G = S^1$ on $M = S^3$ that gives rise to the complex Hopf bundle. Specifically, we consider

$$S^3 = \{(z^1, z^2) \in \mathbb{C}^2;\ |z^1|^2 + |z^2|^2 = 1\}$$

and define a left action of $S^1 = \{e^{\theta i};\ \theta \in \mathbb{R}\}$ on S^3 by

$$e^{\theta i} \cdot (z^1, z^2) = \left(e^{\theta i} z^1, e^{\theta i} z^2\right).$$

The action is clearly free and the orbit space S^3/S^1 is, by definition, the complex projective line $\mathbb{CP}^1$, which is diffeomorphic to S^2. Since S^1 is 1-dimensional, its Lie algebra has a single generator. We choose one such and denote it ξ_1. We denote by x^1 the corresponding dual basis vector so that $\mathbb{C}[\mathfrak{g}]$ can be identified with $\mathbb{C}[x^1]$ (the algebra of polynomials with complex coefficients in the single "variable" x^1). Since S^1 is Abelian, all of these polynomials are S^1-invariant so

$$\left[\mathbb{C}[x^1] \otimes \Omega^*(S^3)\right]^{S^1} = \mathbb{C}[x^1] \otimes \Omega^*(S^3)^{S^1}.$$

Thus, for example, an element of degree 2 (in our grading for $[\mathbb{C}[x^1] \otimes \Omega^*(S^3)]^{S^1}$) can arise either from a polynomial of degree 0 and a form of degree 2, or from a polynomial of degree 1 and a form of degree 0, i. e., the subspace of degree 2 elements is

$$\left(\mathbb{C}^0[x^1] \otimes \Omega^2(S^3)^{S^1}\right) \oplus \left(\mathbb{C}^1[x^1] \otimes \Omega^0(S^3)^{S^1}\right).$$

Any element of degree 2 can therefore be written in the form

$$1 \otimes \omega + x^1 \otimes f,$$

where ω is an S^1-invariant 2-form on S^3 and f is a complex-valued function on S^3 that is constant on each S^1-orbit. Similarly, the subspace of degree 3 elements is

$$\left(\mathbb{C}^0[x^1] \otimes \Omega^3(S^3)^{S^1}\right) \oplus \left(\mathbb{C}^1[x^1] \otimes \Omega^1(S^3)^{S^1}\right).$$

Now we wish to define a cochain complex analogous to the familiar de Rham complex $(\Omega^*(M), \mathrm{d})$ and consider the corresponding cohomology. The analogue of the graded algebra $\Omega^*(M)$ of forms is precisely our algebra $[\mathbb{C}[\mathfrak{g}] \otimes$

$\Omega^*(M)]^G$ which, in this context, we call the algebra of **G-equivariant differential forms** on the G-manifold M and denote

$$\Omega^*_G(M) = [\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)]^G .$$

What is needed now an analogue of the exterior derivative operator d. We begin by defining the **G-equivariant exterior derivative** d_G on all of $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ and will then show that its restriction to $\Omega^*_G(M)$ has the properties required to produce a cochain complex.

Each $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ is regarded as an $\Omega^*(M)$-valued function on $\mathfrak{g}$ and we define $\mathrm{d}_G\alpha$ to be the $\Omega^*(M)$-valued function on $\mathfrak{g}$ whose value at $\xi \in \mathfrak{g}$ is

$$(\mathrm{d}_G\alpha)(\xi) = \mathrm{d}(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi)) \tag{3.5}$$

where $\iota_{\xi^\#}$ is interior multiplication by the vector field $\xi^\#$ defined by (2.2). We note the following alternative description of d_G. Let $\{\xi_1, \ldots, \xi_t\}$ be a basis for $\mathfrak{g}$ and $\{x^1, \ldots, x^t\}$ the dual basis for $\mathfrak{g}^*$. We will write

$$\iota_a = \iota_{\xi_a^\#} \text{ and } \mathcal{L}_a = \mathcal{L}_{\xi_a^\#}$$

for each $a = 1, \ldots, t$. Then $\xi \in \mathfrak{g}$ implies $\xi = x^a(\xi)\xi_a$ and so $\xi^\# = x^a(\xi)\xi_a^\#$. Thus, $\iota_{\xi^\#} = x^a(\xi)\iota_a$ and, for each basic element $\alpha = \mathcal{P} \otimes \varphi$ of $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$,

$$\begin{aligned}\iota_{\xi^\#}(\alpha(\xi)) &= x^a(\xi)\iota_a(\alpha(\xi)) = x^a(\xi)\mathcal{P}(\xi)\iota_a\varphi \\ &= (x^a\mathcal{P})(\xi)\iota_a\varphi = ((x^a \otimes \iota_a)(\mathcal{P} \otimes \varphi))(\xi) \\ &= ((x^a \otimes \iota_a)(\alpha))(\xi) .\end{aligned}$$

Since

$$\begin{aligned}\mathrm{d}(\alpha(\xi)) &= \mathrm{d}(\mathcal{P}(\xi)\varphi) = \mathcal{P}(\xi)\,\mathrm{d}\varphi \\ &= ((1 \otimes \mathrm{d})(\mathcal{P} \otimes \varphi))(\xi) \\ &= ((1 \otimes \mathrm{d})(\alpha))(\xi)\end{aligned}$$

we find that

$$(\mathrm{d}_G\alpha)(\xi) = ((1 \otimes \mathrm{d} - x^a \otimes \iota_a))(\xi)$$

and so

$$\mathrm{d}_G = 1 \otimes \mathrm{d} - x^a \otimes \iota_a . \tag{3.6}$$

Proposition 3.1. *The G-equivariant exterior derivative*

$$\mathrm{d}_G \colon \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M) \to \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$$

is a linear map which increases the ($\mathbb{Z}$-graded) degree of basic elements by 1, preserves the subalgebra $\Omega^*_G(M)$ of G-invariant elements and satisfies, for each $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ and $\xi \in \mathfrak{g}$,

$$((\mathrm{d}_G \circ \mathrm{d}_G)(\alpha))(\xi) = -\mathcal{L}_{\xi^\#}(\alpha(\xi)). \tag{3.7}$$

