

## SINGLE-VALUED AND MULTIVALUED SOLUTIONS FOR THE GENERALIZED HÉNON–HEILES SYSTEM WITH AN ADDITIONAL NONPOLYNOMIAL TERM

SERGEY YU. VERNOV

*Skobeltsyn Institute of Nuclear Physics, Moscow State University  
119992, Moscow, Russia*

**Abstract.** The generalized Hénon–Heiles system with an additional non-polynomial term is considered. The standard method for the search of the elliptic solutions is a transformation of an initial nonlinear polynomial differential equation into a nonlinear algebraic system. It has been demonstrated that the use of the Laurent-series solutions allows to simplify the resulting algebraic system. This procedure has been automatized and generalized on some type of multivalued solutions. To find solutions of the initial equation in the form of the Laurent or Puiseux series we use algorithm of the Painlevé test.

### 1. The Painlevé Property

Let us formulate the Painlevé property for ordinary differential equations (ODE's). Solutions of a system of ODE's are regarded as analytic functions, maybe with isolated singular points [9]. A singular point of a solution is said **critical** (as opposed to **noncritical**) if the solution is multivalued (single-valued) in its neighborhood and *movable* if its location depends on initial conditions. The **general solution** of an ODE of order  $N$  is the set of all solutions mentioned in the existence theorem of Cauchy, i.e. determined by the initial values. It depends on  $N$  arbitrary independent constants. A **special solution** is any solution obtained from the general solution by giving values to the arbitrary constants. A **singular solution** is any solution which is not special, i.e. which does not belong to the general solution. A system of ODE's has **the Painlevé property** if its general solution has no movable critical singular point [9, 14].

There exist two distinctions between the structure of solutions of linear differential equations and nonlinear ones. Linear ODE's have no singular solution and their general solutions have no movable singularity.

Investigations of many dynamical systems show that a system is completely integrable for such values of parameters, at which it has the Painlevé property. At the same time the integrability of an arbitrary system with the Painlevé property has yet to be proved. There is not an algorithm for construction of the additional integral by the Painlevé analysis. There exist many examples of integrable systems without the Painlevé property.

**The Painlevé test** is any algorithm, which checks some necessary conditions for a differential equation to have the Painlevé property. The original algorithm, developed by Painlevé and used by him to find all the second order ODE's with Painlevé property [14], is known as the  $\alpha$ -method. The method of Kovalevskaya [12] is not as general as the  $\alpha$ -method, but much more simple. In 1980, developing the Kovalevskaya method further, Ablowitz, Ramani and Segur [1] constructed a new algorithm of the Painlevé test for ODE's. The remarkable property of these tests is that it can be checked in a finite number of steps. They can only detect the occurrence of logarithmic and algebraic branch points. To date there is no general finite algorithmic method to detect the occurrence of essential singularities. Different variants of the Painlevé test are compared in [4].

## 2. The Hénon–Heiles System

The generalized Hénon–Heiles system with additional nonpolynomial term is described by the Hamiltonian

$$H = \frac{1}{2}(x_t^2 + y_t^2 + \lambda_1 x^2 + \lambda_2 y^2) + x^2 y - \frac{C}{3} y^3 + \frac{\mu}{2x^2} \quad (1)$$

and the corresponding system of the motion equations

$$\begin{cases} x_{tt} = -\lambda_1 x - 2xy + \frac{\mu}{x^3} \\ y_{tt} = -\lambda_2 y - x^2 + Cy^2 \end{cases} \quad (2)$$

where  $x_{tt} \equiv d^2x/dt^2$  and  $y_{tt} \equiv d^2y/dt^2$ ,  $\lambda_1$ ,  $\lambda_2$ ,  $\mu$  and  $C$  are arbitrary numerical parameters. If  $\lambda_2 \neq 0$ , then one can put  $\lambda_2 = \text{sign}(\lambda_2)$  without the loss of generality.

Due to the Painlevé analysis the following integrable cases have been found

- i)  $C = -1$ ,  $\lambda_1 = \lambda_2$ ,
- ii)  $C = -6$ ,  $\lambda_1$  and  $\lambda_2$  are arbitrary,
- iii)  $C = -16$ ,  $\lambda_1 = \lambda_2/16$ .