In particular,

$$\mathrm{d}_G \circ \mathrm{d}_G = 0 \;\; on \;\; \Omega^*_G(M). \tag{3.8}$$

Proof: d_G is obviously linear so suppose $\alpha = \mathcal{P} \otimes \varphi$ is a basic element. Then $\deg \alpha = 2 \deg \mathcal{P} + \deg \varphi$. Moreover,

$$\deg((1 \otimes \mathrm{d})(\alpha)) = 2 \deg \mathcal{P} + (\deg \varphi + 1) = \deg \alpha + 1$$

and

$$\deg((x^a \otimes \iota_a)(\alpha)) = 2(\deg \mathcal{P} + 1) + (\deg \varphi - 1) = \deg \alpha + 1$$

so $\deg(\mathrm{d}_G \alpha) = \deg \alpha + 1$ as required. □

To show that d_G preserves $\Omega^*_G(M)$, assume that $\alpha(g \cdot \xi) = g \cdot \alpha(\xi)$ for all $g \in G$ and all $\xi \in \mathfrak{g}$ (see (3.4)). Then

$$\begin{aligned}
(\mathrm{d}_G\alpha)(g \cdot \xi) &= \mathrm{d}(\alpha(g \cdot \xi)) - \iota_{(g\cdot\xi)^\#}(\alpha(g \cdot \xi)) \\
&= \mathrm{d}(g \cdot \alpha(\xi)) - \iota_{(g\xi g^{-1})^\#}(g \cdot \alpha(\xi)) \\
&= \mathrm{d}\left(\sigma^*_{g^{-1}}\alpha(\xi)\right) - \left(\sigma^*_{g^{-1}} \circ \iota_{\xi^\#} \circ \sigma^*_g\right)\left(\sigma^*_{g^{-1}}\alpha(\xi)\right) \\
&= \sigma^*_{g^{-1}}(\mathrm{d}(\alpha(\xi))) - \sigma^*_{g^{-1}}(\iota_{\xi^\#}(\alpha(\xi))) \\
&= \sigma^*_{g^{-1}}((\mathrm{d}_G\alpha)(\xi)) \\
&= g \cdot ((\mathrm{d}_G\alpha)(\xi))
\end{aligned}$$

so $\mathrm{d}_G\alpha$ is also G-invariant.
Next, for any $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$, we have

$$\begin{aligned}
((\mathrm{d}_G \circ \mathrm{d}_G)(\alpha))(\xi) &= (\mathrm{d}_G(\mathrm{d}_G\alpha))(\xi) \\
&= \mathrm{d}((\mathrm{d}_G\alpha)(\xi)) - \iota_{\xi^\#}((\mathrm{d}_G\alpha)(\xi)) \\
&= \mathrm{d}(\mathrm{d}(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi))) \\
&\quad - \iota_{\xi^\#}(\mathrm{d}(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi))) \\
&= -\,\mathrm{d} \circ \iota_{\xi^\#}(\alpha(\xi)) - \iota_{\xi^\#} \circ \mathrm{d}(\alpha(\xi)) \\
&= -\,(\mathrm{d} \circ \iota_{\xi^\#} + \iota_{\xi^\#} \circ \mathrm{d})(\alpha(\xi)) \\
&= -\mathcal{L}_{\xi^\#}(\alpha(\xi))
\end{aligned}$$

which gives (3.7). To prove (3.8) from (3.7) we assume now that α is G-invariant ($\alpha(g \cdot \xi) = g \cdot \alpha(\xi)$ for all $g \in G$ and $\xi \in \mathfrak{g}$). Then

$$\begin{aligned}
\mathcal{L}_{\xi^\#}(\alpha(\xi)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma^*_{\exp(-t\xi)}(\alpha(\xi))\right)|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\exp(t\xi) \cdot (\alpha(\xi)))|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\alpha(\exp(t\xi) \cdot \xi))|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\alpha(\exp(t\xi)\xi \exp(-t\xi)))|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\alpha(\xi))|_{t=0} = 0 \,.
\end{aligned}$$

The conclusion we draw from Proposition 3.1 is that $(\Omega^*_G(M), \mathrm{d}_G)$ is a cochain complex and we may construct its cohomology in the usual way. Specifically, an $\alpha \in \Omega^*_G(M)$ is said to be **equivariantly closed** if $\mathrm{d}_G\alpha = 0$. This is the case if and only if, for each $i = 1, \ldots, n$,

$$\mathrm{d}(\alpha(\xi)_{[i-1]}) = \iota_{\xi^\#}(\alpha(\xi)_{[i+1]}) \tag{3.9}$$

for each $\xi \in \mathfrak{g}$. An $\alpha \in \Omega^*_G(M)$ is **equivariantly exact** if $\alpha = \mathrm{d}_G\beta$ for some $\beta \in \Omega^*_G(M)$. Thus, for each $\xi \in \mathfrak{g}$,

$$\alpha(\xi) = \mathrm{d}(\beta(\xi)) - \iota_{\xi^\#}(\beta(\xi)) \,.$$

In particular, since $\iota_{\xi^\#}(\beta(\xi))$ can have no top degree part,

$$\alpha(\xi)_{[n]} = \mathrm{d}(\beta(\xi))_{[n]} = \mathrm{d}(\beta(\xi)_{[n-1]}) \tag{3.10}$$

so $\alpha(\xi)_{[n]}$ is exact in the usual de Rham sense. Now, according to (3.8), every equivariantly exact form is equivariantly closed so we may consider, for each $i = 0, \ldots, n$, the quotient space of equivariantly closed forms of degree i modulo the equivariantly exact forms of degree i. This we call the **G-equivariant cohomology group** of degree i for the G-manifold M and we shall denote it $H^i_G(M)$.

$$\Omega^{i-1}_G(M) \xrightarrow{\mathrm{d}^{i-1}_G} \Omega^i_G(M) \xrightarrow{\mathrm{d}^i_G} \Omega^{i+1}_G(M)$$
$$H^i_G(M) = \mathrm{Ker}(\mathrm{d}^i_G)/\,\mathrm{Im}(\mathrm{d}^{i-1}_G)$$

($\Omega^{-1}_G(M)$ is taken to be the trivial vector space 0). As usual, we denote by $H^*_G(M)$ the direct sum of all of the $H^i_G(M)$ and call the elements of $H^*_G(M)$ **equivariant cohomology classes** on M. Just as in the case of de Rham cohomology, the multiplication in the algebra $\Omega^*_G(M)$ gives rise to an algebra structure for $H^*_G(M)$.

To gain some familiarity with these equivariant cohomology groups we will compute a few examples explicitly. First notice that any $\mathcal{P}$ in $\mathbb{C}[\mathfrak{g}]^G$ can be identified with $\mathcal{P} \otimes 1$ in $\Omega^*_G(M)$ and that all such elements clearly satisfy $\mathrm{d}_G(\mathcal{P} \otimes 1) = 0$ and so determine equivariant cohomology classes. If M is a single point (connected, 0-dimensional manifold), then every element of $\Omega^*_G(M)$ is of this form and they all determine distinct cohomology classes (since there are no nonzero exact forms) so

$$H^*_G(pt) \cong \mathbb{C}[\mathfrak{g}]^G. \tag{3.11}$$

Also notice that, if G is trivial, then so is the Lie algebra $\mathfrak{g}$ so there are only constant polynomials on $\mathfrak{g}$. Everything is G-invariant so one can identify $\Omega^*_G(M)$ with $\Omega^*(M)$. Furthermore, $\iota_{\xi^\#} = \iota_0 = 0$ so d_G agrees with d and we conclude that

$$G = \{1\} \Longrightarrow H^*_G(M) \cong H^*_{\text{de Rham}}(M)\,. \tag{3.12}$$

Next let us return to the example of the S^1-action on S^3 described earlier. To compute $H^0_{S^1}(S^3)$ one considers

$$0 \longrightarrow \Omega^0_{S^1}(S^3) \xrightarrow{\mathrm{d}^0_{S^1}} \Omega^1_{S^1}(S^3)\,.$$

Then $H^0_{S^1}(S^3) = \mathrm{Ker}(\mathrm{d}^0_{S^1})$. But any element of $\Omega^0_{S^1}(S^3)$ can be written as $1 \otimes f$, where $f \in \Omega^0(S^3)^{S^1}$ and, for such an element, $\mathrm{d}^0_{S^1}(1 \otimes f) = (1 \otimes \mathrm{d} - x^1 \otimes \iota_1)(1 \otimes f) = 1 \otimes \mathrm{d}f$ and this is zero if and only if $\mathrm{d}f = 0$. Since S^3 is connected, $\mathrm{d}f = 0$ implies that f is a constant function so

$$H^0_{S^1}(S^3) \cong \mathbb{C}\,.$$

We will leave it to the reader to show that $H^1_{S^1}(S^3)$ is trivial. As our final example we will compute $H^2_{S^1}(S^3)$. Thus, we consider

$$\Omega^1_{S^1}(S^3) \xrightarrow{\mathrm{d}^1_{S^1}} \Omega^2_{S^1}(S^3) \xrightarrow{\mathrm{d}^2_{S^1}} \Omega^3_{S^1}(S^3)\,.$$

Then $H^2_{S^1}(S^3) = \mathrm{Ker}(\mathrm{d}^2_{S^1})/\,\mathrm{Im}(\mathrm{d}^1_{S^1})$. Now, we have already seen that any $\tilde{\omega} \in \Omega^2_{S^1}(S^3)$ can be written as

$$\tilde{\omega} = 1 \otimes \omega + x^1 \otimes f$$

where $\omega \in \Omega^2(S^3)^{S^1}$ and $f \in \Omega^0(S^3)^{S^1}$. Thus,

$$\begin{aligned} \mathrm{d}^2_{S^1}\tilde{\omega} &= (1 \otimes \mathrm{d} - x^1 \otimes \iota_1)(1 \otimes \omega - x^1 \otimes f) \\ &= 1 \otimes \mathrm{d}\omega + x^1 \otimes (\mathrm{d}f - \iota_1\omega)\,. \end{aligned}$$

Thus, $\mathrm{d}^2_{S^1}\tilde{\omega} = 0$ implies

$$\mathrm{d}\omega = 0 \text{ and } \mathrm{d}f = \iota_1\omega\,. \tag{3.13}$$

We show first that it follows from (3.13) that there exists an $a \in \mathbb{C}$ and an $\eta \in \Omega^1(S^3)^{S^1}$ such that

$$(1 \otimes \omega + x^1 \otimes f) - a(x^1 \otimes 1) = \mathrm{d}^1_{S^1}(1 \otimes \eta) \tag{3.14}$$

i. e.,

$$1 \otimes \omega + x^1 \otimes (f - a) = 1 \otimes \mathrm{d}\eta - x^1 \otimes \iota_1\eta\,. \tag{3.15}$$

Now, in order for this to be true we must have $\mathrm{d}\eta = \omega$ and $\iota_1\eta = a - f$. We will solve these equations for η and a. Since $\mathrm{d}\omega = 0$ and since $H^2_{\text{de Rham}}(S^3) = 0$, ω must be exact (in the de Rham sense). According to Lemma 3.1, there exists an $\eta \in \Omega^1(S^3)^{S^1}$ with $\mathrm{d}\eta = \omega$. Thus, the first condition is satisfied. Furthermore, since η is S^1-invariant,

$$0 = \mathcal{L}_1\eta = \mathrm{d}(\iota_1\eta) + \iota_1(\mathrm{d}\eta) = \mathrm{d}(\iota_1\eta) + \iota_1\omega = \mathrm{d}(\iota_1\eta + f)\,.$$

Since S^3 is connected this implies that $\iota_1\eta + f$ is some constant function a. For this a we have $\iota_1\eta = a - f$ and the second condition is satisfied as well. This completes the proof of (3.15) and therefore of (3.14). To understand the conclusion to be drawn from (3.14) we observe that $x^1 \otimes 1$ is equivariantly closed ($\mathrm{d}^2_{S^1}(x^1 \otimes 1) = x^1 \otimes \mathrm{d}1 - x^1 \otimes \iota_1(1) = 0$) and so determines an equivariant cohomology class. Thus, (3.14) implies that the cohomology class of $\tilde{\omega} = 1 \otimes \omega + x^1 \otimes f$ is a multiple of the class of $x^1 \otimes 1$. Since $\tilde{\omega}$ was an arbitrary d_{S^1}-closed 2-form we conclude that $H^2_{S^1}(S^3)$ is generated by the class of $x^1 \otimes 1$. If $x^1 \otimes 1$ were d_{S^1}-exact this would imply that $H^2_{S^1}(S^3)$ is trivial, but we now conclude by showing that $x^1 \otimes 1$ is not exact so its cohomology class is not trivial and therefore

$$H^2_{S^1}(S^3) \cong \mathbb{C}\,.$$

To prove this we assume to the contrary that there is an element $\tilde{\eta}$ of $\Omega^1_{S^1}(S^3) \cong \mathbb{C}^0[x^1] \otimes \Omega^1(S^3)^{S^1}$ for which $\mathrm{d}^1_{S^1}\tilde{\eta} = x^1 \otimes 1$. Any such $\tilde{\eta}$ can be written $\tilde{\eta} = 1 \otimes \eta$ for some $\eta \in \Omega^1(S^3)^{S^1}$. Thus, $\mathrm{d}^1_{S^1}(1 \otimes \eta) = x^1 \otimes 1$, i. e.,

$$1 \otimes \mathrm{d}\eta - x^1 \otimes \iota_1\eta = x^1 \otimes 1$$

so we must have

$$\mathrm{d}\eta = 0 \text{ and } \iota_1\eta = -1\,. \tag{3.16}$$

But $d\eta = 0$ and $H^1_{\text{de Rham}}(S^3) = 0$ implies that η is exact and then Lemma 3.1 implies that there is an $f \in \Omega^0(S^3)^{S^1}$ with $\eta = df$. Thus,

$$\iota_1\eta = \iota_1(df) = \mathcal{L}_1 f - d(\iota_1 f) = 0 - d(0) = 0$$

so the second condition in (3.16) could not be satisfied. Thus, $x^1 \otimes 1$ cannot be $d^1_{S^1}$-exact and the proof is complete.

We should point out that, for each of the examples we have computed for $H^*_{S^1}(S^3)$, the S^1-equivariant cohomology group of S^3 agrees with the corresponding ordinary de Rham cohomology group (with complex coefficients) of the orbit space $S^3/S^1 \cong S^2$. That this is no accident is the content of a beautiful theorem of Henri Cartan (see [8] for a proof of a much more general result).