In all above-mentioned cases system (2) is integrable at any value of  $\mu$ . In other cases not only four, but even threeparameter exact solutions have yet to be found.

The function  $y$ , a solution of the system (2), satisfies the following fourth-order equation

$$y_{tttt} = (2C - 8)y_{tt}y - (4\lambda_1 + \lambda_2)y_{tt} + 2(C + 1)y_t^2 + \frac{20C}{3}y^3 + (4C\lambda_1 - 6\lambda_2)y^2 - 4\lambda_1\lambda_2y - 4H \quad (3)$$

where  $H$  is the energy of the system. Note, that the energy  $H$  is not an arbitrary parameter, but a function of initial data  $y_0$ ,  $y_{0t}$ ,  $y_{0tt}$  and  $y_{0ttt}$ . The form of this function depends on  $\mu$

$$H = \frac{y_{0t}^2 + y_0^2}{2} - \frac{C}{3}y_0^3 + \left(\frac{\lambda_1}{2} + y_0\right)(Cy_0^2 - \lambda_2y_0 - y_{0tt}) + \frac{(\lambda_2y_{0t} + 2Cy_0y_{0t} - y_{0ttt})^2 + \mu}{2(Cy_0^2 - \lambda_2y_0 - y_{0tt})}.$$

This formula is correct only if  $x_0 = Cy_0^2 - \lambda_2y_0 - y_{0tt} \neq 0$ . If  $x_0 = 0$ , what is possible only for  $\mu = 0$ , then we can not express  $x_{0t}$  through  $y_0$ ,  $y_{0t}$ ,  $y_{0tt}$  and  $y_{0ttt}$ , so  $H$  is not a function of the initial data. If  $y_{0ttt} = 2Cy_0y_{0t} - \lambda_2y_{0t}$ , then equation (3) with an arbitrary  $H$  corresponds to system (2) with  $\mu = 0$ , in opposite case equation (3) does not correspond to system (2).

### 3. Results of the Painlevé Test

The Ablowitz–Ramani–Segur algorithm of the Painlevé test [1] is very useful for obtaining the solutions as formal Laurent series. Let the behavior of a solution in the neighborhood of the singularity point  $t_0$  be algebraic, i.e.,  $x$  and  $y$  tend to infinity as some powers:  $x = a_\alpha(t - t_0)^\alpha$  and  $y = b_\beta(t - t_0)^\beta$ , where  $\alpha$ ,  $\beta$ ,  $a_\alpha$  and  $b_\beta$  are some constants. If  $\alpha$  and  $\beta$  are negative integer numbers, then substituting the Laurent series expansions for  $x$  and  $y$ , one can transform nonlinear differential equations into a system of linear algebraic equations in coefficients of Laurent series. If a single-valued solution depends on more than one arbitrary parameters then some coefficients of its Laurent series have to be arbitrary and the corresponding systems have to have zero determinants. The numbers of such systems (named **resonances** or **Kovalevskaya exponents**) can be determined due to the Painlevé test.

It is easy to show that solutions of equation (3) tend to infinity as  $b_{-2}/t^2$ . There exist two possible dominant behaviors and resonance structures of these solutions (see Table 3).

The values of  $r$  denote resonances:  $r = -1$  corresponds to arbitrary parameter  $t_0$ , other values of  $r$  determine powers of  $t$  at which new arbitrary parameters can

**Table 1**

<i>Case 1</i>	<i>Case 2</i>
$b_{-2} = -3,$ $r = -1, 10, \frac{5}{2} \pm \frac{\sqrt{1 - 24(1 + C)}}{2}$	$b_{-2} = \frac{6}{C},$ $r = -1, 5, 5 \pm \sqrt{1 - \frac{48}{C}}.$

appear as solutions of the linear systems with zero determinant. Note, that the dominant behavior and the resonance structure depend only on  $C$ .

It is necessary for the integrability of system (2) that all values of  $r$  be integer and that all systems with zero determinants have solutions for any values of all free parameters entering in these systems. This is possible only in the integrable cases i)–iii).