Theorem 3.1. (Cartan) *Let M be a smooth manifold and G a compact, connected Lie group. Suppose there is a smooth, free action of G on M on the left. Then the G-equivariant cohomology algebra $H^*_G(M)$ is isomorphic to the de Rham cohomology $H^*_{\text{de Rham}}(M/G)$ with complex coefficients of the orbit manifold M/G.*

Before turning to the localization theorems we must introduce a notion of integration for equivariant forms and cohomology classes. For this we now assume that M is compact and oriented and that the G-action on M preserves the orientation (each diffeomorphism $\sigma_g \colon M \to M$ is orientation preserving). For each $\alpha \in \Omega^*_G(M)$ we define an element $\int_M \alpha \in \mathbb{C}[\mathfrak{g}]^G$ by setting, for each $\xi \in \mathfrak{g}$,

$$\left(\int_M \alpha\right)(\xi) = \int_M \alpha(\xi) \stackrel{\text{def}}{=\!=} \int_M \alpha(\xi)_{[n]} \tag{3.17}$$

where $n = \dim M$. Note that $\int_M \alpha$ really is G-invariant since

$$\begin{aligned}\left(\int_M \alpha\right)(g\xi g^{-1}) &= \int_M \alpha(g\xi g^{-1})_{[n]} \\ &= \int_M \sigma^*_{g^{-1}}(\alpha(\xi)_{[n]}) \\ &= \int_M \alpha(\xi)_{[n]} \\ &= \left(\int_M \alpha\right)(\xi)\,.\end{aligned}$$

Notice also that if α is d_G-exact (say, $\alpha = \mathrm{d}_G\beta$), then, for each $\xi \in \mathfrak{g}$, (3.10) gives $\alpha(\xi)_{[n]} = \mathrm{d}(\beta(\xi)_{[n-1]})$ so, by Stokes' Theorem, $\int_M \alpha = 0 \in \mathbb{C}[\mathfrak{g}]^G$. The conclusion is that the integration map

$$\int_M : \Omega_G^*(M) \to \mathbb{C}[\mathfrak{g}]^G$$

descends to cohomology:

$$\int_M : H_G^*(M) \to \mathbb{C}[\mathfrak{g}]^G.$$

4. Equivariant Localization

The basic philosophical principle behind all of the equivariant localization theorems is that, in some sense, "G-equivariant cohomology is determined by the fixed point set of the G-action". Our first lemma is an initial indication of what this means and why it is true. Roughly, it says that if $\alpha \in \Omega_G^*(M)$ is G-equivariantly closed, then, for each $\xi \in \mathfrak{g}$, $\alpha(\xi)_{[n]}$ ($n = \dim M$) is cohomologically trivial away from the zero set $Z(\xi^\#)$ (which contains the fixed point set of the G-action and, for S^1-actions and $\xi \neq 0$, coincides with it by Lemma 2.5).

Lemma 4.1. *Let M be a smooth n-manifold and G a compact Lie group that acts smoothly on M on the left. Let $\alpha \in \Omega_G^*(M)$ be G-equivariantly closed. Then, for each $\xi \in \mathfrak{g}$, $\alpha(\xi)_{[n]}$ is (de Rham) exact on $M - Z(\xi^\#)$.*

Proof: Fix a nonzero $\xi \in \mathfrak{g}$ (the result is vacuous if $\xi = 0$). □

Remarks: We will actually prove more than is asserted in the lemma. Since the additional strength will be required in the derivation of the Duistermaat–Heckman Theorem in Section 5 we will elaborate. Note that, with $\xi \in \mathfrak{g}$ held fixed,

$$\{\alpha(\xi);\ \alpha \in \Omega_G^*(M)\} = \Omega^*(M)$$

and, for any $\alpha \in \Omega_G^*(M)$,

$$(\mathrm{d}_G\alpha)(\xi) = (\mathrm{d} - \iota_{\xi^\#})(\alpha(\xi)).$$

Define

$$\mathrm{d}_{\xi^\#} = \mathrm{d} - \iota_{\xi^\#}. \tag{4.1}$$

Then $d_{\xi^\#}$ acts on $\Omega^*(M)$, and, by (3.7),

$$d_{\xi^\#} \circ d_{\xi^\#} = -\mathcal{L}_{\xi^\#} .$$

Consequently, on the subspace

$$\Omega^*_{\xi^\#}(M) = \{\varphi \in \Omega^*(M);\ \mathcal{L}_{\xi^\#}\varphi = 0\} \tag{4.2}$$

of $\xi^\#$-invariant forms

$$d_{\xi^\#} \circ d_{\xi^\#} = 0 \quad \text{on} \quad \Omega^*_{\xi^\#}(M) . \tag{4.3}$$

Applying analogous formulas for d and $\iota_{\xi^\#}$ one obtains the Leibnitz Rule

$$d_{\xi^\#}(\omega \wedge \eta) = (d_{\xi^\#}\omega) \wedge \eta + \left[\omega_{[0]} - \omega_{[1]} + \cdots + (-1)^n \omega_{[n]}\right] \wedge d_{\xi^\#}\eta \tag{4.4}$$

for any ω, $\eta \in \Omega^*(M)$. The proof of Lemma 4.1 will rely only on the fact that $d_{\xi^\#}(\alpha(\xi)) = 0$ and the properties of $d_{\xi^\#}$ just described. In particular, the conclusion will also be true for any $\Omega^*(M)$-valued map $\xi \to \alpha(\xi)$ on $\mathfrak{g}$ even if it is not polynomial in ξ, provided only that $d_{\xi^\#}(\alpha(\xi)) = 0$.

Now we return to the proof of Lemma 4.1. Using the G-invariant Riemannian metric $\langle\,,\rangle_G$ on M we construct a 1-form θ on M dual to $\xi^\#$, i. e., we define

$$\theta(V) = \left\langle \xi^\#, V \right\rangle_G \tag{4.5}$$

for each vector field V on M.

Claim #1: θ is $\xi^\#$-invariant, i. e., $\mathcal{L}_{\xi^\#}\theta = 0$.

To see this we fix a $p \in M$ and $V_p \in T_p(M)$ and show that $(\mathcal{L}_{\xi^\#}\theta)_p(V_p) = 0$. By definition,

$$\mathcal{L}_{\xi^\#}\theta = \frac{d}{dt}\left(\sigma^*_{\exp(-t\xi)}\theta\right)|_{t=0}$$

so

$$\begin{aligned}
(\mathcal{L}_{\xi^\#}\theta)_p(V_p) &= \frac{d}{dt}\left(\left(\sigma^*_{\exp(-t\xi)}\theta\right)_p(V_p)\right)|_{t=0} \\
&= \frac{d}{dt}\left(\theta\left((\sigma_{\exp(-t\xi)})_{*p}(V_p)\right)\right)|_{t=0} \\
&= \frac{d}{dt}\left\langle \xi^\#(\exp(-t\xi)\cdot p), (\sigma_{\exp(-t\xi)})_{*p}(V_p)\right\rangle_G |_{t=0} .
\end{aligned}$$

We leave it to the reader to verify that

$$\xi^\#(\exp(-t\xi)\cdot p) = (\sigma_{\exp(-t\xi)})_{*p}\left(\xi^\#(p)\right)$$

which gives

$$\begin{aligned}(\mathcal{L}_{\xi^\#}\theta)_p(V_p) &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle(\sigma_{\exp(-t\xi)})_{*p}\left(\xi^\#(p)\right),(\sigma_{\exp(-t\xi)})_{*p}(V_p)\right\rangle_G\Big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\xi^\#(p),V_p\right\rangle_G\Big|_{t=0}=0\end{aligned}$$

because $\langle\,,\rangle_G$ is invariant under the G-action. This proves Claim #1 and from it and (4.3) we conclude that

$$\mathrm{d}_{\xi^\#}(\mathrm{d}_{\xi^\#}\theta)=0\,. \tag{4.6}$$

Now notice that

$$\mathrm{d}_{\xi^\#}\theta=\mathrm{d}\theta-\iota_{\xi^\#}\theta=\mathrm{d}\theta-\left\langle\xi^\#,\xi^\#\right\rangle_G=-\left\|\xi^\#\right\|_G^2+\mathrm{d}\theta\,. \tag{4.7}$$

This is a (nonhomogeneous) element of $\Omega^*(M)$ whose scalar (i. e., $\Omega^0(M)$) part is $-\|\xi^\#\|_M^2$ and this *scalar part is nonzero on* $M-Z(\xi^\#)$.

Remark: A nonhomogeneous element of $\Omega^*(M)$ with nonzero scalar part always has a multiplicative inverse (relative to $\wedge$) obtained from the geometric series. Indeed, if we write such an element as $a+\alpha$ with $a\in\Omega^0(M)$, $a\neq 0$, and $\alpha\in\Omega^*(M)$ with $\alpha_{[0]}=0$ and define

$$(a+\alpha)^{-1}=\frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{\alpha}{a}\right)^k$$

(a finite sum), then it is easy to verify that $(a+\alpha)^{-1}\wedge(a+\alpha)=(a+\alpha)\wedge(a+\alpha)^{-1}=1\in\Omega^0(M)$.

We conclude that, on $M-Z(\xi^\#)$, $\mathrm{d}_{\xi^\#}\theta=-\|\xi^\#\|_G^2+\mathrm{d}\theta$ is invertible and

$$(\mathrm{d}_{\xi^\#}\theta)^{-1}=-\left\|\xi^\#\right\|_G^{-2}\left(1+\left\|\xi^\#\right\|_G^{-2}\mathrm{d}\theta\right). \tag{4.8}$$

Thus, on $M-Z(\xi^\#)$, we can define an element β of $\Omega^*(M)$ by

$$\beta=\theta\wedge(\mathrm{d}_{\xi^\#}\theta)^{-1}\,. \tag{4.9}$$

Claim #2: On $M-Z(\xi^\#)$, $\mathrm{d}_{\xi^\#}\beta=1$ and $\mathcal{L}_{\xi^\#}\beta=0$.

To prove this we first compute

$$\begin{aligned}\mathrm{d}_{\xi^\#}\beta&=\mathrm{d}_{\xi^\#}(\theta\wedge(\mathrm{d}_{\xi^\#}\theta)^{-1})\\ &=\mathrm{d}_{\xi^\#}\theta\wedge(\mathrm{d}_{\xi^\#}\theta)^{-1}-\theta\wedge\mathrm{d}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1})\\ &=1-\theta\wedge\mathrm{d}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1})\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{\xi^\#}\beta &= \mathcal{L}_{\xi^\#}(\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \\ &= \mathcal{L}_{\xi^\#}\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1} + \theta \wedge \mathcal{L}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1}) \\ &= 0 + \theta \wedge \mathrm{d}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1}) .\end{aligned}$$

Now we show that $\mathrm{d}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1})$ and $\mathcal{L}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1})$ are both zero. Beginning with

$$(\mathrm{d}_{\xi^\#}\theta) \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1} = 1$$

we compute $\mathrm{d}_{\xi^\#}$ of both sides to obtain

$$\mathrm{d}_{\xi^\#}(\mathrm{d}_{\xi^\#}\theta) \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1} + \left[- \left\|\xi^\#\right\|_G^2 + \mathrm{d}\theta\right] \wedge \mathrm{d}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1}) = 0$$

so, by (4.6) and (4.7),

$$(\mathrm{d}_{\xi^\#}\theta) \wedge \mathrm{d}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1}) = 0 .$$

Now multiply on both sides by $(\mathrm{d}_{\xi^\#}\theta)^{-1}$. The proof for $\mathcal{L}_{\xi^\#}((\mathrm{d}_{\xi^\#}\theta)^{-1})$ is the same so this proves Claim #2.

Finally, we define $\lambda \in \Omega^*(M)$ by

$$\lambda = \beta \wedge \alpha(\xi) = (\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi)$$

and compute

$$\begin{aligned}\mathrm{d}_{\xi^\#}\lambda &= \mathrm{d}_{\xi^\#}(\beta \wedge \alpha(\xi)) \\ &= (\mathrm{d}_{\xi^\#}\beta) \wedge \alpha(\xi) + [\beta_{[0]} - \beta_{[1]} + \cdots] \wedge \mathrm{d}_{\xi^\#}(\alpha(\xi)) \\ &= 1 \wedge \alpha(\xi) + 0 = \alpha(\xi) .\end{aligned}$$

Thus,

$$\mathrm{d}\lambda - \iota_{\xi^\#}\lambda = \alpha(\xi) .$$

Now look at the top n^{th} degree parts. $\iota_{\xi^\#}\lambda$ has none and $(\mathrm{d}\lambda)_{[n]} = \mathrm{d}(\lambda_{[n-1]})$ so

$$\alpha(\xi)_{[n]} = \mathrm{d}\left(\lambda_{[n-1]}\right)$$

and this completes the proof of Lemma 4.1.

For future reference we summarize what we have just proved.

$$\begin{aligned}&\mathrm{d}_{\xi^\#}(\alpha(\xi)) = 0 \text{ and } \theta = \left\langle \xi^\#, \cdot \right\rangle_G \\ &\Longrightarrow \alpha(\xi) = \mathrm{d}_{\xi^\#}\left((\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi)\right)\end{aligned}$$

and

$$\alpha(\xi)_{[n]} = \mathrm{d}\left(\left((\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi)\right)_{[n-1]}\right)$$

on $M - Z(\xi^\#)$.

In order to proceed further we must understand more about the structure of the set $Z(\xi^\#) = \{p \in M\,;\, \xi^\#(p) = 0\}$ of zeros of $\xi^\#$. Notice that it is clear from the definition (2.2) of $\xi^\#$ that any fixed point of the G-action on M is a zero of every $\xi^\#$ so every $Z(\xi^\#)$ contains the fixed point set

$$M^G = \{p \in M\,;\, g \cdot p = p \ \text{ forall } \ g \in G\}\ .$$

If $\xi \in \mathfrak{g}$ has the property that $Z(\xi^\#) = M^G$ (i. e., $\xi^\#$ vanishes only at fixed points of the G-action), then ξ is said to be nondegenerate (for $G = S^1$-actions, every ξ in the Lie algebra of S^1 is nondegenerate by Lemma 2.5). Notice that, in general, one can define, for any $\xi \in \mathfrak{g}$, the subgroup

$$T_\xi = \text{closure}_G\,\{\exp(-t\xi)\,;\, t \in \mathbb{R}\}$$

of G. Then $Z(\xi^\#)$ clearly coincides with the fixed point set of the action on M of T_ξ and, being compact, connected, and Abelian, T_ξ is a torus. Thus, *the zero set of $\xi^\#$ is always the fixed point set of a torus action on M.*