For the search for special solutions it is interesting to consider such values of  $C$ , for which  $r$  are integer numbers either only in *Case 1* or only in *Case 2*. If there exist a negative integer resonance, different from  $r = -1$ , then such Laurent series expansion corresponds rather to special than general solution [16]. We demand that all values of  $r$ , but one, are nonnegative integer numbers and all these values are different. From these conditions we obtain the following values of  $C$ :  $C = -1$  and  $C = -4/3$  (*Case 1*), or  $C = -16/5$ ,  $C = -6$  and  $C = -16$  (*Case 2*), and also  $C = -2$ , in which these two *Cases* coincide.

Let us consider the possibility of existence of the single-valued three-parameter solutions in all these cases. To obtain the result for an arbitrary value of  $\mu$ , we consider equation (3) with an arbitrary  $H$ .

At  $C = -2$  we have a contradiction:  $r = 0$ , but  $b_{-2}$  is not an arbitrary parameter:  $b_{-2} = -3$ . This is the consequence of the fact that, contrary to our assumption, the behavior of the general solution in the neighborhood of a singular point is not algebraic, because its dominant term includes logarithm [16]. At  $C = -6$  and any values of other parameters the exact four-parameter solutions are known. In cases  $C = -1$  and  $C = -16$  the substitution of an unknown function as the Laurent series leads to the conditions  $\lambda_1 = \lambda_2$  or  $\lambda_1 = \lambda_2/16$ , accordingly. Hence, in non-integrable cases three-parameter local solutions have to include logarithmic terms. Single-valued three-parameter solutions can exist only in two above-mentioned nonintegrable cases:  $C = -16/5$  and  $C = -4/3$ .

Using the method of construction of the Laurent series solutions for nonlinear differential equations described in [19], we obtain three-parameter solutions of equation (3) both at  $C = -16/5$  and at  $C = -4/3$ . Values of other parameters are arbitrary.

At  $C = -4/3$  the above-mentioned solutions are

$$\tilde{y} = -\frac{3}{t^2} + \frac{b_{-1}}{t} + \frac{29}{24}b_{-1}^2 + \frac{2\lambda_1 - 3\lambda_2}{4} + \left(\frac{17}{6}b_{-1}^2 + \frac{5}{3}\lambda_1 - \frac{5}{4}\lambda_2\right)b_{-1}t + \dots \quad (4)$$

There exist four possible values of the parameter  $b_{-1}$

$$b_{-1} = \pm \sqrt{\frac{105\lambda_2 - 140\lambda_1 \pm \sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)}}{385}}.$$

The two signs “ $\pm$ ” are independent, while the parameters  $b_2$  and  $b_8$ , coefficients at  $t^2$  and  $t^8$  correspondingly, are arbitrary. The energy  $H$  enters in coefficients beginning from  $b_4$ . We obtain solutions only as formal series, but this series have some domains of convergence, because the convergence of the Laurent- and psi-series solutions of the generalized Hénon–Heiles system has been proved [13]. To search the elliptic solutions we use only some finite number of the Laurent series coefficients, so we have no need of the proof of the Laurent series convergence.

## 4. Global Solutions

### 4.1. Methods of Construction

We have found solutions as formal Laurent series. Of course, existence of such solutions is a necessary, but not sufficient condition to exist global single-valued solutions. Solutions, which are single-valued in the neighborhood of one singular point, can be multivalued in the neighborhood of another singular point. So, we can only assume that global three-parameter solutions are single-valued. If we assume this and moreover that these solutions are elliptic functions (or some degenerations of them), then we can seek them as solutions of some polynomial first order equations. There are a few methods to construct such solutions [5, 10, 15, 22]. Using these methods one represents a solution of a nonlinear ordinary differential equation (ODE) as a polynomial or a rational function of elliptic functions or of degenerate elliptic functions, for example,  $\tanh(t)$ . These methods use results of the Painlevé test, but do not use the obtained Laurent-series solutions. In 2003 Conte and Musette [6] have proposed the method, which uses such solutions.