As in Section 2 for Hamiltonian actions, we define, for each $p \in Z(\xi^\#)$ a linear transformation $L_p(\xi)\colon T_p(M) \to T_p(M)$ by (2.5) and note that it is skew-symmetric with respect to the G-invariant Riemannian metric $\langle\,,\rangle_G$ on M. Now we let $\exp_p$ be the (metric) exponential map on $T_p(M)$ corresponding to $\langle\,,\rangle_G$. This carries a $V_p \in T_p(M)$ onto $\gamma_{V_p}(1)$, where γ_{V_p} is the geodesic of $\langle\,,\rangle_G$ with $\gamma'_{V_p}(0) = V_p$ and it is a local diffeomorphism of some neighborhood of 0 in $T_p(M)$ onto some neighborhood of p in M. The G-invariance of $\langle\,,\rangle_G$ implies that, on some neighborhood of 0 in $T_p(M)$,

$$\exp_p(V_p + tL_p(\xi)(V_p)) = \exp(-t\xi) \cdot \exp_p(V_p)\,. \tag{4.10}$$

Thus, if $\mathcal{L}_p(\xi)$ is the vector field on $T_p(M)$ corresponding to $L_p(\xi)$, i. e.,

$$\mathcal{L}_p(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}[V_p + tL_p(\xi)(V_p)]_{t=0}\,,$$

then

$$\xi^\# = (\exp_p)_*(\mathcal{L}_p(\xi))$$

on some neighborhood of p in M. In particular, integral curves of $\mathcal{L}_p(\xi)$ are (locally) mapped by $\exp_p$ to integral curves of $\xi^\#$.

Now, suppose $V_p \in \mathrm{Ker}(L_p(\xi))$. Then $L_p(\xi)(V_p) = 0$ and this is the case if and only if the integral curve of $\mathcal{L}_p(\xi)$ through V_p is a point, i. e., the integral curve of $\xi^{\#}$ through $\exp_p(V_p)$ is a point. Since this is the case if and only if $\xi^{\#}(\exp_p(V_p)) = 0$ we conclude that, on some neighborhood of 0 in $T_p(M)$

$$V_p \in \mathrm{Ker}(L_p(\xi)) \Longleftrightarrow \exp_p(V_p) \in Z(\xi^{\#})\,.$$

Now $\mathrm{Ker}(L_p(\xi))$ is a linear subspace (and therefore a submanifold) of $T_p(M)$ so the restriction of $\exp_p$ to some open set in $\mathrm{Ker}(L_p(\xi))$ maps diffeomorphically onto a neighborhood of p in $Z(\xi^{\#})$. Thus, $Z(\xi^{\#})$ has a local manifold structure near each of its points p whose dimension is $\dim(\mathrm{Ker}(L_p(\xi)))$. This dimension need not be the same at each $p \in Z(\xi^{\#})$, but is constant on the connected components of $Z(\xi^{\#})$. Thus, we find that $Z(\xi^{\#})$ is a disjoint union of submanifolds of M each of which has dimension $\dim(\mathrm{Ker}(L_p(\xi)))$, where p is any point in the submanifold. In particular, we have the promised generalization of Proposition 2.3.

Proposition 4.1. *Let M be a smooth manifold, G a compact Lie group acting smoothly on M on the left, ξ an element of the Lie algebra $\mathfrak{g}$ of G and $p \in Z(\xi^{\#})$ a zero of $\xi^{\#}$. Then p is an isolated point of $Z(\xi^{\#})$ if and only if $L_p(\xi)\colon T_p(M) \to T_p(M)$ is invertible.*

Henceforth we assume that p is an isolated zero of $\xi^{\#}$. Then $L_p(\xi)$ is invertible and skew-symmetric with respect to $\langle\, , \rangle_G$. It follows that the dimension of M must be even, say,

$$n = 2k$$

and that there exists an oriented, orthonormal basis $\{e_1, \ldots, e_{2k}\}$ for $T_p(M)$ relative to which the matrix of $L_p(\xi)$ has the form (2.9) with $\lambda_j \in \mathbb{R}\backslash\{0\}$ for $j = 1, \ldots, k$. As before we define a square root of the determinant of $L_p(\xi)$ by (2.10). Now, if $V_p \in T_p(M)$ and we write $V_p = V_p^i e_i$ (summation convention), then

$$L_p(\xi)(V_p) = \lambda_1 \left(V_p^2 e_1 - V_p^1 e_2\right) + \cdots + \lambda_k \left(V_p^{2k} e_{2k-1} - V_p^{2k-1} e_{2k}\right).$$

If, as before, we identify $L_p(\xi)$ with a vector field $\mathcal{L}_p(\xi)$ on $T_p(M)$ and recall that, on some neighborhood of p in M, $\xi^{\#}$ agrees with $(\exp_p)_*(\mathcal{L}_p(\xi))$, then, in normal coordinates $x^1, \ldots, x^{2k}$ on that neighborhood determined by $\exp_p$ and $\{e_1, \ldots, e_{2k}\}$, we have

$$\xi^{\#} = \lambda_1 \left(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}\right) + \cdots + \lambda_k \left(x^{2k} \frac{\partial}{\partial x^{2k-1}} - x^{2k-1} \frac{\partial}{\partial x^{2k}}\right). \quad (4.11)$$

Note that if p happens to be a fixed point of the G-action (e. g., if ξ is nondegenerate), then this neighborhood can be chosen G-invariant (restrict to some ϵ-ball relative to $\langle\,,\rangle_G$). With this we are finally prepared to prove our major result.

Theorem 4.1. (Equivariant Localization Theorem) *Let M be a compact, oriented manifold of dimension $n = 2k$ and G a compact Lie group acting smoothly on M on the left. Let α be a G-equivariantly closed differential form on M. Then, for any nondegenerate $\xi \in \mathfrak{g}$ for which $\xi^\#$ has only isolated zeros,*

$$\int_M \alpha(\xi) = \sum_{\substack{p\in M \\ \xi^\#(p)=0}} (-2\pi)^k \left[\det L_p(\xi)\right]^{-\frac{1}{2}} \alpha(\xi)_{[0]}(p)\,. \tag{4.12}$$

Remarks:

1. Since M is compact and $Z(\xi^\#)$ is discrete, the sum in (4.12) is finite. If $Z(\xi^\#)$ happens to be empty, then Lemma 4.1 implies that $\alpha(\xi)_{[n]}$ is exact on all of M so Stokes' Theorem gives $\int_M \alpha(\xi) = 0$ and (4.12) is vacuously satisfied.
2. For S^1-actions Lemma 2.5 implies that the nondegeneracy assumption in Theorem 4.3 is unnecessary.
3. As was the case for Lemma 4.1 our proof of Theorem 4.3 will not use the full strength of the assumption that α is a G-equivariantly closed differential form on M, but only that $\mathrm{d}_{\xi^\#}(\alpha(\xi)) = 0$ for the particular $\xi \in \mathfrak{g}$ referred to in the Theorem.

Proof: By the first remark above we may assume $Z(\xi^\#) \neq \emptyset$. Let $p \in Z(\xi^\#)$. We have shown that we can find a G-invariant neighborhood U_p of p and (normal) coordinates $x^1, \ldots, x^{2k}$ on U_p such that $\xi^\#|U_p$ is given by (4.11), where $[\det(L_p(\xi))]^{\frac{1}{2}} = \lambda_1 \cdots \lambda_{2k} \neq 0$. On U_p we define a 1-form θ^p by

$$\theta^p = \lambda_1^{-1}\left(x^2\,\mathrm{d}x^1 - x^1\,\mathrm{d}x^2\right) + \cdots + \lambda_k^{-1}\left(x^{2k}\,\mathrm{d}x^{2k-1} - x^{2k-1}\,\mathrm{d}x^{2k}\right). \tag{4.13}$$

Then a few simple computations show

$$\theta^p(\xi^\#) = (x^1)^2 + \cdots + (x^{2k})^2. \tag{4.14}$$

$$\mathrm{d}\left(\iota_{\xi^\#}\theta^p\right) = 2x^1\,\mathrm{d}x^1 + \cdots + 2x^{2k}\,\mathrm{d}x^{2k} \tag{4.15}$$

$$\iota_{\xi^\#}(\mathrm{d}\theta^p) = -2x^1\,\mathrm{d}x^1 - \cdots - 2x^{2k}\,\mathrm{d}x^{2k} \tag{4.16}$$

and, from the last two of these,

$$\mathcal{L}_{\xi^\#}\theta^p = \left(\mathrm{d}\circ\iota_{\xi^\#} + \iota_{\xi^\#}\circ\mathrm{d}\right)\theta^p = 0\,. \tag{4.17}$$

Now, each of the sets U_p, $p \in Z(\xi^\#)$, is G-invariant by construction and $M - Z(\xi^\#)$ is G-invariant because ξ is assumed nondegenerate (so $Z(\xi^\#) = M^G$ which is surely G-invariant). Thus

$$\{U_p\}_{p\in Z(\xi^\#)} \cup \{M - Z(\xi^\#)\}$$

is a G-invariant open cover of M. By choosing a partition of unity subordinate to this cover and averaging each of its elements over G (as we did to produce $\langle\,,\rangle_G$ in Section 2 and the map I in Section 3) one can produce a G-invariant partition of unity subordinate to the cover. With this and the 1-forms θ^p on U_p and (as in the proof of Lemma 4.1) $\theta^0 = \langle \xi^\#, \cdot\rangle_G$ on $M - Z(\xi^\#)$, one can piece together a 1-form θ on all of M with the following properties:

1. θ agrees with θ^p on some neighborhood of p.
2. $\mathcal{L}_{\xi^\#}\theta = 0$.
3. $\mathrm{d}_{\xi^\#}\theta$ is invertible on $M - Z(\xi^\#)$.

Exactly as in the proof of Lemma 4.1, properties (2) and (3) together with $\mathrm{d}_{\xi^\#}(\alpha(\xi)) = 0$ imply that

$$\alpha(\xi) = \mathrm{d}_{\xi^\#}\left((\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi)\right) \text{ on } M - Z(\xi^\#)\,. \tag{4.18}$$

Now we compute the integral on the left-hand side of (4.12). For each $p \in Z(\xi^\#)$ and $\epsilon > 0$ sufficiently small we let

$$B_\epsilon(p) = \left\{x = \left(x^1, \ldots, x^{2k}\right)\,;\, |x|_G^2 = \left(x^1\right)^2 + \cdots + \left(x^{2k}\right)^2 \le \epsilon^2\right\} \subseteq U_p$$

and

$$S_\epsilon(p) = \{x;\, |x|_G = \epsilon\}$$

and give both their usual orientations. Since $Z(\xi^\#)$ is a finite set,

$$\begin{aligned}\int_M \alpha(\xi) &= \int_{M - Z(\xi^\#)} \alpha(\xi) = \lim_{\epsilon\to 0} \int_{M - \cup_{p\in Z(\xi^\#)} B_\epsilon(p)} \alpha(\xi)\\
&= \lim_{\epsilon\to 0} \int_{M - \cup_{p\in Z(\xi^\#)} B_\epsilon(p)} \mathrm{d}_{\xi^\#}\left((\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi)\right)\\
&= \lim_{\epsilon\to 0} \int_{M - \cup_{p\in Z(\xi^\#)} B_\epsilon(p)} \mathrm{d}\left((\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi)\right)\end{aligned}$$

(because the $\iota_{\xi^\#}$ term can have no top degree part)

$$= \lim_{\epsilon \to 0} \left(- \sum_{p \in Z(\xi^\#)} \int_{S_\epsilon(p)} (\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi) \right)$$

(the minus sign being due to the switch from boundary to standard orientations)

$$= \sum_{\substack{p \in M \\ \xi^\#(p)=0}} \lim_{\epsilon \to 0} \left(- \int_{S_\epsilon(p)} (\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi) \right).$$

Comparing this with (4.12) we see that it remains only to prove that, for each $p \in Z(\xi^\#)$,

$$\lim_{\epsilon \to 0} \left(- \int_{S_\epsilon(p)} (\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi) \right) = (-2\pi)^k \left[\det L_p(\xi)\right]^{-\frac{1}{2}} \alpha(\xi)_{[0]}(p).$$

Thus, we fix a $p \in Z(\xi^\#)$. For each $\epsilon > 0$ sufficiently small, $\theta = \theta^p$ on $S_\epsilon(p)$. For such an $\epsilon > 0$ we introduce a change of coordinates on U_p by rescaling each x^i by a factor of ϵ, i. e., we replace x^i everywhere with ϵx^i, $i = 1, \ldots, 2k$:

$$x^i \to \epsilon x^i, \qquad i = 1, \ldots, 2k. \tag{4.19}$$

In the new coordinates, $S_\epsilon(p)$ becomes the unit sphere $S_1(p)$. Write $\alpha_\epsilon(\xi)$ for $\alpha(\xi)$ written in these new coordinates, i. e.,

$$\alpha_\epsilon(\xi)(x, \mathrm{d}x) = \alpha(\xi)(\epsilon x, \epsilon\, \mathrm{d}x).$$

Notice that, as $\epsilon \to 0$, all of the $\alpha_\epsilon(\xi)_{[i]}$ with $i > 0$ approach 0, whereas $\alpha_\epsilon(\xi)_{[0]} \to \alpha(\xi)_{[0]}(p)$ since $p = (0, \ldots, 0)$.