The classical theorem, which was established by Briot and Bouquet [2], proves that if the general solution of the polynomial autonomous first order ODE is single-valued, then this solution is either an elliptic function, or a rational function of  $e^{\gamma x}$ ,  $\gamma$  being some constant, or a rational function of  $x$ . Note that the third case is a degeneracy of the second one, which in its turn is a degeneracy of the first one. It has been proved by Painlevé [14] that the necessary form of the polynomial

autonomous first order ODE with the single-valued general solution is

$$\sum_{k=0}^m \sum_{j=0}^{2m-2k} h_{jk} y^j y_t^k = 0, \quad h_{0m} = 1 \quad (5)$$

in which  $m$  is a positive integer number and  $h_{jk}$  are constants.

Rather than to substitute equation (5) in some nonintegrable system, one can substitute the Laurent series of unknown special solutions in equation (5) and obtain a system, which is linear in  $A_{jk}$  and nonlinear in the parameters, including in the Laurent coefficients [6]. There are a few computer algebra algorithms which allow to obtain this system from the given Laurent series. Moreover, it is possible to exclude all  $A_{jk}$  from this system and to obtain a nonlinear system in parameters of nonintegrable system and free parameters from the Laurent series. The main advantage of this method is that the number of unknowns in the resulting nonlinear algebraic system does not depend on number of coefficients of the first order equation. For example, equation (6) with  $m = 8$  includes 60 unknowns  $A_{jk}$ , and it is not possible use the traditional way to find similar solutions. The first computer algebra realization of this algorithm has been written in AMP [8] by Conte. It is based on the  $\alpha$ -method of the Painlevé test. Our realization (in Maple) is based on transformations of the Laurent series [20].

## 4.2. Global Solutions for the Generalized Hénon–Heiles System

To show how the Laurent series solutions can assist to find elliptic ones let us consider equation (3).

To find a special solution of the given equation one can assume that  $y$  satisfies some more simple equation. For example, there exist solutions in terms of the Weierstrass elliptic functions, which satisfy the following first-order differential equation

$$y_t^2 = \tilde{A}y^3 + \tilde{B}y^2 + \tilde{C}y + \tilde{D}, \quad (6)$$

where  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$  and  $\tilde{D}$  are some constants.

Timoshkova [17] generalized equation (6)

$$y_t^2 = A_4 y^3 + A_3 y^{5/2} + A_2 y^2 + A_1 y^{3/2} + A_0 y + \tilde{D} \quad (7)$$

( $A_j$  are constants) and found new one-parameter solutions of the Hénon–Heiles system in two above-mentioned nonintegrable cases ( $C = -4/3$  or  $C = -16/5$ ). These solutions (i.e. solutions with  $A_3 \neq 0$  or  $A_1 \neq 0$ ) are derived only at  $\tilde{D} = 0$ , therefore, substitution  $y(t) = \varrho(t)^2$  gives

$$\varrho_t^2 = \frac{1}{4}(A_4 \varrho^4 + A_3 \varrho^3 + A_2 \varrho^2 + A_1 \varrho + A_0). \quad (8)$$

In [18] we use the substitution

$$y(t) = \varrho(t)^2 + P_0 \quad (9)$$

where  $P_0$  is a constant, and transform equation (3) into

$$\begin{aligned} \varrho_{tttt}\varrho = & -4\varrho_{ttt}\varrho_t - 3\varrho_{tt}^2 + 2(C-4)\varrho_{tt}\varrho^3 + (2P_0(C-4) - 4\lambda_1 - \lambda_2)\varrho_{tt}\varrho \\ & + 2(3C-2)\varrho_t^2\varrho^2 + (2CP_0 - 4\lambda_1 - 8P_0 - \lambda_2)\varrho_t^2 + \frac{10}{3}C\varrho^6 \\ & + (2C\lambda_1 + 10CP_0 - 3\lambda_2)\varrho^4 + 2(2\lambda_1CP_0 + 5CP_0^2 - \lambda_1\lambda_2 \\ & - 3P_0\lambda_2)\varrho^2 + \frac{10}{3}CP_0^3 + 2\lambda_1CP_0^2 - 3P_0^2\lambda_2 - 2\lambda_1\lambda_2P_0 - 2H. \end{aligned} \quad (10)$$

If  $\varrho(t)$  satisfies (8), then equation (10) is equivalent to the following system

$$\left\{ \begin{aligned} (3A_4 + 4)(-3A_4 + 2C) &= 0 \\ A_3(-21A_4 + 9C - 16) &= 0 \\ 96A_4CP_0 - 240A_4A_2 - 192A_4\lambda_1 - 384A_4P_0 - 48A_4\lambda_2 \\ &- 105A_3^2 + 128A_2C - 192A_2 + 128C\lambda_1 + 640CP_0 - 192\lambda_2 = 0 \\ 40A_3CP_0 - 90A_4A_1 - 65A_3A_2 - 80A_3\lambda_1 - 160A_3P_0 \\ &- 20A_3\lambda_2 + 56CA_1 - 64A_1 = 0 \\ 16A_2CP_0 - 36A_4A_0 - 21A_3A_1 - 8A_2^2 - 32A_2\lambda_1 \\ &- 64A_2P_0 - 8\lambda_2A_2 + 24CA_0 + 64\lambda_1CP_0 + 160CP_0^2 \\ &- 16A_0 - 32\lambda_1\lambda_2 - 96P_0\lambda_2 = 0 \\ 10A_3A_0 + (5A_2 + 8CP_0 - 16\lambda_1 - 32P_0 - 4\lambda_2)A_1 &= 0 \\ 384H = -48A_2A_0 + 96CA_0P_0 + 384C\lambda_1P_0^2 + 640CP_0^3 - 9A_1^2 \\ &- 192A_0\lambda_1 - 384A_0P_0 - 48A_0\lambda_2 - 384\lambda_1\lambda_2P_0 - 576\lambda_2P_0^2. \end{aligned} \right. \quad (11)$$

This system is nonlinear in the coefficients of equation (8). The Conte–Musette method allows to obtain the linear system in these coefficients. However we can not use this method for arbitrary  $C$ , because the Laurent series solutions are different for different  $C$ . So, first of all we have to fix value of  $C$ . If  $A_3 \neq 0$ , then from two first equations of system (9) we obtain

$$C = -\frac{4}{3} \quad \text{and} \quad A_4 = -\frac{4}{3} \quad \text{or} \quad C = -\frac{16}{5} \quad \text{and} \quad A_4 = -\frac{32}{15}.$$

Let us choose  $C = -\frac{4}{3}$  and construct the Laurent series solutions for equation (7). The values of resonances are  $-1, 1, 4$  and  $10$ , and we obtain the following Laurent

series, corresponding to  $\tilde{y}$  (functions  $\tilde{\rho}$  and  $-\tilde{\rho}$  correspond to one and the same  $\tilde{y}$ )

$$\tilde{\rho} = \frac{i\sqrt{3}}{t} + c_0 + \frac{i\sqrt{3}}{24}(3\lambda_2 - 2\lambda_1 + 4P_0 + 62c_0^2)t + \dots$$

where

$$c_0 = \pm \frac{\sqrt{161700\lambda_1 - 121275\lambda_2 \pm \sqrt{1155(5481\lambda_2^2 - 12768\lambda_1\lambda_2 + 8512\lambda_1^2)}}}{2310}.$$

The coefficients  $c_3$  and  $c_9$  are arbitrary.

The algorithm of the construction of elliptic solutions from the Laurent-series solutions is the following:

- 1) Choose a positive integer  $m$  and define the first order ODE (5), which contains unknown constants  $h_{jk}$ .
- 2) Choose one of possible values of  $c_0$  and compute coefficients of the Laurent series  $\tilde{\rho}$ . The number of coefficients has to be greater than the number of unknowns.
- 3) Substituting the obtained coefficients, transform equation (5) into a linear and overdetermined system in  $h_{jk}$  with coefficients depending on arbitrary parameters.
- 4) Exclude  $A_{jk}$  and obtain the nonlinear system in parameters.
- 5) Solve the obtained system.