Now we consider the effect of this substitution on $\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}$. Near p, $\theta = \theta^p$ is given by (4.13) so (4.20) introduces an extra factor of ϵ^2. On the other hand,

$$\mathrm{d}_{\xi^\#}\theta = \mathrm{d}\theta - \iota_{\xi^\#}\theta = -2\left(\lambda_1^{-1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots + \lambda_k^{-1}\,\mathrm{d}x^{2k-1} \wedge \mathrm{d}x^{2k}\right) - |x|_G^2$$

so this also picks up a factor of ϵ^2. Consequently, $(\mathrm{d}_{\xi^\#}\theta)^{-1}$ acquires a new factor of $\frac{1}{\epsilon^2}$ and, as a result, $\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}$ is unaffected by the rescaling (4.20). Thus,

$$\int_{S_\epsilon(p)} (\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi) = \int_{S_1(p)} (\theta \wedge (\mathrm{d}_{\xi^\#}\theta)^{-1}) \wedge \alpha_\epsilon(\xi)$$

and so

$$\lim_{\epsilon\to 0}\left(-\int\limits_{S_\epsilon(p)}(\theta\wedge(\mathrm{d}_{\xi^\#}\theta)^{-1})\wedge\alpha(\xi)\right)$$

$$=\left(-\int\limits_{S_1(p)}(\theta\wedge(\mathrm{d}_{\xi^\#}\theta)^{-1})\right)\alpha(\xi)_{[0]}(p)\,.$$

We therefore compute

$$-\int\limits_{\substack{S_1(p)\\(|x|^2_G=1 \text{ on } S_1(p))}}\theta\wedge(\mathrm{d}_{\xi^\#}\theta)^{-1}$$

$$=-\int\limits_{S_1(p)}\theta\wedge(\mathrm{d}\theta-1)^{-1}$$

$$=\int\limits_{S_1(p)}\theta\wedge(1-\mathrm{d}\theta)^{-1}$$

$$=\int\limits_{S_1(p)}\theta\wedge(1+\mathrm{d}\theta+(\mathrm{d}\theta)^2+\cdots+(\mathrm{d}\theta)^{k-1}+(\mathrm{d}\theta)^k)$$

where $(\mathrm{d}\theta)^2=\mathrm{d}\theta\wedge\mathrm{d}\theta$, etc.

$$=\int\limits_{S_1(p)}\theta\wedge(\mathrm{d}\theta)^{k-1}$$

since $\dim S_1(p)=2k-1$

$$=\int\limits_{B_1(p)}(\mathrm{d}\theta)^k$$

by Stokes' Theorem since $\mathrm{d}(\theta\wedge(\mathrm{d}\theta)^{k-1})=\mathrm{d}\theta\wedge(\mathrm{d}\theta)^{k-1}$ $-\,\theta\wedge\mathrm{d}((\mathrm{d}\theta)^{k-1})=(\mathrm{d}\theta)^k-0$

$$=\int\limits_{B_1(p)}\Big((-2)(\lambda_1^{-1}\,\mathrm{d}x^1\wedge\mathrm{d}x^2+\cdots+\lambda_k^{-1}\,\mathrm{d}x^{2k-1}\wedge\mathrm{d}x^{2k})\Big)^k$$

$$=(-2)^k k!\lambda_1^{-1}\cdots\lambda_k^{-1}\int\limits_{B_1(p)}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{2k}$$

$$=(-2)^k k!(\lambda_1\cdots\lambda_k)^{-1}\left(\frac{\pi^k}{k!}\right)$$

$$= (-2\pi)^k [\det L_p(\xi)]^{-\frac{1}{2}}.$$

Substituting this into (4.21) yields (4.19) and so completes the proof of Theorem 4.3. □

5. The Duistermaat–Heckman Theorem

Finally we will derive the Generalized Duistermaat–Heckman Theorem 2.4 from our Localization Theorem 4.3. Recall that the scenario is as follows. We have a compact, symplectic manifold (M, ω) of dimension $2k$ and oriented by the Liouville form $\nu_\omega = \frac{1}{k!}\omega^k$. There is a Hamiltonian action of a compact Lie group G on M with corresponding equivariant moments given by $\mu\colon \mathfrak{g} \to C^\infty(M)$. Finally, we have a $\xi \in \mathfrak{g}$ which is nondegenerate. Notice that, because the action is Hamiltonian, nondegeneracy of ξ implies that $\xi^\#$ has isolated zeros ($\mu(\xi)$ is a Morse function and the zeros of $\xi^\#$ coincide with the critical points of $\mu(\xi)$). Our objective is to prove (2.11).

We consider a (nonpolynomial) map $\mathfrak{g} \to \Omega^*(M)$ called the **G-equivariant symplectic form** ω_G defined by

$$\omega_G = \mu + \omega,$$

i. e.,

$$\omega_G(\xi) = \mu(\xi) + \omega$$

for every $\xi \in \mathfrak{g}$. Of course, this is not a G-equivariant differential form on M (since μ is generally not polynomial in ξ), but we claim that, nevertheless,

$$\mathrm{d}_{\xi^\#}\,(\omega_G(\xi)) = 0 \tag{5.1}$$

for every $\xi \in \mathfrak{g}$. Indeed,

$$\begin{aligned}\mathrm{d}_{\xi^\#}\,(\omega_G(\xi)) &= (\mathrm{d} - \iota_{\xi^\#})\,(\mu(\xi) + \omega)\\ &= \mathrm{d}\,(\mu(\xi) + \omega) - \iota_{\xi^\#}\,(\mu(\xi) + \omega)\\ &= \mathrm{d}\mu(\xi) + 0 - 0 - \iota_{\xi^\#}\omega\\ &= 0 \quad \text{(by (2.3))}\,.\end{aligned}$$

Now consider the element $\mathrm{e}^{\mathrm{i}\omega_G(\xi)} \in \Omega^*(M)$:

$$\mathrm{e}^{\mathrm{i}\omega_G(\xi)} = 1 + \mathrm{i}\omega_G(\xi) - \frac{1}{2}\omega_G(\xi) \wedge \omega_G(\xi) + \cdots$$

(a finite sum). Since (5.1) and the Leibnitz Rule (4.4) imply that

$$\mathrm{d}_{\xi^\#}\,(\omega_G(\xi) \wedge \cdots \wedge \omega_G(\xi)) = 0\,,$$

we conclude that

$$\mathrm{d}_{\xi^\#}\left(\mathrm{e}^{\mathrm{i}\omega_G(\xi)}\right) = 0\,. \tag{5.2}$$

Remark (3) following Theorem 4.3 implies that (5.2) is sufficient to apply the Localization Theorem to $\mathrm{e}^{\mathrm{i}\omega_G(\xi)}$. Since

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\omega_G(\xi)} &= \mathrm{e}^{\mathrm{i}(\mu(\xi)+\omega)} = \mathrm{e}^{\mathrm{i}\mu(\xi)}\,\mathrm{e}^{\mathrm{i}\omega}\\ &= \mathrm{e}^{\mathrm{i}\mu(\xi)}\left(1 + \mathrm{i}\omega - \frac{1}{2}\omega^2 + \cdots\right)\end{aligned}$$

we have

$$\left(\mathrm{e}^{\mathrm{i}\omega_G(\xi)}\right)_{[0]} = \mathrm{e}^{\mathrm{i}\mu(\xi)}\,.$$

Thus, (4.12) gives

$$\begin{aligned}\sum_{\substack{p\in M\\ \xi^\#(p)=0}} (-2\pi)^k[\det L_p(\xi)]^{-\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\mu(\xi)(p)} &= \int_M \mathrm{e}^{\mathrm{i}\omega_G(\xi)}\\ &= \int_M \mathrm{e}^{\mathrm{i}\mu(\xi)}\,\mathrm{e}^{\mathrm{i}\omega}\\ &= \int_M \mathrm{e}^{\mathrm{i}\mu(\xi)}\left(\frac{1}{k!}\mathrm{i}^k\omega^k\right)\\ &= \mathrm{i}^k\int_M \mathrm{e}^{\mathrm{i}\mu(\xi)}\nu_\omega\end{aligned}$$

which is (2.11).

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