On the first step we choose equation (16), which coincides with equation (9). It means that  $m = 2$ , all  $h_{j1}$  are equal to zero and all  $h_{j0} = -A_j/4$ . After the second and the third steps we obtain a linear system on  $A_{j0}$ . This system has the triangular form and linear in  $H$ ,  $c_3$  and  $c_9$  as well. From the first equation we obtain,

$$A_4 = -4/3.$$

After we choose  $c_0$ , for example,

$$c_0 = \frac{\sqrt{1155}\sqrt{140\lambda_1 - 105\lambda_2 - \sqrt{7(783\lambda_2^2 - 1824\lambda_1\lambda_2 + 1216\lambda_1^2)}}}{2310}$$

and obtain from the second equation

$$A_3 = \frac{8\sqrt{1155}}{3465}\sqrt{140\lambda_1 - 105\lambda_2 - \sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)}}$$



and so on:

$$A_2 = \frac{1}{66} \sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2) - \frac{4}{33}\lambda_1 - \frac{31}{22}\lambda_2 - 4P_0}$$

$$A_1 = -\frac{\sqrt{165}}{15246} \sqrt{140\lambda_1 - 105\lambda_2 - \sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)}} \\ \times \left( 13\sqrt{1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2} + \sqrt{7}(136\lambda_1 - 135\lambda_2 - 88P_0) \right).$$

Then we determine the arbitrary parameter  $c_3$ ,

$$A_0 = \frac{1}{2904(-\sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)} + 140\lambda_1 - 105\lambda_2)} \\ \times \left[ (17768\lambda_1^2 - 16653\lambda_1\lambda_2 + 4005\lambda_2^2 + 26664\lambda_1P_0 - 7656\lambda_2P_0 + 16456P_0^2) \right. \\ \times \sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)} + 1070144\lambda_1^3 \\ - 2592360\lambda_1^2\lambda_2 + 1648953\lambda_1\lambda_2^2 - 305937\lambda_2^3 - 492096\lambda_1^2P_0 \\ \left. - 989736\lambda_1\lambda_2P_0 + 816552\lambda_2^2P_0 - 2303840\lambda_1P_0^2 + 1727880\lambda_2P_0^2 \right]$$

and  $H$ .

The remaining equations are satisfied for all values of  $\lambda_1$ ,  $\lambda_2$  and  $P_0$ , so we do not need to solve the nonlinear system. Of course, we can substitute in (5) only a finite number of the Laurent series coefficients, so, we have to check in addition, that equation (8) with obtained values of  $A_j$  gives a solution of (2). The simplest way to do this is to substitute the obtained values of  $A_j$  in system (11). We find a solution of the nonlinear algebraic system solving only linear equations. In principle, any finite system of nonlinear algebraic equations can be solved, using Buchberger algorithm [3, 7] for the Gröbner basis construction. This algorithm has been realized in the standard procedures `solve()` of the computer algebra systems Maple, Mathematica and REDUCE. It diagonalizes the system of nonlinear equations, by constructing the Gröbner basis, i.e. an equivalent system, which consists of an equation in only one variable (except arbitrary parameters, which can not be fixed by the system of equations), an equation in two variables and so on. Therefore, we can obtain solutions of the system, solving only equations in one variable and substitute the obtaining result in the following equations, and so on. The Gröbner basis contains only equations, which are polynomial in all variables, so the obtained equations, which are nonpolynomial in  $\lambda_1$  and  $\lambda_2$ , do not belong to the Gröbner basis of system (11). The operating memory and other parameters of real computers are not enough to solve difficult systems using the Gröbner basis method. The considered method allows to obtain solutions only for fixed values of

$C$ , but is more simple. The Conte–Musette method has also the following preference that one does not need transform system (2) in one differential equation either in  $y$  or in  $x$ . Moreover at  $C = -16/5$  not  $x$ , but  $x^2$  may be an elliptic function. To construct the Laurent series for  $x^2$  is easier than to find the fourth order equation in  $x^2$ . The obtained solution has the form

$$y(t - t_0) = \left( \frac{a\wp(t - t_0) + b}{c\wp(t - t_0) + d} \right)^2 + P_0 \quad (12)$$

where  $\wp(t - t_0)$  is the Weierstrass elliptic function,  $a, b, c$  and  $d$  are some constants. The parameters  $P_0$ , which defines the energy of the system, and  $t_0$  are arbitrary. Solutions of this type exist in both above-mentioned nonintegrable cases  $C = -16/5$  and  $C = -4/3$ . Full list of solutions see in [18].

The function  $y$  satisfies the following equation

$$(y_t^2 - A_4(y - P_0)^3 - A_2(y - P_0)^2 - A_0(y - P_0))^2 = (y - P_0)^3(A_3(y - P_0) + A_2)^2.$$

Surely, we can use the Conte–Musette method without the change of variables (9), but this change allows us to simplify calculations.

The function  $x(t)$  satisfies the first equation of system (2) with

$$\begin{aligned} \mu = & \frac{8}{3}C^2P_0^5 + \left(2\lambda_1C^2 - \frac{14}{3}\lambda_2C\right)P_0^4 + \left(2\lambda_2^2 - \frac{10}{3}CA_0 - 4\lambda_1\lambda_2C\right)P_0^3 \\ & + \left(2\lambda_1\lambda_2^2 - 2\lambda_1CA_0 - 4CH + 3\lambda_2A_0\right)P_0^2 + \left(2\lambda_1\lambda_2A_0 + A_0^2 + 4\lambda_2H\right)P_0 \\ & + 2A_0H + \frac{1}{2}\lambda_1A_0^2 + \frac{9}{128}A_1^2A_0. \end{aligned}$$

The trajectory of the motion can be derived from the second equation of system (2). Substituting  $y_{tt}$ , we obtain

$$\begin{aligned} x^2 = & \left(C - \frac{3}{2}A_4\right)y^2 + (3A_4P_0 - A_2 - 1)y - \frac{1}{4}(5A_1y + 3A_1 - 5A_3P_0)(y - P_0)^{1/2} \\ & - \frac{1}{2}(A_0 + 3A_4P_0^2 - 2A_2P_0). \end{aligned}$$

## 5. Multivalued Solutions

Let us generalize the given method on the search of solutions, which can be expanded in the following Puiseux series

$$y = \sum_{k=-L}^{\infty} S_{k/q} t^{k/q}. \quad (13)$$

We seek solutions as a polynomial

$$y = P_L \rho^L + P_{L-1} \rho^{L-1} + P_{L-2} \rho^{L-2} + P_{L-3} \rho^{L-3} + \dots + P_0.$$

From (13) it follows that

$$\rho = \sum_{j=-1}^{\infty} T_{j/q} t^{j/q}$$

where  $T_{j/q}$  are some constants.

We seek  $\rho(t)$ , which satisfies the following equation

$$\sum_{k=0}^m \sum_{j=0}^{(q+1)(m-k)} h_{jk} \rho^j \rho_t^k = 0, \quad h_{0m} = 1 \quad (14)$$

where  $q$  is a natural number.

To simplify calculations we can put  $P_L = 1$  and  $P_{L-1} = 0$  without the loss of generality, because

$$\tilde{\rho} = \left( \rho - \frac{P_{n-1}}{n} \right) / \sqrt[n]{P_n}$$

satisfies (14) as well. This generalization has been automatized in Maple [21].

## 6. Conclusions

Two nonintegrable cases ( $C = -16/5$  or  $C = -4/3$ ,  $\lambda_1$ ,  $\lambda_2$  and  $\mu$  are arbitrary) of the generalized Hénon–Heiles system with the nonpolynomial term have been considered. Two-parameter elliptic solutions for this system are known in both above-mentioned cases. Two different methods for the search of such solutions have been compared in this paper. It has been demonstrated how the knowledge of the Laurent series solutions of the initial differential equation assists to linearize the obtained nonlinear system of algebraic equations. The Painlevé test does not show any obstacle to the existence of three-parameter single-valued solutions, so, the probability to find exact, for example elliptic, three-parameter solutions, that generalize the obtained solutions, is high. The knowledge of the Laurent series can assist to find such solutions. The Painlevé test is useful to find not only single-valued solution, but also some type of multivalued solutions in the analytic form. The corresponding computer algebra algorithm has been constructed in Maple and REDUCE.

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